

ON BERNOULLI NUMBERS INDEXED BY ROOTED TREES

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Abstract

We investigate Bernoulli numbers defined by rooted tree polylogarithms. We give the explicit formula of the Bernoulli numbers using the Stirling numbers of the second kind, and differential formulas of rooted tree polylogarithms to give recursive formulas of the Bernoulli numbers. We also give some identities between the Bernoulli numbers which come from the Hopf algebra structure of rooted trees. Then we make some observations about the arithmetic properties of the Bernoulli numbers. We also discuss rooted tree polylogarithms with negative exponents and observe that the Bernoulli numbers defined by them are positive integers and satisfy some explicit formulas.

1. Introduction

Bernoulli numbers B_n, first independently studied by Takakazu Seki [17] and Jacob Bernoulli [2], are important numbers, especially for number theorists. They are generalized in various ways, such as: Bernoulli polynomials, Bernoulli numbers with characters, poly-Bernoulli numbers $B_n^{(k)}$ [12], Bernoulli numbers of higher order [5], multi-poly-Bernoulli numbers of Imatomi-Kaneko-Takeda type $B_n^{(k_1,\ldots,k_r)}$ [11], those of Kaneko-Tsumura type [13], multi-indexed poly-Bernoulli numbers [3], etc. Once

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we have a new type of polylogarithm, we may construct certain kinds of Bernoulli numbers.

In [14], we introduced the rooted tree polylogarithm

$$Li_f(z) = \sum_{m:f\text{-order}} \frac{z^{|m|_\infty}}{|m|}$$

for a rooted forest f (see Section 3.1 for its definition.) This new type of polylogarithm allows us to consider the Bernoulli numbers B_n^f defined by

$$\sum_{n=0}^{\infty} B_n^f \frac{z^n}{n!} = \frac{Li_f(1-e^{-z})}{1-e^{-z}}.$$

In this paper, we study the fundamental properties of B_n^f. We first find the following explicit formula for B_n^f, which is shown in Section 3.2.

Theorem 1. *For $n \geq 0$ and a rooted forest f, we have*

$$B_n^f = (-1)^n \sum_{\substack{m:f\text{-}order \\ |m|_\infty \leq n+1}} \frac{(-1)^{|m|_\infty - 1}(|m|_\infty - 1)! \left\{ {n \atop |m|_\infty - 1} \right\}}{|m|},$$

where $\left\{ {n \atop m} \right\}$ denotes the Stirling number of the second kind.

Next, we consider differential formulas for rooted tree polylogarithms and recurrence formulas for B_n^f in Section 3.3. Then we prove the following theorem in Section 3.4 that, due to the Hopf algebra structure of rooted trees, there are some identities among B_n^f and $B_n^{(k_1,\ldots,k_r)}$ (the latter is also indicated by $B_n^{x^{k_1-1}y\cdots x^{k_r-1}y}$ and its upper subscript $x^{k_1-1}y\cdots x^{k_r-1}y$ is linearly extended, as described in Section 2.4 and Section 3.2).

Let S be the antipode of the Hopf algebra of (non-planar) rooted trees $\mathcal{H}$, λ_n the rooted tree of weight n with at most one leaf, e_f (which, to be precise, should be written as $\Phi(e_f)$) for a rooted forest f given in Section 2.3, and $*$ the harmonic product on $\mathfrak{H}^1 = \mathbb{Q} + \mathbb{Q}\langle x, y\rangle y$.

Theorem 2. *For $m \geq 1$ and $n \geq 0$, we have the following identities:*

(i) $B_n^{(m)} = (-1)^{m-1} B_n^{e_{\lambda_m}} = (-1)^m B_n^{S(e_{\lambda_m})}$,

(ii) $B_n^{(\overbrace{1,\ldots,1}^{m})} = (-1)^m B_n^{S(\lambda_m)} = (-1)^m B_n^{S(e_{\lambda_1^m})}$,

(iii) $B_n^{(x+y)^{m-1}y} = B_n^{\lambda_m} = B_n^{e_{\lambda_1^m}}$,

(iv) $B_n^{\overbrace{y * \cdots * y}^{m}} = B_n^{\lambda_1^m} = (-1)^m B_n^{S(\lambda_1^m)}$.

By computation, we obtain the values of B_n^f (Tables 1 and 2) in the Appendix and observe some arithmetic properties of B_n^f's. Let $\mathcal{T}$ be the set of (non-planar) rooted trees.

Theorem 3. *We have $B_n^{e_t} = 0$ for $t \in \mathcal{T} - \{\lambda_n \mid n \geq 1\}$.*

In Section 4, we introduce rooted tree polylogarithms with negative exponents

$$Li_{\overline{f}}(z) = \sum_{m:f\text{-order}} |m| z^{|m|_\infty}$$

for a rooted forest f (see Section 4 for its definition) and define the corresponding Bernoulli numbers $B_n^{\overline{f}}$ by

$$\sum_{n=0}^{\infty} B_n^{\overline{f}} \frac{z^n}{n!} = \frac{Li_{\overline{f}}(1-e^{-z})}{1-e^{-z}}.$$

We describe that each $B_n^{\overline{f}}$ is a positive integer and find in some cases that $B_n^{\overline{f}}$'s satisfy explicit formulas.

2. Preliminaries

To discuss several aspects of Bernoulli numbers indexed by rooted trees, we first review the notion of Hopf algebras briefly for those who are not familiar with the theory. Specific examples, such as Hopf algebras $\mathcal{H}_P, \mathcal{H}$ of rooted trees, and the Hopf algebra $\mathfrak{H}_*^1$ of words, are also described. We also introduce multiple poly-Bernoulli numbers by Imatomi, Kaneko, and Takeda [11].

2.1. A Review of Hopf Algebras

Let K be a field. We denote $\otimes_K$ simply by $\otimes$. A *(K-)algebra* A is a K-vector space with linear maps $m : A \otimes A \to A$ (multiplication or product) and $u : K \to A$ (unit) such that

$$m(m \otimes id) = m(id \otimes m) \quad \text{and} \quad m(u \otimes id) = id = m(id \otimes u)$$

(under identification $K \otimes A \cong A \cong A \otimes K$). Then, we have that $u(1_K)$ is the neutral element in A, i.e., $u(1_K) = 1_A$. A *(K-)coalgebra* C is a K-vector space with linear maps $\Delta : C \to C \otimes C$ (comultiplication or coproduct) and $\varepsilon : C \to K$ (counit) such that

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \quad \text{and} \quad (\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta$$

(under the identification $K \otimes C \cong C \cong C \otimes K$). A 5-tuple $(H, m, u, \Delta, \varepsilon)$ is called a *bialgebra* if the following equivalent conditions hold.

(1) The maps $m : H \otimes H \to H$ and $u : K \to H$ are coalgebra morphisms.

(2) The maps $\Delta : H \to H \otimes H$ and $\varepsilon : H \to K$ are algebra morphisms.

An element $h \in H$ is called *primitive* if $\Delta(h) = h \otimes 1_H + 1_H \otimes h$. A *Hopf algebra* is a bialgebra $(H, m, u, \Delta, \varepsilon)$ with a linear map $S : H \to H$, called the *antipode*, satisfying

$$m(S \otimes id)\Delta = u\varepsilon = m(id \otimes S)\Delta.$$

The antipode S is uniquely determined if it exists. The Hopf algebra $(H, m, u, \Delta, \varepsilon, S)$ is simply denoted by H.

2.2. Hopf Algebras of Rooted Trees: $\mathcal{H}$ and $\mathcal{H}_P$

In this subsection, based on [6, 8], we give a brief review of the Hopf algebras $\mathcal{H}, \mathcal{H}_P$ of rooted trees. Also, see [18, 19].

A *non-planar rooted tree* is a finite, connected, and oriented graph with no cycles and a special vertex called the *root*. Every edge is directed away from the root. Vertices that do not have outgoing edges are sometimes called *leaves*. The number of vertices of a graph g is called its *weight* and is denoted by $\mathrm{wt}(g)$. The set of (isomorphism classes of) rooted trees is denoted by $\mathcal{T}$.

Example 1. We give all elements in $\mathcal{T}$ up to weight 5. The root of every rooted tree is drawn topmost. For example,



We often denote by λ_n the rooted tree that is of weight n and has at most one leaf. The Hopf algebra of rooted trees $\mathcal{H}$ is a commutative Hopf algebra, but not a cocommutative Hopf algebra. The product of rooted trees, called a *non-planar rooted forest*, is given by the disjoint union of the rooted trees. We denote by $\mathbb{I}$ the neutral element called the *empty forest*, which is of weight 0. The set of (isomorphism classes of) rooted forests is denoted by $\mathcal{F}$.

A *planar rooted tree* is a rooted tree t such that each vertex of t has totally ordered children. The set of (isomorphism classes of) planar rooted trees is denoted by $\mathcal{T}_P$. The Hopf algebra of planar rooted trees $\mathcal{H}_P$ is a non-commutative generalization of $\mathcal{H}$. The product of planar rooted trees, called a *planar rooted forest*, is given by concatenation. The set of (isomorphism classes of) planar rooted forests is denoted by $\mathcal{F}_P$.

Let $B_+ : \mathcal{F}_P \to \mathcal{T}_P$ be the *grafting operator* given by connecting all the roots of ordered trees in a forest to a new root. In particular, $B_+(\mathbb{I}) = \bullet$. We extend the definition linearly so that the map $B_+ : \mathcal{H}_P \to \mathbb{Q}\mathcal{T}_P$ is $\mathbb{Q}$-linear.

We define the coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ as an algebra map satisfying

$$\Delta(B_+(f)) = B_+(f) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+) \circ \Delta(f)$$

for $f \in \mathcal{F}$. This gives rise to a connected graded bialgebra structure on $\mathcal{H}$ due to Takeuchi [20], and therefore, $\mathcal{H}$ is equipped with the antipode $S : \mathcal{H} \to \mathcal{H}$. Thus, $\mathcal{H}$ is a commutative and non-cocommutative Hopf algebra. The coproduct Δ (and the counit ε) on $\mathcal{H}_P$ is defined in the same way, and thus, $\mathcal{H}_P$ is a non-commutative and non-cocommutative Hopf algebra.

As shown in [8, Théorème 35], there is a unique bilinear form $(\ ,\ ) : \mathcal{H}_P \times \mathcal{H}_P \to \mathbb{Q}$ satisfying the following three conditions.

1. For all $f \in \mathcal{H}_P$, $(\mathbb{I}, f) = \varepsilon(f)$.
2. For all $f_1, f_2, g \in \mathcal{H}_P$, $(f_1 f_2, g) = (f_1 \otimes f_2, \Delta(g))$.
3. For all $f, g \in \mathcal{H}_P$, $(B_+(f), g) = (f, \gamma(g))$, where $\gamma : \mathcal{H}_P \to \mathcal{H}_P$ is a linear map given by

$$\gamma(\mathbb{I}) = 0 \quad \text{and} \quad \gamma(t_1 \cdots t_n) = \begin{cases} t_1 \cdots t_{n-1} & \text{if } t_n = \bullet \\ 0 & \text{otherwise} \end{cases}$$

for $t_1, \ldots, t_n \in \mathcal{T}_P$.

It has more properties listed below. Let $f, f', g, g' \in \mathcal{H}_P$.

1. $(f \otimes f', g \otimes g') = (f, g)(f', g')$.
2. If f, g are homogeneous in weight and $\mathrm{wt}(f) \neq \mathrm{wt}(g)$, then $(f, g) = 0$.
3. $(f, g) = (g, f)$.
4. If $(f, h) = 0$ for any $h \in \mathcal{H}_P$, then $f = 0$.
5. $(S(f), g) = (f, S(g))$.
6. Let $(e_f)_{f \in \mathcal{F}_P}$ be the basis on $\mathcal{H}_P$ defined by $(e_f, g) = \delta_{f,g}$, where δ denotes Kronecker's delta function. Then we have the following.
   (i) e_f is a homogeneous element having the same weight as f.
   (ii) $(e_t)_{t \in \mathcal{T}_P}$ forms a basis of the space generated by primitive elements of $\mathcal{H}_P$.
   (iii) For $t_1, \ldots, t_n \in \mathcal{T}_P$, $\Delta(e_{t_1 \cdots t_n}) = \sum_{i=0}^{n} e_{t_1 \cdots t_i} \otimes e_{t_{i+1} \cdots t_n}$.

Since the natural projection $\Phi : \mathcal{H}_P \to \mathcal{H}$ is a surjective homomorphism, each $\Phi(e_t)$ for $t \in \mathcal{T}$ (which is also written by e_t in this paper) is a primitive element in $\mathcal{H}$.

2.3. The Hopf Algebra of Words: $\mathfrak{H}^1_*$

In this subsection, based on the theory of harmonic (or quasi-shuffle) products (see [9, 10, 16] for example), we give a brief review of the Hopf algebra $\mathfrak{H}^1_*$ of words.

Let $\mathfrak{H} = \mathbb{Q}\langle x, y\rangle$ be the non-commutative polynomial algebra in two indeterminates x, y. Let $\mathfrak{H}^1 = \mathbb{Q} + \mathfrak{H}y$ be a subalgebra of $\mathfrak{H}$. We introduce another product $* : \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ by $\mathbb{Q}$-bilinearlity, $w * 1 = 1 * w = w$ for any $w \in \mathfrak{H}^1$, and

$$z_k v * z_l w = z_k(v * z_l w) + z_l(z_k v * w) + z_{k+l}(v * w)$$

for $k, l \geq 1$ and $v, w \in \mathfrak{H}^1$, where $z_j = x^{j-1}y$ for $j \geq 1$. The product $*$ is called the *harmonic product*. By definition, the harmonic product is commutative, and hence, $\mathfrak{H}^1_* = (\mathfrak{H}^1, *)$ is a commutative algebra.

We also have recursive rules for the harmonic product in terms of the letters x and y as follows. See [14, Lemma 5.1].

Lemma 1. *Let $v, w \in \mathfrak{H}^1$ and $u = x + y \in \mathfrak{H}$. Then, the following formulas hold.*

(i) $xv * xw = x(v * xw) + x(xv * w) - x^2(v * w)$.

(ii) $xv * yw = x(v * yw) + y(xv * w) - xy(v * w)$.

(iii) $xv * uw = x(v * uw) + u(xv * w) - xu(v * w)$.

(iv) $yv * xw = y(v * xw) + x(yv * w) - xy(v * w)$.

(v) $yv * yw = y(v * yw) + y(yv * w) + xy(v * w)$.

(vi) $yv * uw = y(v * uw) + u(yv * w)$.

(vii) $uv * uw = u(v * uw) + u(uv * w) - xu(v * w)$.

The coproduct $\Delta_* : \mathfrak{H}^1_* \to \mathfrak{H}^1_* \otimes \mathfrak{H}^1_*$ is the algebra map that satisfies

$$\Delta_*(z_{k_1} \cdots z_{k_r}) = \sum_{i=0}^{r} z_{k_1} \cdots z_{k_i} \otimes z_{k_{i+1}} \cdots z_{k_r}$$

for $r, k_1, \ldots, k_r \geq 1$. This gives rise to a connected graded bialgebra structure on $\mathfrak{H}^1_*$, and therefore, $\mathfrak{H}^1_*$ is equipped with the antipode. Thus, $\mathfrak{H}^1_*$ is a commutative and non-cocommutative Hopf algebra.

2.4. Multiple Poly-Bernoulli Numbers by Imatomi, Kaneko and Takeda

Let $r, k_1, \ldots, k_r \geq 1$. The *ordinary multiple polylogarithm* is defined by

$$Li_{k_1,\ldots,k_r}(z) = Li_{z_{k_1}\cdots z_{k_r}}(z) = \sum_{m_1 > \cdots > m_r > 0} \frac{z^{m_1}}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{Q}[[z]].$$

In [11], *multiple poly-Bernoulli numbers* $B_n^{(k_1,\ldots,k_r)}$ are introduced as

$$\sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_r)} \frac{z^n}{n!} = \frac{Li_{k_1,\ldots,k_r}(1-e^{-z})}{1-e^{-z}}.$$

We sometimes write $B_n^{z_{k_1}\cdots z_{k_r}}$ instead of $B_n^{(k_1,\ldots,k_r)}$. We find $B_n^{(1)} = B_n$, the Bernoulli number ($B_1^{(1)} = \frac{1}{2}$). Moreover, $B_n^{(k)}$ is the *poly-Bernoulli number* introduced in [12], and $B_n^{(1,\ldots,1)}$ is the Bernoulli number of higher order introduced in [5].

3. Bernoulli Numbers Indexed by Rooted Trees

3.1. Rooted Tree Polylogarithms

Rooted tree polylogarithms are introduced in [14]. For $f \in \mathcal{F}$, let

$$Li_f(z) = \sum_{m:f\text{-order}} \frac{z^{|m|_\infty}}{|m|} \in \mathbb{Q}[[z]].$$

The labeling map $m : V_f \to \mathbb{Z}_{\geq 1}$ (where V_f is the set of vertices of f) is called an *f-order* if $m(a) \geq m(b)$ whenever there is an edge from a to b, and for the map m, we put

$$|m| = \prod_{a\in V_f} m(a) \quad \text{and} \quad |m|_\infty = \max_{a\in V_f} m(a).$$

For example, $Li_{\lambda_1}(z) = -\log(1-z)$.

By definition, each rooted tree polylogarithm is expressed as a linear combination of the ordinary multiple polylogarithms. Identifying the index $(k_1,\ldots,k_r)$ with the word $z_{k_1}\cdots z_{k_r} \in \mathfrak{H}^1$, we obtain the map $\omega : \mathcal{H} \to \mathfrak{H}^1$ via this interpretation. For example, $\omega(\lambda_2) = z_1^2 + z_2 = y^2 + xy$ since $Li_{\lambda_2}(z) = Li_{1,1}(z) + Li_2(z) = Li_{z_1^2}(z) + Li_{z_2}(z)$.

As explained in [14], the map ω is characterized by the following conditions.

- $\omega(\mathbb{I}) = 1, \omega(\lambda_1) = y$.
- $\omega(B_+(f)) = (x+y)\omega(f) \quad (f \in \mathcal{F} - \{\mathbb{I}\})$.
- $\omega(fg) = \omega(f) * \omega(g) \quad (f, g \in \mathcal{F})$.

The last condition shows that the map $\omega : \mathcal{H} \to \mathfrak{H}^1_*$ is an algebra homomorphism. It is also shown that the map ω is not only an algebra homomorphism, but also a Hopf algebra morphism ([14, Proposition 3.5]).

3.2. Rooted Tree Bernoulli Numbers and the Explicit Formula

For $f \in \mathcal{F}$, we define the numbers B_n^f, which we call the *rooted tree Bernoulli numbers*, by

$$\sum_{n=0}^{\infty} B_n^f \frac{z^n}{n!} = \frac{Li_f(1-e^{-z})}{1-e^{-z}}.$$

The upper index f of B_n^f is linearly extended, as well as the upper index $z_{k_1}\cdots z_{k_r}$ of $B_n^{z_{k_1}\cdots z_{k_r}}$. Then, we have $B_n^f = B_f^{\omega(f)}$ for $f \in \mathcal{H}$.

Proof of Theorem 1. Using $\frac{(e^z-1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{z^n}{n!}$ $(m \geq 0)$, we calculate

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n^f \frac{z^n}{n!} = \frac{Li_f(1-e^{-z})}{1-e^{-z}} &= \sum_{m:f\text{-order}} \frac{(1-e^{-z})^{|m|_\infty - 1}}{|m|} \\
&= \sum_{m:f\text{-order}} (-1)^{|m|_\infty - 1} \frac{(|m|_\infty - 1)!}{|m|} \sum_{n=|m|_\infty - 1}^{\infty} \left\{ {n \atop |m|_\infty - 1} \right\} \frac{(-z)^n}{n!} \\
&= \sum_{n=0}^{\infty} (-1)^n \sum_{\substack{m:f\text{-order} \\ |m|_\infty \leq n+1}} \frac{(-1)^{|m|_\infty - 1}(|m|_\infty - 1)! \left\{ {n \atop |m|_\infty - 1} \right\}}{|m|} \cdot \frac{z^n}{n!}.
\end{aligned}$$

□

By definition of B_n^f, it has an explicit formula in terms of the Stirling numbers of the second kind as shown above. In the case of $f = \lambda_1$, this formula reduces to the Worpitzky identity (also see [1, 15]).

3.3. Recurrence Formulas

To obtain recurrence formulas of the Bernoulli numbers B_n^f, we consider the differential formulas of $Li_f(z)$. In this subsection, we use the symbol t_j for rooted trees.

First, let $f = t_1 = B_+(g)$. If $g = \mathbb{I}$ (that is, $f = \lambda_1$), $Li_f(z) = -\log(1-z)$, and hence, $\frac{d}{dz} Li_f(z) = \frac{1}{1-z}$. Then,

$$\sum_{n=0}^{\infty} B_n^{\lambda_1} \frac{z^n}{n!} = \frac{Li_{\lambda_1}(1-e^{-z})}{1-e^{-z}} = \frac{Li_1(1-e^{-z})}{1-e^{-z}} = \sum_{n=0}^{\infty} B_n^{(1)} \frac{z^n}{n!}.$$

Therefore, $B_n^{\lambda_1} = B_n^{(1)} = B_n$. Otherwise, $Li_f(z) = Li_{(x+y)\omega(g)}(z)$, and hence,

$$\frac{d}{dz} Li_f(z) = \frac{1}{z(1-z)} Li_g(z).$$

Differentiating both sides of the equation

$$Li_f(1-e^{-z}) = (1-e^{-z})\sum_{n=0}^{\infty} B_n^f \frac{z^n}{n!} \tag{1}$$

by z, we have

$$\sum_{n=0}^{\infty} B_n^g \frac{z^n}{n!} = e^{-z}\sum_{n=0}^{\infty} B_n^f \frac{z^n}{n!} + (1-e^{-z})\sum_{n=1}^{\infty} B_n^f \frac{z^{n-1}}{(n-1)!},$$

or equivalently,

$$\sum_{n=0}^{\infty} B_n^f \frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_m^g\right)\frac{z^n}{n!} - \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\binom{n}{m-1}B_m^f\right)\frac{z^n}{n!}.$$

Therefore, for $t_1 = B_+(g) \neq \lambda_1$, we have

$$(n+1)B_n^{t_1} = \sum_{m=0}^{n}\binom{n}{m}B_m^g - \sum_{m=1}^{n-1}\binom{n}{m-1}B_m^{t_1}.$$

If $n = 0$, the second sum on the right-hand side is considered zero.

Next, let $f = t_1t_2$. If $t_1 = t_2 = \lambda_1$, we have

$$\frac{d}{dz}Li_f(z) = \left(\frac{1}{1-z} + \frac{1}{z(1-z)}\right)Li_{\lambda_1}(z)$$

since $\omega(f) = \omega(\lambda_1) * \omega(\lambda_1) = (x+2y)\omega(\lambda_1)$. Differentiating Equation (1) (for $f = \lambda_1\lambda_2$) and performing a calculation similar to the one above, we have

$$(n+1)B_n^{\lambda_1\lambda_1} = 2\sum_{m=0}^{n-1}\binom{n}{m}B_m^{\lambda_1} + B_n^{\lambda_1} - \sum_{m=1}^{n-1}\binom{n}{m-1}B_m^{\lambda_1\lambda_1}.$$

If $t_1 = B_+(g) \neq \lambda_1, t_2 = \lambda_1$, we have

$$\frac{d}{dz}Li_f(z) = \frac{1}{z(1-z)}Li_{g\lambda_1}(z) + \frac{1}{1-z}Li_{t_1}(z)$$

since $\omega(f) = \omega(t_1) * \omega(\lambda_1) = (x+y)\omega(g) * y$. Differentiating Equation (1) (for $f = t_1\lambda_1$) and performing a similar calculation, we have

$$(n+1)B_n^{t_1\lambda_1} = \sum_{m=0}^{n}\binom{n}{m}B_m^{g_1\lambda_1} + \sum_{m=0}^{n-1}\binom{n}{m}B_m^{t_1} - \sum_{m=1}^{n-1}\binom{n}{m-1}B_m^{t_1\lambda_1}.$$

Otherwise, let $t_j = B_+(g_j) \neq \lambda_1$ for $j = 1, 2$. We have

$$\frac{d}{dz}Li_f(z) = \frac{1}{z(1-z)}Li_{g_1t_2+t_1g_2}(z) - \frac{1}{z}Li_{B_+(g_1g_2)}(z)$$

since

$$\begin{aligned}\omega(f) &= \omega(t_1) * \omega(t_2) = (x+y)\omega(g_1) * (x+y)\omega(g_2) \\ &= (x+y)\omega(g_1t_2 + t_1g_2) - x\omega(B_+(g_1g_2)).\end{aligned}$$

Differentiating Equation (1) (for $f = t_1t_2$) and performing a similar calculation, we have

$$(n+1)B_n^{t_1t_2} = \sum_{m=0}^{n} \binom{n}{m} B_m^{g_1t_2+t_1g_2} - B_n^{B_+(g_1g_2)} - \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{t_1t_2}.$$

We can proceed with this calculation recursively, but it seems difficult to obtain explicit recurrence formulas for the B_n^f's in general.

3.4. Identities Coming from Properties of Hopf Algebras

By definition, the rooted tree Bernoulli numbers inherit linear relations from those for polylogarithms. Here, we give proofs of Theorems 2 and 3.

Proof of Theorem 2. The second equality comes from $S(e_{\lambda_m}) = -e_{\lambda_m}$, which is because e_{λ_m} is primitive. For the first equality, it suffices to show that $\omega(e_{\lambda_m}) = (-1)^{m-1}z_m$. Since e_{λ_m} is primitive, $\omega(e_{\lambda_m}) = cz_m$ ($c \in \mathbb{Q}$). By [14, Proposition 3.8],

$$\omega(f) \equiv z_{wt(f)} \pmod{y\text{-degree} \geq 2}$$

for $f \in \mathcal{F}$. Also, by [7, Théorème 83], we have

$$e_{\lambda_m} = \sum_{i=1}^{m} \sum_{\substack{m_1+\cdots+m_i=m \\ m_1,\ldots,m_i\geq 1}} (-1)^{m+i} m_1 \lambda_{m_1} \cdots \lambda_{m_i}.$$

We note that this corresponds to $(-1)^{m-1}D(\lambda_m)$, where D denotes the Dynkin operator; see [4] for details. Hence, the sum of coefficients of e_{λ_m} is

$$\begin{aligned}&(-1)^m \sum_{i=1}^{m} (-1)^i \sum_{\substack{m_1+\cdots+m_i=m \\ m_1,\ldots,m_i\geq 1}} m_1 \\ &= (-1)^m \sum_{i=1}^{m} \frac{(-1)^i}{i} \sum_{\substack{m_1+\cdots+m_i=m \\ m_1,\ldots,m_i\geq 1}} (m_1 + \cdots + m_i) \\ &= (-1)^m \sum_{i=1}^{m} \frac{(-1)^i m}{i} \binom{m-1}{i-1} = (-1)^m \sum_{i=1}^{m} (-1)^i \binom{m}{i} = (-1)^{m-1}.\end{aligned}$$

Therefore, we obtain $c = (-1)^{m-1}$ and complete the proof of Statement (i).

By [8, Proposition 36] or [14, Remark 2.11], we have $e_{\lambda_1^m} = \lambda_m$ $(m \geq 1)$, and hence, the second equality holds. For the first equality, it suffices to show that $\omega(S(\lambda_m)) = (-y)^m$. The proof is based on induction. Since $S(\lambda_1) = -\lambda_1$, it holds if $m = 1$. Since $\Delta(\lambda_m) = \sum_{i=0}^{m} \lambda_i \otimes \lambda_{m-i}$, we have

$$S(\lambda_m) = -\sum_{i=1}^{m} \lambda_i S(\lambda_{m-i})$$

for $m \geq 1$. Hence, for general m, we have

$$\omega(S(\lambda_m)) = -\sum_{i=1}^{m} \omega(\lambda_i) * \omega(S(\lambda_{m-i})) = -\sum_{i=1}^{m} (x+y)^{i-1} y * (-y)^{m-i}$$

by the induction hypothesis. Then, using Proposition 1 below, we obtain the desired identity (ii).

The first equality is satisfied since $\omega(\lambda_m) = (x+y)^{m-1}y$. The second equality holds since $e_{\lambda_1^m} = \lambda_m$ $(m \geq 1)$ which appeared in the proof of Statement (ii). Therefore, we obtain Statement (iii).

Statement (iv) follows from $\omega(\lambda_1^m) = \underbrace{y * \cdots * y}_{m}$ and $S(\lambda_1) = -\lambda_1$. □

Proposition 1. *In $\mathfrak{H}_*^1$, the identity*

$$-\sum_{i=1}^{m} (x+y)^{i-1} y * (-y)^{m-i} = (-y)^m \qquad (m \geq 1)$$

holds.

Proof. By [21, Lemma 2.1 (ii)], we have

$$d(z_k v) * z_l w = \gamma(z_k)(d(v) * z_l w) + z_l(d(z_k v) * w)$$

$(k, l \geq 1, v, w \in \mathfrak{H}^1)$, where $\gamma \in \mathrm{Aut}(\mathfrak{H})$ such that $\gamma(x) = x, \gamma(y) = x + y$, and $d : \mathfrak{H}^1 \to \mathfrak{H}^1$ is a linear map given by $d(wy) = \gamma(w)y$ $(w \in \mathfrak{H})$. Putting $k = 1, v = y^{i-1} (i \geq 1), l = 1$, and $w = y^{j-1} (j \geq 1)$, we have

$$d(y^i) * y^j = (x+y)(d(y^{i-1}) * y^j) + y(d(y^i) * y^{j-1}).$$

Then, for $m \geq 2$,

$$\begin{aligned}
&\sum_{\substack{i+j=m \\ i,j\geq 1}} d(y^i) * (-y)^j \\
&\qquad = (x+y) \sum_{\substack{i+j=m \\ i,j\geq 1}} d(y^{i-1}) * (-y)^j - y \sum_{\substack{i+j=m \\ i,j\geq 1}} d(y^i) * (-y)^{j-1} \\
&\qquad = (x+y)(-y)^{m-1} + x \sum_{i=1}^{m-2} d(y^i) * (-y)^{m-i-1} - y(x+y)^{m-2} y,
\end{aligned}$$

and hence, we have

$$\sum_{\substack{i+j=m\\ i,j\geq 0}} d(y^i) * (-y)^j = x \sum_{\substack{i+j=m-1\\ i,j\geq 0}} d(y^i) * (-y)^j.$$

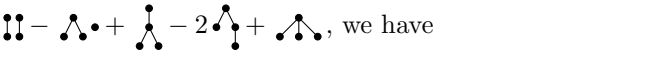
By this, the proposition, or equivalently, the identity

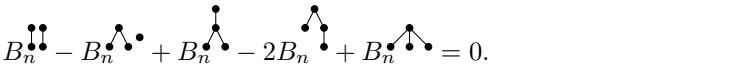
$$\sum_{\substack{i+j=m\\ i,j\geq 0}} d(y^i) * (-y)^j = 0$$

$(m \geq 1)$, is proved by induction on m. □

Remark 1. The proposition can also be proved by using Lemma 1.

Proof of Theorem 3. For $t \in \mathcal{T} - \{\lambda_n \mid n \geq 1\}$, we have $\omega(e_t) = 0$ due to [14, Theorem 3.9], and hence, $Li_{e_t}(z) = 0$. □

Example 2. Since e  $=$ , we have



3.5. Observation

Multiple poly-Bernoulli numbers satisfy the following recurrence formulas (see [11, Proposition 2]): for $r \geq 2, k_2, \ldots, k_r \geq 1$, and $n \geq 0$,

$$(n+1)B_n^{(1,k_2,\ldots,k_r)} = \delta_{r,2} + \sum_{m=1}^{n-1} \left\{ \binom{n}{m} B_m^{(k_2,\ldots,k_r)} - \binom{n}{m-1} B_m^{(1,k_2,\ldots,k_r)} \right\},$$

where δ is Kronecker's delta function, and for $r \geq 1, k_1 > 1, k_2, \ldots, k_r \geq 1$, and $n \geq 0$,

$$(n+1)B_n^{(k_1,\ldots,k_r)} = B_n^{(k_1-1,k_2,\ldots,k_r)} - \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{(k_1,\ldots,k_r)}.$$

Using these formulas, a program for $B_n^{(k_1,\ldots,k_r)}$ can be written. Each of our Bernoulli numbers B_n^f assigned to rooted trees is a linear combination of the $B_n^{(k_1,\ldots,k_r)}$'s. For example, $B_n^{\lambda_1\lambda_2} = B_n^{(3)} + 2B_n^{(2,1)} + 2B_n^{(1,2)} + 3B_n^{(1,1,1)}$. The formula is given via the map ω. Tables 1–4 in the Appendix show some explicit values of the $B_n^{(k_1,\ldots,k_r)}$'s and the B_n^f's, respectively. Tables 1 and 2 in the Appendix give a guess: is it true that, for $f, g \in \mathcal{F}$ with $\mathrm{wt}(f) = \mathrm{wt}(g)$ and for any $n \geq 0$, the sign of B_n^f is the same as that of B_n^g?

According to Tables 1 and 2 in the Appendix, one might have another guess that, for any forest f, the signs of B_n^f appear alternately two by two from a certain n. But this may not be the case because, for example, our calculation shows that the three consecutive terms $B_{19}^f, B_{20}^f, B_{21}^f$ for $f = \lambda_4$ are all negative.

Another observation is on the numerators of the B_n^f's. It is shown in [1, Theorem 14.7] that for any prime p,

$$p^k B_{p-1}^{(k)} \equiv -1 \pmod{p}.$$

This property might be extended as follows: for any $f \in \mathcal{F}$ and any prime p,

$$p^{\mathrm{wt}(f)} B_{p-1}^f \equiv -1 \pmod{p}.$$

4. Rooted Tree Polylogarithms with Negative Exponents and Bernoulli Numbers

In the context of formal power series, we often consider multiple polylogarithms with indices consisting of negative components. For $r, k_1, \ldots, k_r \geq 1$, let

$$Li_{\overline{z_{k_1} \cdots z_{k_r}}}(z) = Li_{(-k_1,\ldots,-k_r)}(z).$$

The corresponding Bernoulli numbers are equal, that is, $B_n^{\overline{z_{k_1} \cdots z_{k_r}}} = B_n^{(-k_1,\ldots,-k_r)}$. By $Li_{a\overline{v}+b\overline{w}}(z) = aLi_{\overline{v}}(z) + bLi_{\overline{w}}(z)$ for the words v, w in $\mathfrak{H}$ and $a, b \in \mathbb{Q}$, we extend the subscript linearly.

We study a rooted tree analogue of this. For $f \in \mathcal{F}$, put

$$Li_{\overline{f}}(z) = \sum_{m:f\text{-order}} |m| z^{|m|_\infty}$$

and define $B_n^{\overline{f}}$ by

$$\sum_{m=0}^{\infty} B_n^{\overline{f}} \frac{z^n}{n!} = \frac{Li_{\overline{f}}(1 - e^{-z})}{1 - e^{-z}}.$$

Since $Li_{\overline{f}}(z) = Li_{\overline{\omega(f)}}(z)$, we have

$$B_n^{\overline{f}} = B_n^{\overline{\omega(f)}}. \tag{2}$$

By [11, Corollary 6] (in which the statement 'positive' should be modified to 'non-negative'), we find $B_n^{\overline{f}} \geq 0$ for any $f \in \mathcal{F}$ and any $n \geq 0$. However, $B_n^{\overline{f}}$ cannot be zero. By Equation (2), we have $B_n^{\overline{f}} = B_n^{(-k)} + \sum c_{k_1,\ldots,k_j} B_n^{(-k_1,\ldots,-k_j)}$, where $k = \mathrm{wt}(f), c_{k_1,\ldots,k_j} \geq 0$, and the sum runs over $2 \leq j \leq k, k_1 + \cdots + k_j = k$. The

leading term, the poly-Bernoulli number $B_n^{(-k)}$, is a positive integer since, for any $n, k \geq 0$,

$$B_n^{(-k)} = \sum_{j=0}^{\min(n,k)} (j!)^2 \left\{ {n+1 \atop j+1} \right\} \left\{ {k+1 \atop j+1} \right\}$$

(see, for example, [1, Theorem 14.14]). The rest of the terms are nonnegative integers. Therefore, we have the following proposition.

Proposition 2. *$B_n^{\overline{f}}$ is a positive integer for any $f \in \mathcal{F}$ and any $n \geq 0$.*

By the way, multiple poly-Bernoulli numbers satisfy the following recurrence formulas:

- For $r \geq 1, k_1, \ldots, k_r \in \mathbb{Z}$, and $n \geq 0$,

$$B_n^{(k_1-1,k_2,\ldots,k_r)} = (n+1)B_n^{(k_1,\ldots,k_r)} + \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{(k_1,\ldots,k_r)}.$$

  See [11, Proposition 2].

- For $r \geq 2$ and $k_2, \ldots, k_r \in \mathbb{Z}$, we have

$$Li_{0,k_2,\ldots,k_r}(z) = \frac{z}{1-z} Li_{k_2,\ldots,k_r}(z),$$

  and hence, for $r \geq 2, k_2, \ldots, k_r \in \mathbb{Z}$, and $n \geq 0$, we have

$$B_n^{(0,k_2,\ldots,k_r)} = \sum_{m=0}^{n-1} \binom{n}{m} B_m^{(k_2,\ldots,k_r)}.$$

Using these formulas together with $B_n^{(0)} = 1$ for any $n \geq 0$, one can calculate multiple poly-Bernoulli numbers $B_n^{(-k_1,\ldots,-k_r)}$(which are also denoted by $B_n^{\overline{(k_1,\ldots,k_r)}}$ or $B_n^{\overline{z_{k_1}\cdots z_{k_r}}}$) for $k_1, \ldots, k_r \geq 1$, and also $B_n^{\overline{f}}$ by Equation (2). Some of these values are given in Tables 5, 6, and 7 in the Appendix. According to them, we expect the following explicit formulas for $B_n^{(-k_1,\ldots,-k_r)}$ and $B_n^{\overline{f}}$:

$$\begin{aligned}
B_n^{(-1)} &= 2^n, \\
B_n^{(-2)} &= 2 \cdot 3^n - 2^n, \\
B_n^{(-1,-1)} &= 3 \cdot 4^n - 4 \cdot 3^n + 2^n, \\
B_n^{(-3)} &= 6 \cdot 4^n - 6 \cdot 3^n + 2^n, \\
B_n^{(-2,-1)} &= 12 \cdot 5^n - 21 \cdot 4^n + 10 \cdot 3^n - 2^n, \\
B_n^{(-1,-2)} &= 8 \cdot 5^n - 15 \cdot 4^n + 8 \cdot 3^n - 2^n, \\
B_n^{(-1,-1,-1)} &= 15 \cdot 6^n - 40 \cdot 5^n + 36 \cdot 4^n - 12 \cdot 3^n + 2^n, \\
B_n^{(-4)} &= 24 \cdot 5^n - 36 \cdot 4^n + 14 \cdot 3^n - 2^n, \\
B_n^{(-3,-1)} &= 60 \cdot 6^n - 132 \cdot 5^n + 93 \cdot 4^n - 22 \cdot 3^n + 2^n,
\end{aligned}$$

$$
\begin{aligned}
B_n^{(-2,-2)} &= 40\cdot 6^n - 92\cdot 5^n + 69\cdot 4^n - 18\cdot 3^n + 2^n,\\
B_n^{(-1,-3)} &= 30\cdot 6^n - 72\cdot 5^n + 57\cdot 4^n - 16\cdot 3^n + 2^n,\\
B_n^{(-2,-1,-1)} &= 90\cdot 7^n - 275\cdot 6^n + 304\cdot 5^n - 144\cdot 4^n + 26\cdot 3^n - 2^n,\\
B_n^{(-1,-2,-1)} &= 72\cdot 7^n - 225\cdot 6^n + 256\cdot 5^n - 126\cdot 4^n + 24\cdot 3^n - 2^n,\\
B_n^{(-1,-1,-2)} &= 48\cdot 7^n - 155\cdot 6^n + 188\cdot 5^n - 96\cdot 4^n + 20\cdot 3^n - 2^n,\\
B_n^{(-1,-1,-1,-1)} &= 105\cdot 8^n - 420\cdot 7^n + 655\cdot 6^n - 496\cdot 5^n + 183\cdot 4^n - 28\cdot 3^n + 2^n,
\end{aligned}
$$

$B_n^{\overline{\bullet}} = 2^n,$

$B_n^{\overline{\bullet\!-\!\bullet}} = 3\cdot 4^n - 2\cdot 3^n,$

$B_n^{\overline{\bullet\ \bullet}} = 6\cdot 4^n - 6\cdot 3^n + 2^n,$

$B_n^{\overline{\bullet\!-\!\bullet\!-\!\bullet}} = 15\cdot 6^n - 20\cdot 5^n + 6\cdot 4^n,$

$B_n^{\overline{\bullet\!-\!(\bullet,\bullet)}} = 30\cdot 6^n - 48\cdot 5^n + 21\cdot 4^n - 2\cdot 3^n,$

$B_n^{\overline{(\bullet\!-\!\bullet)\ \bullet}} = 45\cdot 6^n - 80\cdot 5^n + 42\cdot 4^n - 6\cdot 3^n,$

$B_n^{\overline{\bullet\ \bullet\ \bullet}} = 90\cdot 6^n - 180\cdot 5^n + 114\cdot 4^n - 24\cdot 3^n + 2^n,$

$B_n^{\overline{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet}} = 105\cdot 8^n - 210\cdot 7^n + 130\cdot 6^n - 24\cdot 5^n,$

$B_n^{\overline{(\bullet\!-\!\bullet\!-\!\bullet)\!-\!\bullet}} = 210\cdot 8^n - 468\cdot 7^n + 345\cdot 6^n - 92\cdot 5^n + 6\cdot 4^n,$

$B_n^{\overline{(\bullet\!-\!\bullet)\ (\bullet\!-\!\bullet)}} = 315\cdot 8^n - 750\cdot 7^n + 610\cdot 6^n - 192\cdot 5^n + 18\cdot 4^n,$

$B_n^{\overline{(\bullet\!-\!(\bullet,\bullet))\!-\!\bullet}} = 630\cdot 8^n - 1620\cdot 7^n + 1470\cdot 6^n - 552\cdot 5^n + 75\cdot 4^n - 2\cdot 3^n,$

$B_n^{\overline{\bullet\!-\!\bullet\!-\!(\bullet,\bullet)}} = 420\cdot 8^n - 1050\cdot 7^n + 915\cdot 6^n - 320\cdot 5^n + 36\cdot 4^n,$

$B_n^{\overline{\bullet\!-\!(\bullet,\bullet,\bullet)}} = 630\cdot 8^n - 1680\cdot 7^n + 1610\cdot 6^n - 660\cdot 5^n + 105\cdot 4^n - 4\cdot 3^n,$

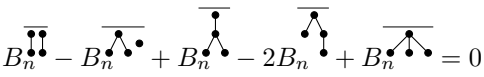
$B_n^{\overline{(\bullet\!-\!\bullet\!-\!\bullet)\ \bullet}} = 840\cdot 8^n - 2268\cdot 7^n + 2205\cdot 6^n - 920\cdot 5^n + 150\cdot 4^n - 6\cdot 3^n,$

$B_n^{\overline{\bullet\!-\!(\bullet,\bullet)\ \bullet}} = 1260\cdot 8^n - 3570\cdot 7^n + 3705\cdot 6^n - 1700\cdot 5^n + 324\cdot 4^n - 18\cdot 3^n,$

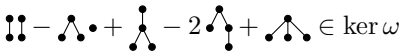
$B_n^{\overline{\bullet\ \bullet\ \bullet\ \bullet}} = 2520\cdot 8^n - 7560\cdot 7^n + 8460\cdot 6^n - 4320\cdot 5^n + 978\cdot 4^n - 78\cdot 3^n + 2^n.$

Let $B(4)$ be the space spanned by the sequences $\{B_n^{\overline{f}} \mid (f) = 4\}$. According to the explicit formulas above, we find $\dim B(4) = 6$, and hence, we have three linearly

independent relations for the $B_n^{\overline{f}}$'s with $\mathrm{wt}(f)=4$. One of the three is



because



(see [14]), and hence, the sum of the corresponding polylogarithms is 0.

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Appendix

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$B_n^{\mathbb{I}}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$B_n^{\bullet}$	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{-1}{30}$	0	$\frac{5}{66}$	0	$\frac{-691}{2730}$
$B_n^{\bullet\bullet}$	1	$\frac{3}{4}$	$\frac{17}{36}$	$\frac{5}{24}$	$\frac{7}{450}$	$\frac{-7}{120}$	$\frac{-38}{2205}$	$\frac{3}{56}$	$\frac{11}{350}$	$\frac{-11}{120}$	$\frac{-3263}{38115}$	$\frac{65}{264}$	$\frac{13399637}{40990950}$
$B_n^{\bullet\bullet}$	1	$\frac{5}{4}$	$\frac{35}{36}$	$\frac{11}{24}$	$\frac{7}{450}$	$\frac{-17}{120}$	$\frac{-38}{2205}$	$\frac{23}{168}$	$\frac{11}{350}$	$\frac{-29}{120}$	$\frac{-3263}{38115}$	$\frac{175}{264}$	$\frac{13399637}{40990950}$
$B_n^{\bullet\bullet}$	1	$\frac{7}{8}$	$\frac{151}{216}$	$\frac{137}{288}$	$\frac{12493}{54000}$	$\frac{161}{7200}$	$\frac{-291703}{3704400}$	$\frac{-1207}{35280}$	$\frac{659921}{7938000}$	$\frac{1331}{16800}$	$\frac{-12129947}{78262800}$	$\frac{-79547}{304920}$	$\frac{2134239258113}{4923832914000}$
B_n	1	$\frac{9}{8}$	$\frac{241}{216}$	$\frac{263}{288}$	$\frac{28243}{54000}$	$\frac{511}{7200}$	$\frac{-754753}{3704400}$	$\frac{-1219}{11760}$	$\frac{1872671}{7938000}$	$\frac{3949}{16800}$	$\frac{-37565357}{78262800}$	$\frac{-233701}{304920}$	$\frac{7263231876863}{4923832914000}$
B_n	1	$\frac{13}{8}$	$\frac{385}{216}$	$\frac{419}{288}$	$\frac{41743}{54000}$	$\frac{371}{7200}$	$\frac{-1125193}{3704400}$	$\frac{-3013}{35280}$	$\frac{2849171}{7938000}$	$\frac{33}{160}$	$\frac{-19478479}{26087600}$	$\frac{-212537}{304920}$	$\frac{11552934794363}{4923832914000}$
$B_n^{\bullet\bullet\bullet}$	1	$\frac{19}{8}$	$\frac{691}{216}$	$\frac{845}{288}$	$\frac{88993}{54000}$	$\frac{581}{7200}$	$\frac{-2576083}{3704400}$	$\frac{-4819}{35280}$	$\frac{6802421}{7938000}$	$\frac{5599}{16800}$	$\frac{-143872327}{78262800}$	$\frac{-49361}{43560}$	$\frac{29178018520613}{4923832914000}$

Table 1: B_n^f ($\mathrm{wt}(f)= 0, 1, 2, 3$).

n	0	1	2	3	4	5	6	7	8	9	10	11	12
B_n	1	$\frac{15}{16}$	$\frac{1085}{1296}$	$\frac{2375}{3456}$	$\frac{1567541}{3240000}$	$\frac{105707}{432000}$	$\frac{35635723}{1555848000}$	$\frac{-1905199}{19756800}$	$\frac{-231609829}{5000940000}$	$\frac{455543}{3969000}$	$\frac{1280051188949}{9762501672000}$	$\frac{-2570318231}{11269843200}$	$\frac{-1116589595846267 33}{2217940536111300 00}$
B_n	1	$\frac{17}{16}$	$\frac{1427}{1296}$	$\frac{3689}{3456}$	$\frac{2915291}{3240000}$	$\frac{241957}{432000}$	$\frac{164517973}{1555848000}$	$\frac{-14582147}{59270400}$	$\frac{-954739579}{5000940000}$	$\frac{1198187}{3969000}$	$\frac{5029040835659}{9762501672000}$	$\frac{-271667329}{450793728}$	$\frac{-4307440535420064 83}{2217940536111300 00}$
B_n	1	$\frac{21}{16}$	$\frac{2039}{1296}$	$\frac{5741}{3456}$	$\frac{4674791}{3240000}$	$\frac{374057}{432000}$	$\frac{148774273}{1555848000}$	$\frac{-2775223}{6585600}$	$\frac{-1016700079}{5000940000}$	$\frac{4633}{7938}$	$\frac{5815241532479}{9762501672000}$	$\frac{-15257180573}{11269843200}$	$\frac{-5235981049536279 83}{2217940536111300 00}$
B_n	1	$\frac{27}{16}$	$\frac{3137}{1296}$	$\frac{10115}{3456}$	$\frac{9168041}{3240000}$	$\frac{804407}{432000}$	$\frac{397740823}{1555848000}$	$\frac{-19273739}{19756800}$	$\frac{-2641454329}{5000940000}$	$\frac{5610707}{3969000}$	$\frac{14968719704009}{9762501672000}$	$\frac{-7701904183}{2253968640}$	$\frac{-1345292060315157 233}{2217940536111300 00}$
B_n	1	$\frac{29}{16}$	$\frac{3011}{1296}$	$\frac{8309}{3456}$	$\frac{6384791}{3240000}$	$\frac{469177}{432000}$	$\frac{115496413}{1555848000}$	$\frac{-29519783}{59270400}$	$\frac{-877218079}{5000940000}$	$\frac{766441}{1134000}$	$\frac{5265519712139}{9762501672000}$	$\frac{-1170852619}{751322880}$	$\frac{-4869088354267179 83}{2217940536111300 00}$
B_n	1	$\frac{33}{16}$	$\frac{3839}{1296}$	$\frac{11657}{3456}$	$\frac{9656291}{3240000}$	$\frac{745277}{432000}$	$\frac{146057713}{1555848000}$	$\frac{-17009281}{19756800}$	$\frac{-1198486579}{5000940000}$	$\frac{202387}{162000}$	$\frac{7410088697159}{9762501672000}$	$\frac{-34621542073}{11269843200}$	$\frac{-6960490487682774 83}{2217940536111300 00}$
B_n	1	$\frac{35}{16}$	$\frac{4325}{1296}$	$\frac{13979}{3456}$	$\frac{12390041}{3240000}$	$\frac{1043527}{432000}$	$\frac{410767963}{1555848000}$	$\frac{-73477193}{59270400}$	$\frac{-2761280329}{5000940000}$	$\frac{291199}{162000}$	$\frac{15777366171869}{9762501672000}$	$\frac{-5489820563}{1252204800}$	$\frac{-1424888952718185 233}{2217940536111300 00}$
B_n	1	$\frac{47}{16}$	$\frac{6557}{1296}$	$\frac{22055}{3456}$	$\frac{19369541}{3240000}$	$\frac{1540747}{432000}$	$\frac{270988603}{1555848000}$	$\frac{-109159469}{59270400}$	$\frac{-2318610829}{5000940000}$	$\frac{10815857}{3969000}$	$\frac{14607579926549}{9762501672000}$	$\frac{-10958370641}{1609977600}$	$\frac{-1388687552969170 733}{2217940536111300 00}$
B_n	1	$\frac{65}{16}$	$\frac{10895}{1296}$	$\frac{41129}{3456}$	$\frac{38870291}{3240000}$	$\frac{3224317}{432000}$	$\frac{471550993}{1555848000}$	$\frac{-48082759}{11854080}$	$\frac{-4234519579}{5000940000}$	$\frac{49128091}{7938000}$	$\frac{27161662847159}{9762501672000}$	$\frac{-178445015449}{11269843200}$	$\frac{-2607854898870425 483}{2217940536111300 00}$

Table 2: B_n^f ($\mathrm{wt}(f)= 4$).

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$B_n^{(0)}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$B_n^{(1)}$	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{-1}{30}$	0	$\frac{5}{66}$	0	$\frac{-691}{2730}$
$B_n^{(2)}$	1	$\frac{1}{4}$	$\frac{-1}{36}$	$\frac{-1}{24}$	$\frac{7}{450}$	$\frac{1}{40}$	$\frac{-38}{2205}$	$\frac{-5}{168}$	$\frac{11}{350}$	$\frac{7}{120}$	$\frac{-3263}{38115}$	$\frac{-15}{88}$	$\frac{13399637}{40990950}$
$B_n^{(1,1)}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{-1}{12}$	0	$\frac{1}{12}$	0	$\frac{-3}{20}$	0	$\frac{5}{12}$	0
$B_n^{(3)}$	1	$\frac{1}{8}$	$\frac{-11}{216}$	$\frac{-1}{288}$	$\frac{1243}{54000}$	$\frac{-49}{7200}$	$\frac{-75613}{3704400}$	$\frac{599}{35280}$	$\frac{234671}{7938000}$	$\frac{-803}{16800}$	$\frac{-4955857}{78262800}$	$\frac{53443}{304920}$	$\frac{921931911863}{4923832914000}$
$B_n^{(2,1)}$	0	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{-1}{16}$	$\frac{-1}{24}$	$\frac{7}{144}$	$\frac{1}{24}$	$\frac{-5}{72}$	$\frac{-5}{72}$	$\frac{187}{1200}$	$\frac{7}{40}$	$\frac{-91}{180}$	$\frac{-5}{8}$
$B_n^{(1,2)}$	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{24}$	$\frac{-1}{12}$	$\frac{-7}{360}$	$\frac{1}{15}$	$\frac{23}{1260}$	$\frac{-25}{252}$	$\frac{-121}{4200}$	$\frac{7}{30}$	$\frac{481}{6930}$	$\frac{-35}{44}$
$B_n^{(1,1,1)}$	0	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{-1}{6}$	0	$\frac{2}{9}$	0	$\frac{-1}{2}$	0	$\frac{5}{3}$

Table 3: $B_n^{(k_1,\ldots,k_r)}$ $(k_1+\cdots+k_r=0,1,2,3)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$B_n^{(4)}$	1	$\frac{1}{16}$	$\frac{-49}{1296}$	$\frac{41}{3456}$	$\frac{26291}{3240000}$	$\frac{-1921}{144000}$	$\frac{845233}{1555848000}$	$\frac{1048349}{59270400}$	$\frac{-60517579}{5000940000}$	$\frac{-50233}{1587600}$	$\frac{506605371959}{9762501672000}$	$\frac{823605863}{11269843200}$	$\frac{-53797712101337483}{2217940536111300000}$
$B_n^{(3,1)}$	0	$\frac{1}{8}$	$\frac{-1}{72}$	$\frac{-7}{192}$	$\frac{31}{1440}$	$\frac{41}{1728}$	$\frac{-23}{672}$	$\frac{-85}{3456}$	$\frac{487}{6480}$	$\frac{103}{3000}$	$\frac{-54217}{237600}$	$\frac{-4283}{86400}$	$\frac{18341}{19656}$
$B_n^{(2,2)}$	0	$\frac{1}{4}$	$\frac{1}{36}$	$\frac{-7}{96}$	$\frac{-1}{720}$	$\frac{241}{4320}$	$\frac{-47}{5040}$	$\frac{-4559}{60480}$	$\frac{139}{4536}$	$\frac{10001}{63000}$	$\frac{-90929}{831600}$	$\frac{-230233}{475200}$	$\frac{52763}{108108}$
$B_n^{(1,3)}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{-1}{144}$	$\frac{-1}{18}$	$\frac{8}{675}$	$\frac{133}{3600}$	$\frac{-19381}{1058400}$	$\frac{-199}{4410}$	$\frac{34093}{882000}$	$\frac{737}{8400}$	$\frac{-14460679}{128066400}$	$\frac{-9568}{38115}$
$B_n^{(2,1,1)}$	0	0	$\frac{1}{9}$	$\frac{1}{24}$	$\frac{-13}{180}$	$\frac{-1}{24}$	$\frac{11}{126}$	$\frac{1}{16}$	$\frac{-68}{405}$	$\frac{-5}{36}$	$\frac{281}{594}$	$\frac{7}{16}$	$\frac{-22696}{12285}$
$B_n^{(1,2,1)}$	0	0	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{-1}{30}$	$\frac{-1}{12}$	$\frac{5}{168}$	$\frac{5}{48}$	$\frac{-7}{135}$	$\frac{-5}{24}$	$\frac{551}{3960}$	$\frac{49}{80}$	$\frac{-2147}{4095}$
$B_n^{(1,1,2)}$	0	0	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{7}{60}$	$\frac{-1}{8}$	$\frac{-37}{420}$	$\frac{31}{240}$	$\frac{118}{945}$	$\frac{-5}{21}$	$\frac{-785}{2772}$	$\frac{161}{240}$	$\frac{42376}{45045}$
$B_n^{(1,1,1,1)}$	0	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{12}$	0	$\frac{-7}{24}$	0	$\frac{1}{2}$	0	$\frac{-11}{8}$	0

Table 4: $B_n^{(k_1,\ldots,k_r)}$ $(k_1+\cdots+k_r=4)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$B_n^{\bullet}$	1	2	4	8	16	32	64	128	256	512	1024	2048	4096
$B_n^{\bullet\bullet}$	1	6	30	138	606	2586	10830	44778	183486	747066	3027630	12228618	49268766
$B_n^{\bullet\bullet\bullet}$	1	8	46	230	1066	4718	20266	85310	354106	1455278	5938186	24104990	97478746
$B_n^{\bullet\bullet\bullet\bullet}$	1	14	136	1124	8476	60284	411916	2734844	17774956	113675804	717971596	4490559164	27869585836
$B_n^{\bullet\bullet\bullet\bullet}$	1	18	198	1770	14094	104298	734238	4987770	33001614	214046538	1367137278	8627887770	53935978734
$B_n^{\bullet\bullet\bullet\bullet}$	1	20	238	2246	18586	141470	1017178	7022126	47045866	308138270	1983413818	12594715406	79125409546
$B_n^{\bullet\bullet\bullet\bullet}$	1	26	352	3596	31396	248276	1836052	12947156	88166836	584842676	3802264852	24336573716	153864949876

Table 5: $B_n^{\overline{f}}$ $(\mathrm{wt}(f)=1,2,3)$.

$B_n^{\overline{f}}$ \ n	0	1	2	3	4	5	6	7	8	9	10	11	12
$B_n^{\overline{\bullet\bullet\bullet\bullet}}$	1	30	510	6810	79350	847050	8509110	81773610	759974550	6881829450	61049317110	532696258410	4585997073750
$B_n^{\overline{\bullet\bullet\bullet\bullet}}$	1	38	724	10400	127648	1416968	14673904	144472520	1369211728	12598925288	113256051184	999195311240	8681997336208
$B_n^{\overline{\bullet\bullet\bullet\bullet}}$	1	42	858	12942	164658	1878462	19872498	199001502	1911967698	17790492222	161400479538	1434841040862	12546933985938
$B_n^{\overline{\bullet\bullet\bullet\bullet}}$	1	54	1242	20166	270018	3198534	34824402	356671446	3488981058	32940034854	302420624562	2715062235126	23936746715298
$B_n^{\overline{\bullet\bullet\bullet\bullet}}$	1	44	946	14834	194326	2267114	24406726	247814954	2407597606	22605687434	206611854406	1848064749674	16243063485286
$B_n^{\overline{\bullet\bullet\bullet\bullet}}$	1	48	1104	18192	247416	2971488	32731224	338499552	3338004696	31726583328	292914025944	2642066255712	23385154076376
$B_n^{\overline{\bullet\bullet\bullet\bullet}}$	1	56	1354	22874	315766	3830066	42484534	441640514	4372262086	41684559026	385789742614	3486641720354	30910030156006
$B_n^{\overline{\bullet\bullet\bullet\bullet}}$	1	62	1612	28640	410056	5111672	57906472	611982440	6138982936	59154091832	552242354632	5026772159240	44828511124216
$B_n^{\overline{\bullet\bullet\bullet\bullet}}$	1	80	2350	45014	678586	8781950	102335290	1105449614	11282322826	110230055870	1040693765530	9560458802414	85910207866666

Table 6: $B_n^{\overline{f}}$ $(\mathrm{wt}(f) = 4)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$B_n^{\overline{(0)}}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$B_n^{\overline{(1)}}$	1	2	4	8	16	32	64	128	256	512	1024	2048	4096
$B_n^{\overline{(2)}}$	1	4	14	46	146	454	1394	4246	12866	38854	117074	352246	1058786
$B_n^{\overline{(1,1)}}$	0	2	16	92	460	2132	9436	40532	170620	708212	2910556	11876372	48209980
$B_n^{\overline{(3)}}$	1	8	46	230	1066	4718	20266	85310	354106	1455278	5938186	24104990	97478746
$B_n^{\overline{(2,1)}}$	0	4	50	418	2918	18394	108710	615178	3376598	18128794	95756870	499626538	2582676278
$B_n^{\overline{(1,2)}}$	0	2	28	248	1792	11552	69328	396608	2194192	11849792	62867728	329125568	1705714192
$B_n^{\overline{(1,1,1)}}$	0	0	12	228	2700	25620	213612	1637748	11850060	82241940	553408812	3637702068	23483716620
$B_n^{\overline{(4)}}$	1	16	146	1066	6902	41506	237686	1315666	7107302	37712866	197451926	1023358066	5262831302
$B_n^{\overline{(3,1)}}$	0	8	154	1826	17302	143978	1101814	7959386	55165222	370796138	2435127574	15708685946	99928971142
$B_n^{\overline{(2,2)}}$	0	4	86	1078	10562	89854	698306	5101198	35651282	241154494	1591499426	10305915118	65758422002
$B_n^{\overline{(1,3)}}$	0	2	52	704	7192	62792	496552	3672104	25894312	176333672	1169685352	7604529704	48673147432
$B_n^{\overline{(2,1,1)}}$	0	0	36	948	14916	183060	1937076	18583908	166420836	1416970740	11613863316	92434590468	718917349956
$B_n^{\overline{(1,2,1)}}$	0	0	24	672	10944	137280	1474464	14297472	129051744	1105419840	9102623904	72713955072	567195474144
$B_n^{\overline{(1,1,2)}}$	0	0	12	372	6396	82980	911772	8985732	82084476	709532580	5883946332	47263402692	370302410556
$B_n^{\overline{(1,1,1,1)}}$	0	0	0	144	5136	105600	1651440	21858144	258599376	2823909120	29055119280	285641821344	2709958467216

Table 7: $B_n^{\overline{(k_1,\ldots,k_r)}}$ $((k_1 + \cdots + k_r = 0, 1, 2, 3, 4)$.