

LEVEL TOTIENTS FOR INTEGER MOSAICS

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Abstract

The mosaic of a positive integer n, also called its prime tower factorization, is obtained by factoring n into prime powers and then factoring the exponents recursively. The primes appearing at the rth iteration of this process form level r. Let $P_i(n)$ be the set of primes appearing in the first i levels of the mosaic of n, and let $\varphi_i(n)$ count the integers $m \leq n$ for which $P_i(m) \cap P_i(n) = \varnothing$. For $i = 1$, this is Euler's totient function. We prove a Möbius divisor-sum formula for $\varphi_i(n)$, reduce it to a sum over a certain set of powerful integers, and count the terms of the obtained sum. For a finite set of primes S, the density $\delta_i(S)$ of integers whose first i levels avoid S exists and has an Euler product. Taking $S = P_i(n)$ gives

$$\frac{\varphi_i(n)}{n} = \delta_i(P_i(n)) + O\left(n^{-1/2} \exp\left(\frac{c\sqrt{\log n}}{\log\log n}\right)\right)$$

for $n \geq 16$, where c is an absolute constant.

1. Introduction

An *integer mosaic* is obtained by repeatedly applying the fundamental theorem of arithmetic to the exponents in the prime factorization of a positive integer. Mullin

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introduced this construction in the 1960s [11, 12]. Subsequent work studied arithmetic functions on mosaics, generalized multiplicativity, and mosaic divisors [2, 13]. The same recursive construction is also called the *prime tower factorization* in recent work on height, prime occurrence, and tree shape [3, 4, 5, 6].

If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the canonical prime factorization of $n > 1$, then the mosaic of n is obtained by writing

$$\mathcal{M}(1) = 1, \qquad \mathcal{M}(n) = \prod_{j=1}^{k} p_j^{\mathcal{M}(\alpha_j)},$$

and factoring the exponents recursively. Here $\mathcal{M}(n)$ denotes the unevaluated product produced by this recursion rather than its numerical value, which is n itself. For example,

$$995328 = 2^{12}3^5 \quad \text{gives} \quad \mathcal{M}(995328) = 2^{\mathcal{M}(12)}3^{\mathcal{M}(5)} = 2^{2^2 3}3^5.$$

The first level consists of the ordinary prime divisors of n, the second level consists of the prime divisors of the exponents, and later levels are obtained by continuing this process. We write $P_i(n)$ for the set of primes appearing in the first i levels. Thus, $P_1(n)$ is the usual set of prime divisors, while $P_2(n)$ also records primes dividing the exponents.

Gillman introduced the notion of i-relative primality [8, 9]. Two positive integers a and b are i-relatively prime if $P_i(a) \cap P_i(b) = \varnothing$. The associated level totient is

$$\varphi_i(n) = \#\{m \le n : P_i(m) \cap P_i(n) = \varnothing\}.$$

For $i = 1$, this is Euler's totient function. For $i \ge 2$, the condition also uses primes that occur in the exponents. For instance,

$$P_2(8) = P_2(2^3) = \{2, 3\}, \qquad P_2(9) = P_2(3^2) = \{2, 3\},$$

so 8 and 9 are ordinarily coprime but not 2-relatively prime. Thus, the level totient differs from Euler's function already at level 2.

Bildhauser, Erickson, Tacoma, and Gillman introduced the functions φ_i in their study of integer mosaics and asked for a method to compute them [2]. The rest of this introduction states our results and records the formulas that they provide. In Sections 2 through 8, we state the definitions formally and prove all the statements.

We compute $\varphi_i(n)$ by introducing the indicator

$$\chi_{i,S}(m) = \mathbf{1}\{P_i(m) \cap S = \varnothing\},$$

where S is a finite set of primes. Since $P_i(ab) = P_i(a) \cup P_i(b)$ whenever $(a, b) = 1$, the function $\chi_{i,S}$ is multiplicative. Let

$$h_{i,S} = \chi_{i,S} * \mu,$$

where $*$ denotes Dirichlet convolution and μ is the Möbius function. Möbius inversion gives $\chi_{i,S} = h_{i,S} * \mathbf{1}$, where $\mathbf{1}$ is the constant function $\mathbf{1}(x) = 1$. Equivalently, $\chi_{i,S}(m) = \sum_{d|m} h_{i,S}(d)$. Taking $S = P_i(n)$ and summing over $m \le n$ gives

$$\varphi_i(n) = \sum_{m=1}^{n} \chi_{i,P_i(n)}(m) = \sum_{d=1}^{n} h_{i,P_i(n)}(d) \left\lfloor \frac{n}{d} \right\rfloor.$$

We next note that the prime-power values of $h_{i,S}$ are explicit. If $p \in S$, then

$$h_{i,S}(p) = -1, \qquad h_{i,S}(p^a) = 0 \quad (a \ge 2).$$

If $p \notin S$, then $h_{i,S}(p) = 0$, while for $a \ge 2$,

$$h_{i,S}(p^a) = \chi_{i-1,S}(a) - \chi_{i-1,S}(a-1).$$

Thus, primes in S that contribute nonzero terms occur only to the first power, while primes outside S can occur only with exponents $a \ge 2$ at which $\chi_{i-1,S}(a)$ and $\chi_{i-1,S}(a-1)$ differ.

This motivates the definition

$$E_{i,S} = \{a \ge 2 : \chi_{i-1,S}(a) \ne \chi_{i-1,S}(a-1)\}.$$

We define $\mathcal{V}_{i,S}$ to be the set of integers all of whose prime factors lie outside S, and whose prime exponents lie in $E_{i,S}$. Every element of $\mathcal{V}_{i,S}$ is powerful. After the contribution from primes in S is summed separately, the divisor formula becomes a sum over $v \in \mathcal{V}_{i,S}$. Write $R_S = \prod_{p \in S} p$ and

$$\Phi_S(x) = \sum_{u | R_S} \mu(u) \left\lfloor \frac{x}{u} \right\rfloor$$

for the number of positive integers at most x with no prime factor in S. For $S = P_i(n)$, Theorem 3 states that

$$\varphi_i(n) = \sum_{\substack{v \in \mathcal{V}_{i,S} \\ v \le n}} h_{i,S}(v) \Phi_S\left(\left\lfloor \frac{n}{v} \right\rfloor\right).$$

When $S \ne \varnothing$ and $i \ge 2$, the least element of $E_{i,S}$ is $q = \min S$. We prove the asymptotic $|\mathcal{V}_{i,S} \cap [1,x]| \sim C_{i,S} x^{1/q}$, where $C_{i,S} > 0$ is given by an explicit Euler product.

For fixed S, we prove that the density

$$\delta_i(S) = \lim_{N \to \infty} \frac{1}{N} \#\{m \le N : P_i(m) \cap S = \varnothing\}$$

exists and has an Euler product. The number of integers up to N whose first i levels avoid S equals

$$\delta_i(S) N + O_{i,S}\left(N^{1/q}\right).$$

Because $P_i(n)$ varies with n, the estimate above does not give a bound uniform in n. Using the bound $\prod_{p\in P_i(n)} p \le n$ gives the uniform estimate

$$\frac{\varphi_i(n)}{n} = \delta_i(P_i(n)) + O\left(n^{-1/2}\exp\left(\frac{c\sqrt{\log n}}{\log\log n}\right)\right)$$

for $n \ge 16$, where c is an absolute constant.

The paper is organized as follows. Section 2 records the level notation and proves multiplicativity of the fixed-set indicator. Section 3 proves the Möbius formula. Section 4 separates the squarefree part in S from the powerful part outside S. Section 5 studies the sets $E_{i,S}$. Section 6 counts the integers in $\mathcal{V}_{i,S}$. Section 7 proves the fixed-set density theorem, and Section 8 applies it to $S = P_i(n)$.

2. Mosaic Levels and i-Relative Primality

Throughout the paper, S denotes a finite set of primes, and i denotes a nonnegative integer unless otherwise stated.

Definition 1. Let $n > 1$ have canonical prime factorization $n = \prod_{j=1}^{k} p_j^{\alpha_j}$. The sets $\Lambda_r(n)$ of primes appearing at level r of the mosaic of n are defined recursively by $\Lambda_0(n) = \varnothing$, $\Lambda_1(n) = \{p_1, \dots, p_k\}$, and, for $r \ge 2$,

$$\Lambda_r(n) = \bigcup_{j=1}^{k} \Lambda_{r-1}(\alpha_j).$$

We also set $\Lambda_r(1) = \varnothing$ for every $r \ge 0$.

Example 1. For $n = 995328 = 2^{12}3^5 = 2^{2^2}3^5$, the first three levels are

$$\Lambda_1(n) = \{2, 3\}, \qquad \Lambda_2(n) = \{2, 3, 5\}, \qquad \Lambda_3(n) = \{2\}.$$

Definition 2. For $n \in \mathbb{Z}^+$, let $P_i(n)$ be the set of primes appearing in the first i levels of the mosaic of n:

$$P_i(n) = \bigcup_{r=1}^{i} \Lambda_r(n).$$

Equivalently, $P_0(n) = \varnothing$, $P_i(1) = \varnothing$, and for $i \ge 1$,

$$P_i(n) = \{p_1, \dots, p_k\} \cup \bigcup_{j=1}^{k} P_{i-1}(\alpha_j),$$

where $n = \prod_{j=1}^{k} p_j^{\alpha_j}$ is the canonical prime factorization of $n > 1$.

Definition 3. Two positive integers a and b are called *i-relatively prime* if

$$P_i(a) \cap P_i(b) = \varnothing.$$

The next lemma records how P_i behaves on products of coprime integers.

Lemma 1. *If* $\gcd(a, b) = 1$*, then* $P_i(ab) = P_i(a) \cup P_i(b)$ *for every* $i \geq 0$.

Proof. For $i = 0$, both sides are empty. Assume $i \geq 1$, and write $a = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and $b = q_1^{\beta_1} \cdots q_r^{\beta_r}$ in canonical prime factorization. Since $\gcd(a, b) = 1$, the primes p_j and q_ℓ are distinct for all j and ℓ, so

$$ab = \prod_{j=1}^{k} p_j^{\alpha_j} \prod_{\ell=1}^{r} q_\ell^{\beta_\ell}$$

is the canonical factorization of ab. By Definition 2,

$$\begin{aligned} P_i(ab) &= \{p_1, \ldots, p_k, q_1, \ldots, q_r\} \cup \bigcup_{j=1}^{k} P_{i-1}(\alpha_j) \cup \bigcup_{\ell=1}^{r} P_{i-1}(\beta_\ell) \\ &= P_i(a) \cup P_i(b). \end{aligned}$$

□

Definition 4. For a finite set of primes S, define

$$\chi_{i,S}(m) = \begin{cases} 1, & P_i(m) \cap S = \varnothing, \\ 0, & P_i(m) \cap S \neq \varnothing. \end{cases}$$

Lemma 2. *For every finite set of primes* S*, the function* $\chi_{i,S}$ *is multiplicative.*

Proof. Suppose $\gcd(a, b) = 1$. By Lemma 1,

$$P_i(ab) \cap S = \big(P_i(a) \cap S\big) \cup \big(P_i(b) \cap S\big).$$

Thus, $P_i(ab) \cap S = \varnothing$ if and only if both $P_i(a) \cap S$ and $P_i(b) \cap S$ are empty. Hence, $\chi_{i,S}(ab) = \chi_{i,S}(a)\chi_{i,S}(b)$. Also $\chi_{i,S}(1) = 1$, since $P_i(1) = \varnothing$. □

Lemma 3. *For* $i \geq 1$*, prime* p*, and integer* $k \geq 1$*,*

$$\chi_{i,S}(p^k) = \begin{cases} 0, & p \in S, \\ \chi_{i-1,S}(k), & p \notin S. \end{cases}$$

Proof. By Definition 2,

$$P_i(p^k) = \{p\} \cup P_{i-1}(k).$$

If $p \in S$, then $P_i(p^k) \cap S \neq \varnothing$, so $\chi_{i,S}(p^k) = 0$. If $p \notin S$, then

$$P_i(p^k) \cap S = P_{i-1}(k) \cap S,$$

and the claim follows. □

3. A Möbius Formula for the Level Totient

We now define the i-level totient function $\varphi_i(n)$ and write it as a divisor sum. Since the fixed-set indicator $\chi_{i,S}$ is multiplicative, its Möbius transform has prime-power values that can be computed explicitly. The rest of the paper studies the nonzero terms of this transform.

Definition 5. For $n \in \mathbb{Z}^+$ and $i \geq 0$, define the *i-level totient* by

$$\varphi_i(n) = \sum_{m=1}^{n} \chi_{i,P_i(n)}(m) = \sum_{m \leq n} \mathbf{1}\{P_i(m) \cap P_i(n) = \varnothing\}.$$

For $i = 1$, this is Euler's totient function. For $i \geq 2$, it need not be multiplicative as a function of n. For example, one computes $\varphi_i(72) = 22$, while $\varphi_i(8)\varphi_i(9) = 9$ for every $i \geq 2$.

Definition 6. Let $h_{i,S} = \chi_{i,S} * \mu$, where $*$ denotes Dirichlet convolution and μ is the ordinary Möbius function. Equivalently,

$$h_{i,S}(n) = \sum_{d|n} \chi_{i,S}(d)\mu\left(\frac{n}{d}\right).$$

As the Dirichlet convolution of multiplicative functions $\chi_{i,S}$ and μ, the function $h_{i,S}$ is multiplicative.

Theorem 1. *For every $n \in \mathbb{Z}^+$ and $i \geq 1$,*

$$\varphi_i(n) = \sum_{d=1}^{n} h_{i,P_i(n)}(d) \left\lfloor \frac{n}{d} \right\rfloor.$$

Proof. Let $S = P_i(n)$. Since $h_{i,S} = \chi_{i,S} * \mu$, Möbius inversion gives $\chi_{i,S} = h_{i,S} * \mathbf{1}$, where $\mathbf{1}$ is the constant function $\mathbf{1}(x) = 1$. Hence,

$$\chi_{i,S}(m) = \sum_{d|m} h_{i,S}(d).$$

Substituting into Definition 5 gives

$$\varphi_i(n) = \sum_{m \leq n} \sum_{d|m} h_{i,S}(d).$$

Reindexing by writing $m = dk$ yields

$$\varphi_i(n) = \sum_{d \leq n} h_{i,S}(d) \sum_{k \leq n/d} 1 = \sum_{d \leq n} h_{i,S}(d) \left\lfloor \frac{n}{d} \right\rfloor. \qquad \square$$

We finish this section by proving a property of $h_{i,S}$ on prime powers that will be used for the reduction of the Möbius sum in Section 4.

Lemma 4. *For $i \geq 1$, prime p, and integer $k \geq 1$,*

$$h_{i,S}(p^k) = \begin{cases} -1, & p \in S \text{ and } k = 1, \\ 0, & p \in S \text{ and } k \geq 2, \\ 0, & p \notin S \text{ and } k = 1, \\ \chi_{i-1,S}(k) - \chi_{i-1,S}(k-1), & p \notin S \text{ and } k \geq 2. \end{cases}$$

Proof. By definition,

$$h_{i,S}(p^k) = \sum_{j=0}^{k} \chi_{i,S}(p^j)\mu(p^{k-j}).$$

Since $\mu(p^r) = 0$ for $r \geq 2$, only the terms $j = k$ and $j = k-1$ can contribute. Thus,

$$h_{i,S}(p^k) = \chi_{i,S}(p^k) - \chi_{i,S}(p^{k-1}).$$

For $k = 1$, this is $\chi_{i,S}(p) - 1$, which equals -1 if $p \in S$ and 0 otherwise. For $k \geq 2$, Lemma 3 gives 0 when $p \in S$, and gives

$$\chi_{i-1,S}(k) - \chi_{i-1,S}(k-1)$$

when $p \notin S$. □

4. Reduction of the Möbius Sum

We now identify the nonzero terms in the Möbius formula. By Lemma 4 the primes in S that contribute nonzero terms occur only to the first power, while primes outside S can occur only with exponents $a \geq 2$ at which $\chi_{i-1,S}(a)$ and $\chi_{i-1,S}(a-1)$ differ. We begin by recalling the notion of a powerful integer.

Definition 7. A positive integer v is *powerful* if $p^2 \mid v$ for every prime $p \mid v$. By convention, 1 is powerful. We denote the set of all powerful integers by $\mathbb{P}$.

Since the prime-power formula in Lemma 4 depends only on the difference between consecutive values of $\chi_{i-1,S}$ for primes outside of S, we isolate this difference through the following notation.

For $i \geq 1$ and $k \geq 2$, define

$$\Delta_{i,S}(k) = \chi_{i-1,S}(k) - \chi_{i-1,S}(k-1),$$

and set

$$E_{i,S} = \{k \geq 2 : \Delta_{i,S}(k) \neq 0\}.$$

By Lemma 4, if $p \notin S$ and $k \geq 2$, then $h_{i,S}(p^k) = \Delta_{i,S}(k)$.

Definition 8. For $i \geq 1$ and a finite set of primes S, let $\mathcal{V}_{i,S}$ be the set of positive integers v whose canonical prime factorization $v = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ satisfies $p_j \notin S$ and $\alpha_j \in E_{i,S}$ for every j. We include 1 in $\mathcal{V}_{i,S}$.

Note that $\mathcal{V}_{i,S} \subseteq \mathbb{P}$, since $E_{i,S} \subseteq \{2, 3, 4, \ldots\}$.

Theorem 2. *Let $i \geq 1$ and let S be a finite set of primes. If $h_{i,S}(d) \neq 0$, then d factors uniquely as $d = uv$, where $u \mid R_S$, $R_S = \prod_{p \in S} p$, and $v \in \mathcal{V}_{i,S}$. Moreover,*

$$h_{i,S}(d) = \mu(u) h_{i,S}(v).$$

Conversely, if $u \mid R_S$, $v \in \mathcal{V}_{i,S}$, and $\gcd(u, v) = 1$, then $h_{i,S}(uv) = \mu(u) h_{i,S}(v)$ and this value is nonzero.

Proof. Write $d = \prod_{p \mid d} p^{k_p}$. Since $h_{i,S}$ is multiplicative,

$$h_{i,S}(d) = \prod_{p \mid d} h_{i,S}(p^{k_p}).$$

If $h_{i,S}(d) \neq 0$, then every local factor is nonzero. By Lemma 4, this forces $k_p = 1$ for $p \in S$, and $k_p \in E_{i,S}$ for $p \notin S$. Hence, with

$$u = \prod_{\substack{p \mid d \\ p \in S}} p, \qquad v = \prod_{\substack{p \mid d \\ p \notin S}} p^{k_p},$$

we have $u \mid R_S$, $v \in \mathcal{V}_{i,S}$, and $d = uv$. The factorization is unique because it separates the prime divisors of d according to membership in S.

Since $\gcd(u, v) = 1$, multiplicativity gives $h_{i,S}(d) = h_{i,S}(u) h_{i,S}(v)$. The integer u is squarefree and all of its prime divisors lie in S, so Lemma 4 gives

$$h_{i,S}(u) = \prod_{p \mid u} (-1) = \mu(u).$$

This proves the first assertion. The converse follows from the same local factor calculation: if $u \mid R_S$ and $v \in \mathcal{V}_{i,S}$, then every local factor in $h_{i,S}(uv)$ is nonzero, and multiplicativity gives the displayed formula. □

Definition 9. For a finite set of primes S, define

$$\Phi_S(x) = \sum_{u \mid R_S} \mu(u) \left\lfloor \frac{x}{u} \right\rfloor, \qquad R_S = \prod_{p \in S} p.$$

Equivalently, $\Phi_S(x)$ is the number of positive integers at most x with no prime factor in S.

Theorem 3. *For every $n \in \mathbb{Z}^+$ and $i \geq 1$, writing $S = P_i(n)$, we have*

$$\varphi_i(n) = \sum_{\substack{v \in \mathcal{V}_{i,S} \\ v \leq n}} h_{i,S}(v) \Phi_S\left(\left\lfloor \frac{n}{v} \right\rfloor\right).$$

Proof. By Theorem 1,

$$\varphi_i(n) = \sum_{d \leq n} h_{i,S}(d) \left\lfloor \frac{n}{d} \right\rfloor.$$

By Theorem 2, each nonzero term factors uniquely as $d = uv$ with $u \mid R_S$, $v \in \mathcal{V}_{i,S}$, and $h_{i,S}(uv) = \mu(u) h_{i,S}(v)$, so

$$\varphi_i(n) = \sum_{\substack{v \in \mathcal{V}_{i,S} \\ v \leq n}} h_{i,S}(v) \sum_{\substack{u \mid R_S \\ uv \leq n}} \mu(u) \left\lfloor \frac{n}{uv} \right\rfloor.$$

For fixed v, the terms with $uv > n$ are zero, and $\lfloor n/(uv) \rfloor = \lfloor \lfloor n/v \rfloor / u \rfloor$, so the inner sum equals

$$\sum_{\substack{u \mid R_S \\ uv \leq n}} \mu(u) \left\lfloor \frac{n}{uv} \right\rfloor = \sum_{u \mid R_S} \mu(u) \left\lfloor \frac{\lfloor n/v \rfloor}{u} \right\rfloor = \Phi_S\left(\left\lfloor \frac{n}{v} \right\rfloor\right). \qquad \square$$

5. The Exponent Sets

The reduced formula depends on the exponent set $E_{i,S}$. We prove two structural results: the first identifies the smallest possible exponent when $S \neq \varnothing$ and $i \geq 2$, and the second describes $E_{2,S}$ through congruence conditions modulo $R_S = \prod_{p \in S} p$.

Proposition 1. *For $S \neq \varnothing$ and $i \geq 2$, the smallest element of $E_{i,S}$ is $q = \min S$.*

Proof. We first prove, by induction on j, that $P_j(m) \cap S = \varnothing$ for every positive integer $m < q$.

The case $j = 0$ is immediate from $P_0(m) = \varnothing$. Suppose the claim holds for $j = a$. Let $m < q$ and write $m = \prod_{\ell=1}^{r} p_\ell^{\alpha_\ell}$ in canonical prime factorization. Each prime divisor p_ℓ satisfies $p_\ell < q$, so $p_\ell \notin S$. Also, since $p_\ell^{\alpha_\ell} \leq m < q$, we have $\alpha_\ell < q$. By the inductive hypothesis, $P_a(\alpha_\ell) \cap S = \varnothing$ for every ℓ. Hence, by Definition 2,

$$P_{a+1}(m) \cap S = \left(\{p_1, \ldots, p_r\} \cup \bigcup_{\ell=1}^{r} P_a(\alpha_\ell) \right) \cap S = \varnothing.$$

Now we show $q = \min E_{i,S}$. Suppose $2 \leq t < q$. Since both t and $t-1$ are less than q, the induction claim gives $\chi_{i-1,S}(t) = \chi_{i-1,S}(t-1) = 1$. Hence, $\Delta_{i,S}(t) = 0$, so

no t with $2 \le t < q$ lies in $E_{i,S}$. It remains to show that q lies in $E_{i,S}$. Since $i \ge 2$, Definition 2 gives

$$P_{i-1}(q) = \{q\} \cup P_{i-2}(1) = \{q\}.$$

Thus, $\chi_{i-1,S}(q) = 0$. Since $q - 1 < q$, the induction claim gives $\chi_{i-1,S}(q-1) = 1$. Therefore,

$$\Delta_{i,S}(q) = \chi_{i-1,S}(q) - \chi_{i-1,S}(q-1) = -1 \neq 0.$$

Thus, $q \in E_{i,S}$, while no smaller integer lies in $E_{i,S}$. □

Theorem 4. *For $k \ge 2$, we have $k \in E_{2,S}$ if and only if exactly one of k and $k-1$ is relatively prime to R_S. Equivalently,*

$$E_{2,S} = \{k \ge 2 : \mathbf{1}\{\gcd(k, R_S) = 1\} \neq \mathbf{1}\{\gcd(k-1, R_S) = 1\}\}.$$

In particular, $E_{2,S}$ is eventually periodic modulo R_S.

Proof. For $i = 2$, we have $\Delta_{2,S}(k) = \chi_{1,S}(k) - \chi_{1,S}(k-1)$. Since $P_1(m)$ is the set of prime divisors of m, we have $\chi_{1,S}(m) = 1$ if and only if $\gcd(m, R_S) = 1$. Therefore, $\Delta_{2,S}(k) \neq 0$ exactly when the two conditions

$$\gcd(k, R_S) = 1 \qquad \text{and} \qquad \gcd(k-1, R_S) = 1$$

have different truth values. This proves the characterization. Since both conditions depend only on k modulo R_S, the set $E_{2,S}$ is periodic modulo R_S apart from the initial restriction $k \ge 2$. □

Corollary 1. *The natural density of $E_{2,S}$ is*

$$\frac{2\left(\prod_{p\in S}(p-1) - \prod_{p\in S}(p-2)\right)}{\prod_{p\in S} p}.$$

Proof. By Theorem 4, we count residue classes a mod R_S for which exactly one of a and $a-1$ is relatively prime to R_S.

The number of classes a mod R_S with $\gcd(a, R_S) = 1$ is $\varphi(R_S) = \prod_{p\in S}(p-1)$. The translation $a \mapsto a - 1$ permutes the residue classes modulo R_S, so the number of classes with $\gcd(a-1, R_S) = 1$ is also $\varphi(R_S)$. The number of classes for which both a and $a-1$ are relatively prime to R_S is

$$\prod_{p\in S}(p-2),$$

because modulo each $p \in S$ the class a mod p must avoid both 0 and 1. Hence, the number of classes for which a is relatively prime to R_S but $a-1$ is not, and the number for which $a-1$ is relatively prime to R_S but a is not, both equal

$$\prod_{p\in S}(p-1) - \prod_{p\in S}(p-2).$$

Thus, the total number of admissible classes is

$$2\left(\prod_{p\in S}(p-1)-\prod_{p\in S}(p-2)\right).$$

Dividing by $R_S=\prod_{p\in S}p$ gives the density. □

Example 2. If $S=\{2\}$, then $R_S=2$, and $\chi_{1,S}(k)=1$ exactly when k is odd. Hence, $\chi_{1,S}(k)$ differs for consecutive integers, so

$$E_{2,\{2\}}=\{2,3,4,\ldots\}.$$

Thus, $\mathcal{V}_{2,\{2\}}$ is the set of odd powerful integers.

6. Counting the Reduced Sum

The reduced formula of Theorem 3 sums over $\mathcal{V}_{i,S}$ in place of all divisors $d\le n$, so the number of its terms is $|\mathcal{V}_{i,S}\cap[1,n]|$. We estimate this count. Every element of $\mathcal{V}_{i,S}$ is q-full, which already gives the order of magnitude $x^{1/q}$. We then sharpen it to the asymptotic $C_{i,S}x^{1/q}$ with an explicit positive constant.

Proposition 2. *For $S\neq\varnothing$, $q=\min S$, and $i\ge 2$,*

$$|\mathcal{V}_{i,S}\cap[1,x]|=O_{i,S}(x^{1/q}).$$

Proof. By Proposition 1, every exponent appearing in an element of $\mathcal{V}_{i,S}$ is at least q. Thus, each element of $\mathcal{V}_{i,S}$ is q-full, and the number of q-full integers up to x is $O_q(x^{1/q})$ by the Erdős–Szekeres estimate [7]. The claim follows. □

We now record the density of squarefree integers with no prime factor in S. For $S=\varnothing$ this is the classical estimate $\sum_{m\le y}\mu^2(m)\sim y/\zeta(2)$, proved in the book of Tenenbaum [15, Section I.3]. We state the general case in the form given by Sanna [14, Lemma 2.4].

Lemma 5. *Let S be a finite set of primes and let $Q_S(y)$ denote the number of squarefree integers at most y that are relatively prime to R_S. Then*

$$Q_S(y)\sim\kappa_S y,\qquad \kappa_S=\frac{1}{\zeta(2)}\prod_{p\in S}\left(1+\frac{1}{p}\right)^{-1}.$$

Proof. Applying Sanna's lemma [14, Lemma 2.4] with $h=R_S$ gives

$$Q_S(y)\sim\frac{6}{\pi^2}\frac{R_S}{\psi(R_S)}y.$$

Since $6/\pi^2=1/\zeta(2)$ and $\psi(R_S)=R_S\prod_{p\in S}(1+1/p)$ for the Dedekind function ψ, the main term is $\kappa_S y$. □

To obtain the leading constant of the asymptotic, we isolate the contribution of the least exponent q in the Dirichlet series of $\mathcal{V}_{i,S}$. For real $s > 1/q$, define

$$F_{i,S}(s) = \sum_{v \in \mathcal{V}_{i,S}} \frac{1}{v^s} = \prod_{p \notin S} \left(1 + \sum_{k \in E_{i,S}} \frac{1}{p^{ks}}\right).$$

Since $q = \min E_{i,S}$ by Proposition 1, the factor $1 + p^{-qs}$ occurs at each prime $p \notin S$. We write

$$F_{i,S}(s) = B_S(s) H_{i,S}(s),$$

where

$$B_S(s) = \prod_{p \notin S} (1 + p^{-qs})$$

and

$$H_{i,S}(s) = \prod_{p \notin S} \frac{1 + \sum_{k \in E_{i,S}} p^{-ks}}{1 + p^{-qs}}.$$

The factor $B_S(s)$ is the Dirichlet series for qth powers of squarefree integers with no prime factor in S. Thus, the squarefree asymptotic in Lemma 5 supplies the main growth term, while $H_{i,S}(s)$ contributes to the leading constant.

Theorem 5. *Let $S \neq \varnothing$, let $i \geq 2$, and let $q = \min S$. Then*

$$|\mathcal{V}_{i,S} \cap [1, x]| \sim C_{i,S} x^{1/q},$$

where

$$C_{i,S} = \frac{1}{\zeta(2)} \prod_{p \in S} \left(1 + \frac{1}{p}\right)^{-1} \prod_{p \notin S} \frac{1 + \sum_{k \in E_{i,S}} p^{-k/q}}{1 + p^{-1}}.$$

In particular, $C_{i,S} > 0$.

Proof. For real $s > 1/q$, let

$$B_S(s) = \prod_{p \notin S} (1 + p^{-qs}) = \sum_{\substack{m \geq 1 \\ \mu^2(m) = 1 \\ \gcd(m, R_S) = 1}} \frac{1}{m^{qs}}.$$

Then $F_{i,S}(s) = B_S(s) H_{i,S}(s)$.

We first justify the coefficient expansion of $H_{i,S}$ at $s = 1/q$. For $p \notin S$, put

$$H_p(s) = \frac{1 + \sum_{k \in E_{i,S}} p^{-ks}}{1 + p^{-qs}}.$$

Since q is the least element of $E_{i,S}$, we have

$$H_p(s) = 1 + \frac{\sum_{\substack{k \in E_{i,S} \\ k>q}} p^{-ks}}{1 + p^{-qs}}.$$

At $s = 1/q$, the expansion

$$\frac{1}{1+p^{-qs}} = \sum_{r=0}^{\infty} (-1)^r p^{-qrs}$$

is absolutely convergent. Therefore, the sum of the absolute values of the nonconstant local coefficients of H_p at $s = 1/q$ is bounded by

$$\left(\sum_{\substack{k \in E_{i,S} \\ k>q}} p^{-k/q} \right) \left(\sum_{r=0}^{\infty} p^{-r} \right) \leq \frac{\sum_{k \geq q+1} p^{-k/q}}{1 - p^{-1}} \ll_q p^{-(q+1)/q}.$$

Since $\sum_p p^{-(q+1)/q} < \infty$, the Euler product for $H_{i,S}$ expands into a Dirichlet series

$$H_{i,S}(s) = \sum_{b=1}^{\infty} \frac{c_b}{b^s}$$

such that $\sum_{b=1}^{\infty} |c_b|/b^{1/q} < \infty$.

For $s > 1/q$, the products above converge absolutely, so comparing Dirichlet coefficients in $F_{i,S}(s) = B_S(s)H_{i,S}(s)$ gives

$$|\mathcal{V}_{i,S} \cap [1,x]| = \sum_{b \leq x} c_b Q_S\left((x/b)^{1/q}\right),$$

where Q_S is as in Lemma 5. By that lemma, $Q_S(y) \sim \kappa_S y$. For each fixed b,

$$\frac{Q_S((x/b)^{1/q})}{x^{1/q}} \longrightarrow \kappa_S b^{-1/q} \qquad (x \to \infty).$$

Moreover,

$$0 \leq \frac{Q_S((x/b)^{1/q})}{x^{1/q}} \leq b^{-1/q}.$$

Since $\sum_{b=1}^{\infty} |c_b|/b^{1/q} < \infty$, dominated convergence gives

$$\frac{|\mathcal{V}_{i,S} \cap [1,x]|}{x^{1/q}} \longrightarrow \kappa_S \sum_{b=1}^{\infty} \frac{c_b}{b^{1/q}} = \kappa_S H_{i,S}(1/q).$$

Thus, $|\mathcal{V}_{i,S} \cap [1,x]| \sim \kappa_S H_{i,S}(1/q) x^{1/q}$. Finally,

$$H_{i,S}(1/q) = \prod_{p \notin S} \frac{1 + \sum_{k \in E_{i,S}} p^{-k/q}}{1 + p^{-1}}.$$

Therefore, $|\mathcal{V}_{i,S} \cap [1,x]| \sim C_{i,S} x^{1/q}$, where

$$C_{i,S} = \frac{1}{\zeta(2)} \prod_{p \in S} \left(1 + \frac{1}{p}\right)^{-1} \prod_{p \notin S} \frac{1 + \sum_{k \in E_{i,S}} p^{-k/q}}{1 + p^{-1}}.$$

All factors in this product are positive, and the convergence established above shows that the product is finite and nonzero. Hence, $C_{i,S} > 0$. □

Corollary 2. *Let $S \neq \varnothing$, let $i \geq 2$, and let $q = \min S$. For fixed S, the number of nonzero v-summands with $v \leq x$ in the reduced fixed-set sum is*

$$|\mathcal{V}_{i,S} \cap [1,x]| \sim C_{i,S} x^{1/q}.$$

In particular, for a given n, the number of v-summands in Theorem 3 is exactly $|\mathcal{V}_{i,P_i(n)} \cap [1,n]|$.

Proof. For fixed S, the first assertion is exactly Theorem 5. In the reduced formula for a given n, the summands are indexed by $v \in \mathcal{V}_{i,P_i(n)}$ for $v \leq n$. For each such v, we have $h_{i,P_i(n)}(v) \neq 0$ by the definition of $\mathcal{V}_{i,P_i(n)}$, and

$$\Phi_{P_i(n)}\left(\left\lfloor \frac{n}{v} \right\rfloor\right) \geq 1,$$

because $\lfloor n/v \rfloor \geq 1$ and 1 has no prime factor in $P_i(n)$. Hence, the number of v-summands is exactly $|\mathcal{V}_{i,P_i(n)} \cap [1,n]|$. □

7. Densities for Fixed Prime Sets

Fix a finite set of primes S. We now study the density of positive integers whose first i mosaic levels avoid S.

Definition 10. For $N \in \mathbb{Z}^+$, define

$$A_{i,S}(N) = \sum_{m \leq N} \chi_{i,S}(m), \qquad \delta_i(S) = \lim_{N \to \infty} \frac{A_{i,S}(N)}{N}.$$

We show in Lemma 6 that the limit exists.

Lemma 6. *For every finite set of primes S and every $i \geq 1$, the density $\delta_i(S)$ exists and satisfies*

$$\delta_i(S) = \sum_{d=1}^{\infty} \frac{h_{i,S}(d)}{d},$$

with absolute convergence.

Proof. By Möbius inversion of $\chi_{i,S}(m)$, we have

$$A_{i,S}(N) = \sum_{d \le N} h_{i,S}(d) \left\lfloor \frac{N}{d} \right\rfloor.$$

We first prove that $\sum_{d \ge 1} |h_{i,S}(d)|/d$ converges. By Lemma 4, each prime-power value of $h_{i,S}$ has absolute value at most 1. Since $h_{i,S}$ is multiplicative, $|h_{i,S}(d)| \le 1$ for every d.

By Theorem 2, every d with $h_{i,S}(d) \neq 0$ can be written uniquely as $d = uv$, with $u \mid R_S$ and $v \in \mathcal{V}_{i,S}$. Therefore,

$$\sum_{d=1}^{\infty} \frac{|h_{i,S}(d)|}{d} \le \sum_{u \mid R_S} \frac{1}{u} \sum_{v \in \mathcal{V}_{i,S}} \frac{1}{v}.$$

Since $\mathcal{V}_{i,S} \subseteq \mathbb{P}$, and since Golomb proved the identity [10]

$$\sum_{v \in \mathbb{P}} \frac{1}{v} = \frac{\zeta(2)\zeta(3)}{\zeta(6)},$$

the series is convergent.

Writing $\lfloor N/d \rfloor = N/d - \theta_{N,d}$ with $0 \le \theta_{N,d} < 1$,

$$\frac{A_{i,S}(N)}{N} = \sum_{d \le N} \frac{h_{i,S}(d)}{d} - \frac{1}{N} \sum_{d \le N} h_{i,S}(d)\theta_{N,d}.$$

The first term tends to $\sum_{d \ge 1} h_{i,S}(d)/d$ since $\sum_{d \ge 1} |h_{i,S}(d)|/d$ converges. For the error term,

$$\left| \frac{1}{N} \sum_{d \le N} h_{i,S}(d)\theta_{N,d} \right| \le \frac{1}{N} \sum_{d \le N} |h_{i,S}(d)|.$$

Let $\varepsilon > 0$. Choose M such that

$$\sum_{d > M} \frac{|h_{i,S}(d)|}{d} < \varepsilon/2.$$

For $N > M$,

$$\frac{1}{N} \sum_{d \le N} |h_{i,S}(d)| \le \frac{1}{N} \sum_{d \le M} |h_{i,S}(d)| + \sum_{M < d \le N} \frac{|h_{i,S}(d)|}{d}.$$

The first term is less than $\varepsilon/2$ for all sufficiently large N, while the second is less than $\varepsilon/2$ by the choice of M. Hence, the error term tends to 0, and the density exists with the claimed value. □

Since $h_{i,S}$ is multiplicative, the series of Lemma 6 factors as an Euler product, which we now compute.

Theorem 6. *For every finite set of primes S and every $i \geq 1$, we have*

$$\delta_i(S) = \prod_{p \in S} \left(1 - \frac{1}{p}\right) \prod_{p \notin S} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{k=1}^{\infty} \frac{\chi_{i-1,S}(k)}{p^k}\right).$$

Proof. By Lemma 6 and the multiplicativity of $h_{i,S}$, it follows that $\delta_i(S)$ has the Euler product

$$\delta_i(S) = \prod_{p} \left(1 + \sum_{k=1}^{\infty} \frac{h_{i,S}(p^k)}{p^k}\right).$$

For $p \in S$, Lemma 4 gives $h_{i,S}(p) = -1$ and $h_{i,S}(p^k) = 0$ for $k \geq 2$, so the local factor is $1 - 1/p$. For $p \notin S$, Lemma 4 gives $h_{i,S}(p) = 0$ and $h_{i,S}(p^k) = \Delta_{i,S}(k)$ for $k \geq 2$. Thus, the local factor is

$$1 + \sum_{k=2}^{\infty} \frac{\Delta_{i,S}(k)}{p^k}.$$

Since $\Delta_{i,S}(k) = \chi_{i-1,S}(k) - \chi_{i-1,S}(k-1)$ and $\chi_{i-1,S}(1) = 1$, we have

$$\begin{aligned} 1 + \sum_{k=2}^{\infty} \frac{\Delta_{i,S}(k)}{p^k} &= 1 + \left(\sum_{k=1}^{\infty} \frac{\chi_{i-1,S}(k)}{p^k} - \frac{1}{p}\right) - \frac{1}{p} \sum_{k=1}^{\infty} \frac{\chi_{i-1,S}(k)}{p^k} \\ &= \left(1 - \frac{1}{p}\right) \left(1 + \sum_{k=1}^{\infty} \frac{\chi_{i-1,S}(k)}{p^k}\right). \end{aligned}$$

Combining the local factors gives the stated product. □

Corollary 3. *Let $R_S = \prod_{\ell \in S} \ell$. Then*

$$\delta_2(S) = \prod_{p \in S} \left(1 - \frac{1}{p}\right) \prod_{p \notin S} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{d \mid R_S} \frac{\mu(d)}{p^d - 1}\right).$$

Proof. Taking $i = 2$ in Theorem 6, it remains to compute $\sum_{k=1}^{\infty} \chi_{1,S}(k)/p^k$. Since $\chi_{1,S}(k) = 1$ if and only if $\gcd(k, R_S) = 1$, we have

$$\chi_{1,S}(k) = \sum_{d \mid \gcd(k, R_S)} \mu(d).$$

Substituting and switching the order of summation gives

$$\sum_{k=1}^{\infty} \frac{\chi_{1,S}(k)}{p^k} = \sum_{d \mid R_S} \mu(d) \sum_{\substack{k \geq 1 \\ d \mid k}} \frac{1}{p^k}. \tag{1}$$

For each fixed $d \mid R_S$, the inner sum over k on the right-hand side of (1) is

$$\sum_{\substack{k\geq 1\\ d\mid k}} \frac{1}{p^k} = \sum_{j=1}^{\infty} p^{-dj} = \frac{1}{p^d-1}.$$

Hence,

$$\sum_{k=1}^{\infty} \frac{\chi_{1,S}(k)}{p^k} = \sum_{d\mid R_S} \frac{\mu(d)}{p^d-1}.$$

Substituting into Theorem 6 gives the result. □

Theorem 7. *Let $S \neq \varnothing$, let $i \geq 2$, and let $q = \min S$. Then*

$$A_{i,S}(N) = \delta_i(S)N + O_{i,S}(N^{1/q}).$$

Proof. By Lemma 6 and the fact that $\lfloor N/d \rfloor = N/d + O(1)$,

$$|A_{i,S}(N) - N\delta_i(S)| \leq \sum_{d\leq N} |h_{i,S}(d)| + N \sum_{d>N} \frac{|h_{i,S}(d)|}{d}.$$

By Theorem 2, every d with $h_{i,S}(d) \neq 0$ has the form $d = uv$, where $u \mid R_S$ and $v \in \mathcal{V}_{i,S}$. Since S is fixed and $|h_{i,S}(d)| \leq 1$, it is enough to estimate

$$\sum_{v\in\mathcal{V}_{i,S}} \mathbf{1}\{v \leq X\} = |\mathcal{V}_{i,S} \cap [1, X]| \quad \text{and} \quad \sum_{\substack{v>X\\ v\in\mathcal{V}_{i,S}}} \frac{1}{v}.$$

By Proposition 2, $|\mathcal{V}_{i,S} \cap [1, X]| = O_{i,S}(X^{1/q})$, and partial summation gives

$$\sum_{\substack{v>X\\ v\in\mathcal{V}_{i,S}}} \frac{1}{v} = O_{i,S}\left(X^{1/q-1}\right).$$

Therefore, $\sum_{d\leq N} |h_{i,S}(d)| = O_{i,S}(N^{1/q})$ and

$$N \sum_{d>N} \frac{|h_{i,S}(d)|}{d} = O_{i,S}(N^{1/q}).$$

Combining these estimates gives $A_{i,S}(N) = \delta_i(S)N + O_{i,S}(N^{1/q})$. □

8. A Uniform Approximation for the Level Totient

The preceding section studies the density $\delta_i(S)$ for a fixed set S. To estimate $\varphi_i(n)/n$, we set $S = P_i(n)$. Because this set varies with n, this section makes the estimate uniform in n.

Lemma 7. *For every $i \geq 0$ and every $n \in \mathbb{Z}^+$,*

$$\prod_{p \in P_i(n)} p \leq n.$$

Proof. We argue by induction on i. For $i = 0$, the product is empty and equals 1, so the result is immediate. For $i \geq 1$, write $n = \prod_{j=1}^{k} p_j^{\alpha_j}$. By Definition 2,

$$P_i(n) = \{p_1, \ldots, p_k\} \cup \bigcup_{j=1}^{k} P_{i-1}(\alpha_j).$$

By the induction hypothesis $\prod_{p \in P_{i-1}(\alpha_j)} p \leq \alpha_j$, and $\alpha_j \leq p_j^{\alpha_j - 1}$ for every prime p_j and integer $\alpha_j \geq 1$, so

$$\prod_{p \in P_i(n)} p \leq \left(\prod_{j=1}^{k} p_j \right) \left(\prod_{j=1}^{k} \prod_{p \in P_{i-1}(\alpha_j)} p \right) \leq \prod_{j=1}^{k} p_j \alpha_j \leq \prod_{j=1}^{k} p_j^{\alpha_j} = n. \qquad \square$$

Remark 1. For $i \geq 1$, equality holds in Lemma 7 if and only if n is squarefree, so the bound is optimal. If n is squarefree, every exponent in its factorization equals 1. Since $P_{i-1}(1) = \varnothing$, we have $P_i(n) = P_1(n)$, and the product equals n. Conversely, if $\prod_{p \in P_i(n)} p = n$, then n is a product of distinct primes.

Lemma 8. *There is an absolute constant $c > 0$ such that, for every $N \geq 16$ and every finite set of primes S with $R_S \leq N$,*

$$\prod_{p \in S} \left(1 + p^{-1/2}\right) \leq \exp\left(\frac{c\sqrt{\log N}}{\log \log N} \right).$$

Proof. Since $1 + x \leq e^x$ for every real x, we have

$$\prod_{p \in S} \left(1 + p^{-1/2}\right) \leq \exp\left(\sum_{p \in S} p^{-1/2} \right). \tag{2}$$

It suffices to show that the exponent $\sum_{p \in S} p^{-1/2}$ on the right-hand side of (2) is at most $c\sqrt{\log N}/\log \log N$ for some absolute constant $c > 0$.

Let $z \geq 2$. Since $\log R_S = \sum_{p \in S} \log p$, and every $p \in S$ with $p > z$ satisfies $\log p > \log z$, there are at most $(\log R_S)/\log z$ primes in S that are larger than z, each with $p^{-1/2} < z^{-1/2}$. Since the remaining primes of S are at most z,

$$\sum_{p \in S} p^{-1/2} \leq \sum_{p \leq z} p^{-1/2} + \frac{\log R_S}{\sqrt{z}\, \log z}.$$

We first show that

$$\sum_{p \leq z} p^{-1/2} \ll \frac{\sqrt{z}}{\log z}$$

for every $z \geq 2$. For $2 \leq z < 4$ the claim is immediate, so we may assume $z \geq 4$. By partial summation and Chebyshev's estimate $\pi(t) \ll t/\log t$ (see [1]),

$$\sum_{p \leq z} p^{-1/2} = \frac{\pi(z)}{\sqrt{z}} + \frac{1}{2}\int_2^z \frac{\pi(t)}{t^{3/2}}\,dt \ll \frac{\sqrt{z}}{\log z} + \int_2^z \frac{dt}{\sqrt{t}\,\log t}.$$

For $2 \leq t \leq \sqrt{z}$ we have $\log t \geq \log 2$, so

$$\int_2^{\sqrt{z}} \frac{dt}{\sqrt{t}\,\log t} \ll \int_2^{\sqrt{z}} \frac{dt}{\sqrt{t}} \ll z^{1/4}.$$

For $\sqrt{z} \leq t \leq z$ we have $\log t \geq \frac{1}{2}\log z$, so

$$\int_{\sqrt{z}}^z \frac{dt}{\sqrt{t}\,\log t} \ll \frac{1}{\log z}\int_{\sqrt{z}}^z \frac{dt}{\sqrt{t}} \ll \frac{\sqrt{z}}{\log z}.$$

Since $z^{1/4} \ll \sqrt{z}/\log z$, the claim follows.

Take $z = \log N$. Since $N \geq 16$ we have $z \geq 2$, and $R_S \leq N$ gives $\log R_S \leq \log N$. Therefore,

$$\sum_{p \in S} p^{-1/2} \ll \frac{\sqrt{\log N}}{\log\log N} + \frac{\log N}{\sqrt{\log N}\,\log\log N} \ll \frac{\sqrt{\log N}}{\log\log N},$$

and there is an absolute constant $c > 0$ such that

$$\prod_{p \in S}\left(1 + p^{-1/2}\right) \leq \exp\left(\frac{c\sqrt{\log N}}{\log\log N}\right). \qquad \square$$

Theorem 8. *There is an absolute constant $c > 0$ such that, for every $i \geq 1$ and every integer $n \geq 16$,*

$$\frac{\varphi_i(n)}{n} = \delta_i(P_i(n)) + O\left(n^{-1/2}\exp\left(\frac{c\sqrt{\log n}}{\log\log n}\right)\right).$$

Proof. Let $S = P_i(n)$. By Lemma 7, we have $R_S \leq n$. Recall that $\mathbb{P}$ denotes the set of powerful integers. By Theorem 1 and Lemma 6, we have

$$|\varphi_i(n) - n\delta_i(S)| \leq \sum_{d \leq n} |h_{i,S}(d)| + n\sum_{d > n} \frac{|h_{i,S}(d)|}{d}.$$

By Theorem 2, every d with $h_{i,S}(d) \neq 0$ factors as $d = uv$ with $u \mid R_S$ and $v \in \mathcal{V}_{i,S} \subseteq \mathbb{P}$. Since $|h_{i,S}(d)| \leq 1$, we have

$$\sum_{d \leq n} |h_{i,S}(d)| + n\sum_{d > n} \frac{|h_{i,S}(d)|}{d} \leq \sum_{u \mid R_S}\left(\sum_{v \leq n/u} \mathbf{1}\{v \in \mathbb{P}\} + \frac{n}{u}\sum_{\substack{v > n/u \\ v \in \mathbb{P}}} \frac{1}{v}\right).$$

The number of powerful integers up to t is $O(t^{1/2})$ (see [10]), and partial summation gives

$$\sum_{\substack{v>t \\ v\in\mathbb{P}}} \frac{1}{v} = O(t^{-1/2}).$$

Thus, the expression in parentheses is $O((n/u)^{1/2})$, and hence

$$|\varphi_i(n) - n\delta_i(S)| = O\left(n^{1/2} \sum_{u|R_S} u^{-1/2}\right).$$

Now

$$\sum_{u|R_S} u^{-1/2} = \prod_{p\in S}(1+p^{-1/2}).$$

Since $R_S \le n$, Lemma 8 applies with $N = n$ and yields

$$\sum_{u|R_S} u^{-1/2} \le \exp\left(\frac{c\sqrt{\log n}}{\log\log n}\right).$$

Consequently,

$$\left|\frac{\varphi_i(n)}{n} - \delta_i(P_i(n))\right| = O\left(n^{-1/2}\exp\left(\frac{c\sqrt{\log n}}{\log\log n}\right)\right). \qquad \square$$

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