

A MACMAHON–STREHL TYPE IDENTITY FOR THE FRANEL NUMBERS OF ORDER FOUR

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Abstract

In this paper, we establish a new MacMahon–Strehl type identity for the Franel numbers of order four. We also present a bijective proof of this identity using a double counting argument.

1. Introduction

Let n and k be integers. The binomial coefficient is defined by

$$\binom{n}{k} := \begin{cases} \dfrac{n!}{k!(n-k)!}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

This convention allows us to write sums without specifying explicit bounds; for example, we can write $\sum_k \binom{n}{k}$ instead of $\sum_{k=0}^{n} \binom{n}{k}$.

It is well known that

$$\sum_k \binom{n}{k} = 2^n.$$

Moreover, the sum of squares of binomial coefficients is given by the Vandermonde identity

$$\beta_n := \sum_k \binom{n}{k}^2 = \binom{2n}{n},$$

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which satisfies the recurrence relation (see, for instance, [2])

$$n\beta_n = 2(2n-1)\beta_{n-1}, \qquad \beta_0 = 1.$$

In 1894, Franel [6] introduced the sequence $(f_n)_{n\geq 0}$ defined by

$$f_n := \sum_k \binom{n}{k}^3,$$

and showed that it satisfies the recurrence relation

$$(n+1)^2 f_{n+1} = (7n^2+7n+2)f_n + 8n^2 f_{n-1}, \qquad f_0 = 1,\ f_1 = 2.$$

Later, in 1895, Franel [7] studied the sequence

$$\varphi_n := \sum_k \binom{n}{k}^4, \tag{1}$$

and proved that it satisfies the three-term recurrence

$$(n+1)^3\varphi_{n+1} = 2(2n+1)(3n^2+3n+1)\varphi_n + 4n(4n-1)(4n+1)\varphi_{n-1},$$

with initial values $\varphi_0 = 1$ and $\varphi_1 = 2$. Further information on recurrences for sums of powers of binomial coefficients can be found in [4, 10].

In 1905, MacMahon [11] established the identity

$$\sum_k \binom{n}{k}^3 x^k y^{n-k} = \sum_k \binom{n}{k}\binom{n-k}{k}\binom{n+k}{k}(xy)^k(x+y)^{n-2k},$$

which yields

$$f_n = \sum_k \binom{n}{k}\binom{n-k}{k}\binom{n+k}{k}2^{n-2k}. \tag{2}$$

Later, in 1993, by applying the Chu–Vandermonde convolution [1], Strehl [14] obtained the identity

$$f_n = \sum_k \binom{n}{k}^2\binom{2k}{n}. \tag{3}$$

In 2008, Gould [9] obtained the following identity via Carlitz's formula [3]:

$$f_n = \sum_k \binom{n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}. \tag{4}$$

Furthermore, Zhao [15] provided a combinatorial proof of the equivalence between (2) and (4) using free 2-Motzkin paths, while Mikić [12] gave a double counting proof of identity (3).

The purpose of this paper is to establish a new form for the Franel numbers (1). Moreover, by developing Mikić's double counting approach, we provide a combinatorial proof of the new identity (8), which is stated below. The paper is organized as follows. In Section 2, we state our main results. In Section 3, we present a combinatorial proof of Equation (8) based on a double counting argument. In Section 4, we generalize this combinatorial approach to obtain a two-parameter identity, from which identity (8) is recovered as a special case.

2. Main Theorems

We first recall the following extension of the binomial coefficient. For a complex variable x and an integer n, we define

$$\binom{x}{n} := \begin{cases} \dfrac{1}{n!}\, x(x-1)\cdots(x-n+1), & \text{if } n \geq 1, \\ 1, & \text{if } n = 0, \\ 0, & \text{if } n < 0. \end{cases}$$

We will also need the following binomial identities.

Lemma 1 ([1, 8])**.** *Let n, m, r be nonnegative integers and let x, y be complex variables. We have*

$$\binom{x}{n}\binom{n}{m} = \binom{x}{m}\binom{x-m}{n-m}, \tag{5}$$

$$\sum_k \binom{n}{k}\binom{m}{n-k}\binom{x+n-k}{n+m} = \binom{x}{n}\binom{x}{m}, \tag{6}$$

and

$$\sum_k \binom{n}{k}\binom{x}{k}\binom{y}{r+k}\binom{x+y+n}{k}^{-1} = \binom{x+r+n}{n}\binom{y+n}{n+r}\binom{x+y+n}{n}^{-1}. \tag{7}$$

Remark 1. Identity (6) is the classical Riordan identity, which can be found in [8, (6.45)]. Also, (7) is [8, (11.1)].

Theorem 1 (MacMahon–Strehl type identity of order four)**.** *Let n be a nonnegative integer. Then*

$$\varphi_n = \sum_k \binom{n}{k}^2 \binom{2k}{n}\binom{2n-k}{n}. \tag{8}$$

Proof. By the Vandermonde identity, we have

$$\binom{2k}{n} = \sum_i \binom{k}{i}\binom{k}{n-i}. \tag{9}$$

Hence,

$$\sum_k \binom{n}{k}^2 \binom{2k}{n}\binom{2n-k}{n} = \sum_{k,i} \binom{n}{k}^2 \binom{k}{i}\binom{k}{n-i}\binom{2n-k}{n}.$$

Applying identity (5), we obtain

$$\sum_k \binom{n}{k}^2 \binom{2k}{n}\binom{2n-k}{n} = \sum_{k,i} \binom{n}{i}\binom{n-i}{k-i}\binom{n}{n-i}\binom{i}{k+i-n}\binom{2n-k}{n}.$$

Rearranging the sums gives

$$\sum_k \binom{n}{k}^2 \binom{2k}{n}\binom{2n-k}{n} = \sum_i \binom{n}{i}\binom{n}{n-i} \sum_k \binom{n-i}{k-i}\binom{i}{n-k}\binom{2n-k}{n}.$$

Setting $k-i=j$, this becomes

$$\sum_k \binom{n}{k}^2 \binom{2k}{n}\binom{2n-k}{n} = \sum_i \binom{n}{i}\binom{n}{n-i} \sum_j \binom{n-i}{j}\binom{i}{n-i-j}\binom{2n-i-j}{n}.$$

Finally, an application of the Riordan identity (6) yields the desired result. □

More generally, we establish the following result.

Theorem 2. *Let x be a complex variable and n be a nonnegative integer. Then*

$$\sum_k \binom{n}{k}^2 \binom{x}{k}\binom{x}{n-k} = \sum_k \binom{x}{k}\binom{n}{k}\binom{2k}{n}\binom{x+n-k}{n-k}. \tag{10}$$

Proof. In view of (9) and (5), we obtain

$$\begin{aligned}
\sum_k \binom{x}{k}\binom{x+n-k}{n-k}\binom{n}{k}\binom{2k}{n} &= \sum_{k,i} \binom{x}{k}\binom{n}{k}\binom{k}{i}\binom{k}{n-i}\binom{x+n-k}{n-k} \\
&= \sum_{k,i} \binom{x}{i}\binom{x-i}{k-i}\binom{n}{n-i}\binom{i}{k+i-n}\binom{x+n-k}{n-k} \\
&= \sum_i \binom{x}{i}\binom{n}{n-i} \sum_k \binom{x-i}{k-i}\binom{i}{n-k}\binom{x+n-k}{n-k} \\
&= \sum_i \binom{x}{i}\binom{n}{i} \sum_j \binom{x-i}{j}\binom{i}{n-i-j}\binom{x+n-i-j}{n-i-j}.
\end{aligned}$$

Also, by (5), we have

$$\sum_j \binom{x-i}{j}\binom{i}{n-i-j}\binom{x+n-i-j}{n-i-j} = \binom{x+n-i}{n-i} \sum_j \frac{\binom{x-i}{j}\binom{i}{n-i-j}\binom{n-i}{j}}{\binom{x+n-i}{j}}.$$

It follows from (7) that

$$\sum_j \binom{x-i}{j}\binom{i}{n-i-j}\binom{x+n-i-j}{n-i-j} = \binom{x}{n-i}\binom{n}{i}.$$

Therefore,

$$\sum_j \binom{x}{j}\binom{n}{j}\binom{2k}{n}\binom{x+n-j}{n-j} = \sum_k \binom{n}{k}^2\binom{x}{k}\binom{x}{n-k},$$

as desired. □

Remark 2. Setting $x = n$ in (10) recovers (8).

Remark 3. Let k be a nonnegative integer. Note that

$$\begin{aligned}\binom{-\frac{1}{2}}{k} &= \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-k+1)}{k!}\\ &= \frac{(-1)^k}{2^k}\frac{1\cdot 3\cdot\cdots\cdot(2k-1)}{k!}\\ &= \frac{(-1)^k(2k)!}{2^{2k}k!k!} = \frac{(-1)^k}{2^{2k}}\binom{2k}{k}.\end{aligned} \tag{11}$$

Moreover, with the same steps one obtains

$$\binom{\frac{1}{2}}{k} = (-1)^{k+1}\binom{2k}{k}\frac{1}{2^{2k}(2k-1)}. \tag{12}$$

Corollary 1. *Let n be a nonnegative integer. Then*

$$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k} = \sum_k \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\binom{2(n-k)}{n}(-1)^k, \tag{13}$$

and

$$\begin{aligned}&\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}\frac{1}{(2k-1)(2(n-k)-1)}\\ &\quad= \sum_k \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\binom{2(n-k)}{n}\frac{(2k+1)(-1)^{k+1}}{2(n-k)-1}.\end{aligned} \tag{14}$$

Proof. Identities (13) and (14) follow by taking $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ in (10) and applying identities (11) and (12), respectively.

□

Remark 4. The right-hand side of (13) is known as the Domb numbers [5], which is listed as A002895 in the OEIS [13].

Corollary 2. *Let n be a nonnegative integer. Then*

$$\sum_k \binom{n}{k}^2 \binom{n+k}{k}\binom{2n-k}{n} = \sum_k \binom{n}{k}^2 \binom{2n-k}{n}\binom{2(n-k)}{n}. \tag{15}$$

Proof. Let n, k be nonnegative integers. Observe that

$$\binom{-1-n}{k} = \binom{n+k}{k}(-1)^k.$$

Hence, (15) follows by substituting $x = -n-1$ into (10), and applying the above binomial identity. □

Remark 5. Note that (15) corresponds to the sequence A183204 in the OEIS [13].

3. Combinatorial Interpretation

We begin by introducing the combinatorial objects required for the proof of identity (8). Let n be a nonnegative integer. For $n > 0$, set

$$[n] := \{1, 2, \ldots, n\},$$

and let $[0]$ denote the empty set $\emptyset$. For two finite sets A and B, we denote by $|A|$ the cardinality of A, we define $A \setminus B = A \cap \overline{B}$, and we set

$$A\Delta B := (A \setminus B) \cup (B \setminus A).$$

Let Y be the set defined by

$$Y := \left\{ (A, B, C) \;\middle|\; \begin{array}{l} A, B \subset [2n],\ C \subset [3n] \setminus [n],\ |A| = |B| = |C| = n, \\ A \subset B\Delta[n],\ B \setminus [n] \subset C \end{array} \right\}.$$

We first compute the cardinality of Y by counting according to the set B. Let $k = |B \setminus [n]|$, where $0 \le k \le n$. The elements of $B \setminus [n]$ can be chosen from $[2n] \setminus [n]$ in $\binom{n}{k}$ ways, and the remaining $n-k$ elements of B belong to $B \cap [n]$, which can be chosen from $[n]$ in $\binom{n}{n-k}$ ways, as in [12]. Since the k elements of $B \setminus [n]$ must belong to C, the remaining $n-k$ elements of C can be chosen from $\big([3n] \setminus [n]\big) \setminus \big(B \setminus [n]\big)$ in $\binom{2n-k}{n-k}$ ways. Moreover, since $|B\Delta[n]| = 2k$, the set A can be chosen in $\binom{2k}{n}$ ways. Therefore, multiplying the independent choices and summing over k, we obtain

$$|Y| = \sum_k \binom{n}{k}\binom{n}{n-k}\binom{2k}{n}\binom{2n-k}{n-k}. \tag{16}$$

Observe that the condition $A \subset B\Delta[n]$ is equivalent to

$$A \setminus [n] \subset B \quad \text{and} \quad A \cap B \cap [n] = \emptyset.$$

Therefore, the set Y can also be described as

$$Y := \left\{ (A,B,C) \;\middle|\; \begin{array}{l} A,B \subset [2n],\ C \subset [3n] \setminus [n],\ |A| = |B| = |C| = n, \\ A \setminus [n] \subset B,\ B \setminus [n] \subset C,\ A \cap B \cap [n] = \emptyset \end{array} \right\}.$$

We now compute $|Y|$ by counting according to the set A. Let $k = |A \cap [n]|$, where $0 \le k \le n$. The elements of $A \cap [n]$ can be chosen from $[n]$ in $\binom{n}{k}$ ways, and the remaining $n-k$ elements of $A \setminus [n]$ can be chosen from $[2n] \setminus [n]$ in $\binom{n}{n-k}$ ways. Since $A \setminus [n] \subset B$, these $n-k$ elements already belong to B. The remaining k elements of B may be chosen either from $[n] \setminus (A \cap [n])$ or from $([2n] \setminus [n]) \setminus (A \setminus [n])$. Let j denote the number of these k elements chosen from the former set. There are $\binom{n-k}{j}$ ways to choose these j elements, and $\binom{k}{j}$ ways to choose the remaining $k-j$ elements from the latter set. Observe that $|B \setminus [n]| = n - j$. Since $B \setminus [n] \subset C$, the remaining j elements of C can be chosen from $([3n] \setminus [n]) \setminus (B \setminus [n])$, a set of size $n+j$, in $\binom{n+j}{j}$ ways. Summing over j, and in view of identity (6), we obtain

$$\sum_j \binom{n-k}{j}\binom{k}{j}\binom{n+j}{j} = \binom{n}{k}^2.$$

Consequently,

$$|Y| = \sum_k \binom{n}{k}\binom{n}{n-k}\binom{n}{k}^2 = \sum_k \binom{n}{k}^4 = \varphi_n. \tag{17}$$

Comparing (16) and (17) completes the combinatorial proof of identity (8).

4. A Generalized MacMahon–Strehl Type Identity of Order Four

We will need the following binomial identity.

Lemma 2 ([8, (6.43)])**.** *Let a, b, c, d be nonnegative integers. We have*

$$\sum_k \binom{b}{k}\binom{c}{d-k}\binom{a+k}{b+c} = \binom{a}{b+c-d}\binom{a-c+d}{d}.$$

We now consider a more general setting. Let m be an integer such that $m > n > 0$, and define the set

$$Z := \left\{ (A,B,C) \;\middle|\; \begin{array}{l} A,B \subset [m],\ C \subset [m+n] \setminus [n],\ |A| = |B| = |C| = n, \\ A \subset B\Delta[n],\ B \setminus [n] \subset C \end{array} \right\}.$$

By following the same counting arguments as in the previous section, together with Lemma 2, we obtain the identity

$$\sum_k \binom{m-n}{k}\binom{n}{n-k}\binom{2k}{n}\binom{m-k}{n-k} = \sum_k \binom{n}{k}\binom{m-n}{n-k}\binom{m-n}{k}\binom{n}{k}.$$

Observe that, when $m = 2n$, the above identity reduces to (8).

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