

ON PRIME AND COMPOSITE RECURSIVE DIVISORS

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Abstract

In this paper, we define prime and composite recursive divisor functions to investigate some properties of prime and composite recursive divisors of a positive integer. We derive explicit expressions that characterize these functions and establish their relationship. In addition, we prove their relationship to the recursive divisor function.

1. Introduction

The number of positive divisors d of a positive integer n, as well as the sum of these divisors, can both be found by the *divisor function* $\sigma_x(n) = \sum_{d|n} d^x$, where x is a nonnegative integer [1]; this function is of extreme importance and is closely related to interesting problems such as the Riemann Hypothesis [5]. Modern developments have also considered arithmetic functions related to prime factors, such as $\omega(n)$, the number of distinct prime factors of n, and $\Omega(n)$, the total number of prime factors of n. Classical results show that integers satisfying constraints on these functions have well-defined density properties [6], and later generalizations study their distribution and ergodic behavior, which are closely tied to the distribution of prime numbers and hence to results such as the Prime Number Theorem [8].

Further research has explored refined aspects of divisors such as their distribution, consecutive prime factors, analytic behavior, and the occurrence of prime divisors

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in structured sequences such as Beatty sequences [2], as well as the distribution of prime factors of values of the sum-of-proper-divisors function [7].

Recursive divisors are a recursive analogue of standard divisors [4]. Let n be a positive integer. We consider the set of all positive divisors of n. For each nontrivial divisor, we consider its proper divisors. We repeat this process recursively for all newly obtained divisors, continuing until the only divisor that remains is 1. All divisors obtained through this process are called *recursive divisors* of n. For instance, let $n = 8$. We find the recursive divisors as follows.

$$\begin{array}{cccc} 1 & 2 & 4 & 8 \\ & \downarrow & \downarrow & \\ & 1 & 1,2 & \\ & & \downarrow & \\ & & 1. & \end{array}$$

Thus, 8 has eight recursive divisors, namely: $1, 2, 4, 8, 1, 1, 2$, and 1. To study the recursive divisors of a given positive integer n, Fink [4] defined the *recursive divisor function* $\kappa_x(n)$ as

$$\kappa_x(n) = n^x + \sum_{d \lfloor n} \kappa_x(d), \tag{1}$$

where x is any nonnegative integer, and $d \lfloor n$ means $d|n$ and $d < n$. We note that $\kappa_0(n)$ counts the total number of recursive divisors of n, while $\kappa_1(n)$ sums up the recursive divisors of n, $\kappa_2(n)$ sums up the squares of the recursive divisors of n and so on. In his recent work, Fink derived explicit closed-form expressions for the function $\kappa_0(n)$, revealing deeper structural and analytic properties of this recursively defined function [3]. These expressions depend on the prime signature of n and are therefore closely related to classical quantities such as $\omega(n)$ and $\Omega(n)$. However, for the more general function $\kappa_x(n)$, obtaining similar closed-form expressions remains an open problem, motivating further investigation in this direction. We observe that any recursive divisor greater than 1 is either a prime or a composite. In this paper, we study the prime and composite recursive divisors separately in detail and establish the relationships between them.

The outline of the paper is as follows. Section 2 provides preliminary results. Section 3 introduces the prime recursive divisor function. Section 4 presents the composite recursive divisor function. Section 5 exhibits the relationship between the prime and composite divisor functions of a positive integer. Section 6 concludes the paper.

2. Preliminaries

In this section, we list a few results from [4] which are useful for the present study. In Equation (1), $\kappa_0(n)$ gives the number of recursive divisors of positive integer n.

To get the number of recursive divisors which are of a specific value j, we use the function $\kappa_0(n, j)$ [4], defined by

$$\kappa_0(n,j) = \begin{cases} \sum_{d \lfloor n} \kappa_0(d,j), & j \lfloor n, \\ 1, & j = n, \\ 0, & \text{otherwise.} \end{cases}$$

For example, the recursive divisors of 8 are $1, 2, 4, 8, 1, 1, 2$, and 1. The number of recursive divisors which are 1, is given by $\kappa_0(8, 1) = 4$.

Lemma 1. ([4]). *Let $j \in \mathbb{N}$ be a recursive divisor of some positive integer n. Then, $\kappa_0(jn, j) = \kappa_0(n, 1)$.*

Lemma 2. ([4]). *For any positive integer $n \geq 2$, the number of recursive divisors is twice the number of recursive divisors of size 1, that is, $\kappa_0(n) = 2\kappa_0(n, 1)$.*

Theorem 1. ([4]). *Let $a \in \mathbb{N}$ and p be a prime. Then, $\kappa_0(p^a) = 2^a$.*

Corollary 1. ([4]). *Let $\alpha \in \mathbb{N}$ be the maximum exponent in the prime factorization of some positive integer n. Then, 2^α divides $\kappa_0(n)$.*

3. Prime Recursive Divisors

To find the number or sum of the recursive divisors that are primes, the recursive divisor function $\kappa_x(n)$ is insufficient. Hence, we introduce a new function that considers only the primes from the collection of all recursive divisors. We also use the fact that the only prime recursive divisor of a prime number is itself.

Definition 1. Let n be a positive integer and x be a nonnegative integer. Then, the *prime recursive divisor function* is defined as

$$P_x(n) = \begin{cases} n^x, & n \text{ is prime}, \\ \sum_{d \lfloor n} P_x(d), & n \text{ is composite}. \end{cases}$$

We define $P_x(1) = 0$.

To get the number of prime recursive divisors of n, we set $x = 0$. When $x = 1$, we obtain the sum of prime recursive divisors of n and when $x = 2$, we obtain the sum

of squares of prime recursive divisors of n and so on. To illustrate this function, we consider $n = 12$. Then,

$$\begin{aligned} P_x(12) &= \sum_{d\lfloor 12} P_x(d) \\ &= P_x(1) + P_x(2) + P_x(3) + P_x(4) + P_x(6) \\ &= 2^x + 3^x + \sum_{d\lfloor 4} P_x(d) + \sum_{d\lfloor 6} P_x(d) \\ &= 3 \cdot 2^x + 2 \cdot 3^x. \end{aligned}$$

We have $P_0(12) = 5$, that is, the number of prime recursive divisors of 12 is five and $P_1(12) = 12$, that is, the sum of the prime recursive divisors of 12 is twelve. We compute the prime recursive divisor function $P_x(n)$ for the first twenty positive integers with $x = 0, 1$, and 2 and display the results in Table 1.

n	$P_0(n)$	$P_1(n)$	$P_2(n)$	n	$P_0(n)$	$P_1(n)$	$P_2(n)$
1	0	0	0	**11**	1	11	121
2	1	2	4	**12**	5	12	30
3	1	3	9	**13**	1	13	169
4	1	2	4	**14**	2	9	53
5	1	5	25	**15**	2	8	34
6	2	5	13	**16**	4	8	16
7	1	7	49	**17**	1	17	289
8	2	4	8	**18**	5	13	35
9	1	3	9	**19**	1	19	361
10	2	7	29	**20**	5	16	62

Table 1: Values of $P_0(n)$, $P_1(n)$ and $P_2(n)$ for $n = 1$ to 20.

We observe that $P_2(2^a) = 2^a$ for some integer $a > 1$. Now, we derive a general expression for the prime recursive divisor function $P_x(n)$ in terms of $\kappa_0(n, j)$, that is, the number of recursive divisors of size of j.

Theorem 2. *Let n be composite and $d < n$ be prime. Then,*

$$P_x(n) = \frac{1}{2} \sum_{\substack{d|n \\ d:prime}} d^x \kappa_0\left(\frac{n}{d}\right).$$

Proof. The prime recursive divisor function $P_x(n)$ can be written as

$$P_x(n) = \sum_{\substack{d|n \\ d:\text{prime}}} d^x \kappa_0(n, d).$$

Using Lemma 1,

$$P_x(n) = \sum_{\substack{d|n \\ d:\text{prime}}} d^x \kappa_0(n/d, 1).$$

Since $\frac{n}{d} \geq 2$, we apply Lemma 2 to arrive at

$$P_x(n) = \frac{1}{2} \sum_{\substack{d|n \\ d:\text{prime}}} d^x \kappa_0\left(\frac{n}{d}\right).$$

□

Remark 1. Let $a \in \mathbb{N},\ a > 1$, and $n = p^a$ for some prime p. Then, $P_x(n) = p^x 2^{a-2}$.

From Theorem 2,

$$P_x(n) = \frac{1}{2} \sum_{d|n} d^x \kappa_0\left(\frac{n}{d}\right),$$

where the sum runs over the prime divisors of n. Since p is the only prime divisor of n, we obtain

$$\begin{aligned} P_x(n) &= \frac{1}{2} p^x \kappa_0\left(\frac{p^a}{p}\right) \\ &= \frac{1}{2} p^x \kappa_0(p^{a-1}). \end{aligned}$$

From Theorem 1, we have $\kappa_0(p^a) = 2^a$. Hence,

$$P_x(n) = p^x 2^{a-2}.$$

To illustrate this, we consider $n = 81 = 3^4$. Then, $P_x(81) = 3^x 2^2$. Therefore, $P_0(81) = 2^2 = 4$ and $P_1(81) = 3 \cdot 2^2 = 12$.

Theorem 3. *Let m and n be positive integers. Then,*

$$P_0(mn) \geq P_0(m) P_0(n).$$

Proof. When m or n is equal to 1, the result holds as $P_0(1) = 0$. Suppose m and n are primes. If $m = n$, then $P_0(mn) = 1 = P_0(m)P_0(n)$. If $m \neq n$, then the only prime recursive divisors of mn are m and n. Hence,

$$P_0(mn) = 2 \geq P_0(m)P_0(n) = 1.$$

Now, we show that the result holds when one of them is prime and the other is composite. Let m be prime and n be composite. Then,

$$\begin{aligned} P_0(mn) &= \sum_{d \lfloor mn} P_0(d) \\ &> \sum_{d \lfloor n} P_0(d) \\ &= P_0(m)P_0(n). \end{aligned}$$

Finally, we let both m and n be composites. We now proceed by induction on m. Assume that $P_0(kn) \geq P_0(k)P_0(n)$ for all natural numbers $k < m$. Let $m_1, m_2, ..., m_i$ be the proper divisors of m. Then,

$$\begin{aligned} P_0(mn) &= \sum_{d \lfloor mn} P_0(d) \\ &> P_0(m_1 n) + P_0(m_2 n) + \cdots + P_0(m_i n) \\ &\geq P_0(m_1)P_0(n) + P_0(m_2)P_0(n) + \cdots + P_0(m_i)P_0(n) \\ &= [P_0(m_1) + P_0(m_2) + \cdots + P_0(m_i)]\, P_0(n) \\ &= \left(\sum_{d \lfloor m} P_0(d) \right) P_0(n) \\ &= P_0(m)P_0(n). \end{aligned}$$

□

4. Composite Recursive Divisors

In this section, we introduce the composite recursive divisor function to study composite recursive divisors.

Definition 2. Let n be a positive integer and x be a nonnegative integer. Then, the *composite recursive divisor function* is defined as

$$C_x(n) = \begin{cases} n^x + \sum_{d \lfloor n} C_x(d), & n \text{ is composite,} \\ \\ 0, & \text{otherwise.} \end{cases}$$

To get the number of composite recursive divisors of n, we set $x = 0$. When $x = 1$, we obtain the sum of composite recursive divisors of n, and when $x = 2$, we obtain the sum of squares of composite recursive divisors of n and so on. To illustrate this function, we consider $n = 20$. Then,

$$\begin{aligned} C_x(20) &= 20^x + \sum_{d \lfloor 20} C_x(d) \\ &= 20^x + C_x(1) + C_x(2) + C_x(4) + C_x(5) + C_x(10) \\ &= 20^x + 4^x + 10^x. \end{aligned}$$

We have $C_0(20) = 3$, that is, there are three composite recursive divisors for 20 and $C_1(20) = 34$, that is, the sum of those composite recursive divisors is thirty

four. We compute the prime recursive divisor function $C_x(n)$ for the first twenty positive integers with $x = 0, 1$, and 2 and display the results in the Table 2.

n	$C_0(n)$	$C_1(n)$	$C_2(n)$	n	$C_0(n)$	$C_1(n)$	$C_2(n)$
1	0	0	0	**11**	0	0	0
2	0	0	0	**12**	3	22	196
3	0	0	0	**13**	0	0	0
4	1	4	16	**14**	1	14	196
5	0	0	0	**15**	1	15	225
6	1	6	36	**16**	4	32	352
7	0	0	0	**17**	0	0	0
8	2	12	80	**18**	3	33	441
9	1	9	81	**19**	0	0	0
10	1	10	100	**20**	3	34	516

Table 2: Values of $C_0(n)$, $C_1(n)$ and $C_2(n)$ for $n = 1$ to 20.

Theorem 4. *Let n and d be composites such that $d < n$. Then,*

$$C_x(n) = \frac{n^x}{2} + \frac{1}{2} \sum_{\substack{d|n \\ d:composite}} d^x \kappa_0\left(\frac{n}{d}\right).$$

Proof. The composite recursive divisor function $C_x(n)$ can be written as

$$C_x(n) = \sum_{\substack{d|n \\ d:\text{composite}}} d^x \kappa_0(n, d).$$

Using Lemma 1,

$$\begin{aligned} C_x(n) &= \sum_{\substack{d|n \\ d:\text{composite}}} d^x \kappa_0(n/d, 1) \\ &= n^x + \sum_{\substack{d\lfloor n \\ d:\text{composite}}} d^x \kappa_0(n/d, 1). \end{aligned}$$

Using Lemma 2,

$$C_x(n) = n^x + \sum_{\substack{d\lfloor n \\ d:\text{composite}}} d^x \frac{\kappa_0(n/d)}{2}$$

$$C_x(n) = \frac{n^x}{2} + \frac{1}{2} \sum_{\substack{d|n \\ d:\text{composite}}} d^x \kappa_0\left(\frac{n}{d}\right).$$

□

Remark 2. Let $a \in \mathbb{N},\ a > 2$, and $n = p^a$ for some prime p. Then,

$$C_x(n) = n^x + 2^{a-1} \sum_{k=2}^{a-1} \left(\frac{p^x}{2} \right)^k.$$

From Theorem 4,

$$C_x(n) = \frac{n^x}{2} + \frac{1}{2} \sum_{d|n} d^x \kappa_0 \left(\frac{n}{d} \right),$$

where the sum runs over the composite divisors d of n. The composites that divide $n = p^a$ are exactly $p^2, p^3, \ldots, p^a$. Hence, we obtain

$$\begin{aligned} C_x(n) &= \frac{n^x}{2} + \frac{1}{2} \sum_{k=2}^{a} p^{kx} \kappa_0 \left(\frac{p^a}{p^k} \right) \\ &= n^x + \frac{1}{2} \sum_{k=2}^{a-1} p^{kx} \kappa_0(p^{a-k}). \end{aligned}$$

Using Theorem 1, we have $\kappa_0(p^{a-k}) = 2^{a-k}$. Therefore,

$$C_x(n) = n^x + \frac{1}{2} \sum_{k=2}^{a-1} p^{kx} 2^{a-k}.$$

Hence, we have

$$C_x(n) = n^x + 2^{a-1} \sum_{k=2}^{a-1} \left(\frac{p^x}{2} \right)^k.$$

To illustrate this, we consider $n = 32 = 2^5$. Then, $C_x(32) = 32^x + 2^4 \sum_{k=2}^{4} 2^{(x-1)k}$. Therefore, we get $C_0(32) = 1 + 2^4 \sum_{k=2}^{4} 2^{-k} = 1 + 2^4(2^{-2} + 2^{-3} + 2^{-4}) = 8$, and further $C_1(32) = 32 + 2^4 \sum_{k=2}^{4} 1 = 32 + 2^4(3) = 80$.

Theorem 5. *Let m and n be positive integers. Then,*

$$C_0(mn) \geq C_0(m) C_0(n).$$

Proof. We prove the result for composite values of m and n, as it is trivial in all other cases. Assume that $C_0(kn) \geq C_0(k) C_0(n)$ for all natural numbers $k < m$ and

for all n. Let $m_1, m_2, \ldots, m_i$ be the proper divisors of m. Then,

$$\begin{aligned} C_0(mn) &= 1 + \sum_{d \lfloor mn} C_0(d) \\ &> 1 + C_0(m_1 n) + \cdots + C_0(m_i n) + \sum_{d \lfloor n} C_0(d) \\ &\geq C_0(m_1)C_0(n) + \cdots + C_0(m_i)C_0(n) + C_0(n) \\ &= [C_0(m_1) + \cdots + C_0(m_i) + 1]\, C_0(n) \\ &= \left(1 + \sum_{d \lfloor m} C_0(d)\right) C_0(n) \\ &= C_0(m)C_0(n). \end{aligned}$$

Thus, the proof follows by induction. □

5. Relationship between Prime and Composite Recursive Divisors

The functions $P_0(n)$ and $C_0(n)$ represent the number of prime recursive divisors and composite recursive divisors of n, respectively. The following theorem demonstrates that, for any positive integer n, the number of prime recursive divisors is always greater than or equal to the number of composite recursive divisors.

Theorem 6. *For $n \geq 1$, we have $P_0(n) \geq C_0(n)$.*

Proof. Let $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ be the prime factorization of n and $\Omega(n) = a_1 + a_2 + \cdots + a_k$.

We prove by induction on $\Omega(n)$. For base cases $\Omega(n) = 0$ and 1, the theorem holds trivially. Assume that $P_0(n) \geq C_0(n)$ for all n with $\Omega(n) < i$.

For any proper divisor d of n, we have $\Omega(d) < i$, and thus $P_0(d) \geq C_0(d)$. Further, there must always be a prime divisor for $n > 1$, and therefore we have $P_0(d) > C_0(d)$ for that prime divisor d. Taking the sum over all the proper divisors of n gives

$$\sum_{d \lfloor n} P_0(d) > \sum_{d \lfloor n} C_0(d).$$

that is,

$$P_0(n) > C_0(n) - 1.$$

Thus, we obtain

$$P_0(n) \geq C_0(n).$$

□

The following theorem provides a sufficient condition under which the number of prime and composite recursive divisors are equal.

Theorem 7. *Let $a \in \mathbb{N},\ a > 1$, and $n = p^a$ for some prime p. Then,*

$$P_0(n) = C_0(n).$$

Proof. When $a = 2$, we have $P_0(n) = C_0(n) = 1$. For $a > 2$, using Remark 1, we have $P_0(n) = 2^{a-2}$. Similarly, for $a > 2$, using Remark 2, we have

$$C_0(n) = 1 + 2^{a-1} \sum_{k=2}^{a-1} \left(\frac{1}{2}\right)^k = 2^{a-2}.$$

□

Theorem 8. *Let n be composite. For $x \geq 1$,*

$$P_x(n) < C_x(n).$$

Proof. Let $p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be the prime factorization of n and $\Omega(n) = a_1 + a_2 + \cdots + a_k$. When $\Omega(n) = 2$, that is, $n = p^2$ or pq for some primes p and q, we have $P_x(n) < C_x(n)$. Assume that $P_x(n) < C_x(n)$ for all composite n with $\Omega(n) < i$. Now, for $x \geq 1$, we have

$$\begin{aligned} p_1^x + p_2^x + \cdots + p_k^x &< p_1^x p_2^x \cdots p_k^x \\ &\leq p_1^{a_1 x} p_2^{a_2 x} \cdots p_k^{a_k x} \\ &= n^x. \end{aligned}$$

For any proper composite divisor d of n, by the inductive hypothesis, we have, $P_x(d) < C_x(d)$. Therefore, for $x \geq 1$,

$$p_1^x + p_2^x + \cdots + p_k^x + \sum_{\substack{d \lfloor n \\ d \text{ is composite}}} P_x(d) < n^x + \sum_{\substack{d \lfloor n \\ d \text{ is composite}}} C_x(d),$$

which implies that

$$\sum_{d \lfloor n} P_x(d) < n^x + \sum_{d \lfloor n} C_x(d).$$

Hence, for $x \geq 1$, we have $P_x(n) < C_x(n)$. □

We observe that the recursive divisor function $\kappa_x(n)$ can be expressed in terms of $\kappa_0(n, 1)$, $P_x(n)$ and $C_x(n)$ as $\kappa_x(n) = \kappa_0(n, 1) + P_x(n) + C_x(n)$, where $\kappa_0(n, 1)$ is the number of recursive divisors of size 1, $P_x(n)$ is the prime recursive divisor function and $C_x(n)$ is the composite recursive divisor function.

Remark 3. For $n \geq 2$, we have

$$\kappa_0(n) = 2\left[P_0(n) + C_0(n)\right] \quad \text{and} \quad \kappa_0(n,1) = P_0(n) + C_0(n).$$

We found that $\kappa_0(n)$ can be written as $\kappa_0(n) = \kappa_0(n,1) + P_0(n) + C_0(n)$. For $n \geq 2$, applying Lemma 2 gives

$$\kappa_0(n) = \frac{\kappa_0(n)}{2} + P_0(n) + C_0(n).$$

Therefore, for $n \geq 2$, we get $\kappa_0(n) = 2\left[P_0(n) + C_0(n)\right]$ and $\kappa_0(n,1) = P_0(n) + C_0(n)$.

Theorem 9. *Let $a \in \mathbb{N},\ a > 1$, and $n = p^a$ for some prime p. Then,*

$$\kappa_0(n) = 4P_0(n) = 4C_0(n).$$

Proof. Let $a > 1$ and $n = p^a$ for some prime p. Since $n \geq 2$, from Remark 3,

$$\kappa_0(n) = 2\left[P_0(n) + C_0(n)\right].$$

Using Theorem 7, we obtain

$$\kappa_0(n) = 4P_0(n) = 4C_0(n).$$

□

Theorem 10. *Let $p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be the prime factorization of n and $\max\{a_1, a_2, ..., a_k\} > 1$. Then, $P_0(n)$ and $C_0(n)$ must be both odd or both even.*

Proof. Let $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ and $a = \max\{a_1, a_2, ..., a_k\} > 1$. Since $n \geq 2$, from Remark 3,

$$\kappa_0(n) = 2\left[P_0(n) + C_0(n)\right].$$

Using Corollary 1, we have $2^{a-1} \mid P_0(n) + C_0(n)$. This implies that $P_0(n) + C_0(n)$ must be even. Hence, $P_0(n)$ and $C_0(n)$ must be both odd or both even. □

We illustrate this for $n = 500$. The prime factorization of $n = 500$ is $2^2 5^3$. The maximum exponent in the prime factorization is 3 which is greater than 1. Therefore, $P_0(500)$ and $C_0(500)$ must be both even or both odd. We have $P_0(500) = 46$ and $C_0(500) = 30$ which are even.

6. Conclusion

In this paper, we introduced two new functions to study the prime and composite recursive divisors among the recursive divisors of a positive integer. We proved the relationship between these prime and composite recursive divisor functions and further proved their relationship to the recursive divisor function. We observed that the number of prime and composite recursive divisors together equals half the total number of recursive divisors.

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