

PROOF OF A CONJECTURE ON OVERCOLORED PARTITIONS RESTRICTED BY PARITY OF THE PARTS

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Abstract

In a recent paper, Thejitha and Fathima introduced the overcolored partition function $\bar{a}_{r,s}(n)$, which enumerates overpartitions in which even parts may appear in one of r colors and odd parts in one of s colors, for fixed integers $r, s \geq 1$. They also proposed several conjectures concerning families of congruences modulo powers of 2 for specific arithmetic progressions of $\bar{a}_{r,s}(n)$. In this paper, we provide an elementary proof of this conjecture that relies only on classical q-series manipulations and properties of Ramanujan's theta function.

1. Introduction

A *k-colored partition* of a positive integer n is a partition of n in which each part is assigned one of k colors. We denote the number of *k-colored partitions* of n by $p_k(n)$. For $q \in \mathbb{C}$ with $|q| < 1$, we use the notation

$$f_n^k := \prod_{j \geq 1} (1 - q^{jn})^k,$$

for integers n, k with $n > 0$. We simply write f_n^k as f_n when $k = 1$. The generating function for $p_k(n)$ is given by

$$\sum_{n=0}^{\infty} p_k(n) q^n = \frac{1}{f_1^k}.$$

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In particular, when $k = 1$, the function $p_1(n)$ reduces to the unrestricted partition function $p(n)$.

Agarwal and Andrews [1] were the first to introduce colored partitions in the context of unrestricted partitions. Later, several mathematicians studied different aspects of colored partitions. In particular, Thejitha, Sellers, and Fathima [9] considered colored partitions in which the even parts may appear in one of r colors and the odd parts may appear in one of s colors, where $r, s \geq 1$ are fixed integers. They denoted the corresponding counting function by $a_{r,s}(n)$. The generating function for $a_{r,s}(n)$ is given by

$$\sum_{n=0}^{\infty} a_{r,s}(n)q^n = \frac{f_2^{s-r}}{f_1^s}. \tag{1}$$

The special case $r = 1$ was first examined by Hirschhorn and Sellers [7], while the case $s = 1$ was studied by Amdeberhan, Sellers, and Singh [2].

An *overpartition* of n is a partition of n in which the first occurrence of each distinct part may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n, where $\overline{p}(0) = 1$. The generating function for $\overline{p}(n)$ was introduced by Corteel and Lovejoy [4] and is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = \frac{f_2}{f_1^2}.$$

A similar generalization appears in the work of Amdeberhan, Sellers, and Singh [2], where they considered overpartitions in a colored setting. In particular, they extended their earlier study of the case $s = 1$ of Equation (1) to overpartitions. They investigated partitions of n in which each even part may occur in one of $r \geq 1$ colors, and the first occurrence of each part may be overlined. Let $\bar{a}_r^*(n)$ denote the number of such partitions. The generating function for $\bar{a}_r^*(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{a}_r^*(n)q^n = \frac{f_4^{r-1}}{f_1^2 f_2^{2r-3}}.$$

Recently, Das, Maity, and Saikia [5] investigated the divisibility properties of $\bar{a}_r^*(n)$ for various values of r. In particular, they established generalizations of Sellers' results (see [8, Theorems 2.5 and 2.7]).

Very recently, Thejitha and Fathima [10] introduced $\bar{a}_{r,s}(n)$, which denotes the number of partitions of n in which even parts may appear in one of r colors and odd parts may appear in one of s colors, for fixed integers $r, s \geq 1$, with the additional condition that the first occurrence of each part may be overlined. They obtained the generating function for $\bar{a}_{r,s}(n)$ as

$$\bar{A}_{r,s}(q) := \sum_{n=0}^{\infty} \bar{a}_{r,s}(n)q^n = \frac{f_2^{3s-2r}}{f_1^{2s} f_4^{s-r}}.$$

In the same paper, they established several infinite families of congruences for $\bar{a}_{r,s}(n)$ modulo primes and powers of 2. In particular, their results yield various Ramanujan-type congruences modulo powers of 2 and modulo primes $p \geq 3$ satisfied by $\bar{a}_{r,s}(n)$. In the concluding remarks, they [10, Conjecture 6.1] stated that they would be pleased to see an elementary proof of the following congruences. Specifically, they conjectured that for $k \geq 1$ and $j, i \geq 0$, we have

$$\begin{aligned}
\bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(3n+2) &\equiv 0 \pmod{2^{k+1}},\\
\bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(9n+3) &\equiv 0 \pmod{2^{k+2}},\\
\bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(9n+6) &\equiv 0 \pmod{2^{k+2}}.
\end{aligned}$$

The purpose of this paper is to give an elementary proof of the above congruences.

2. Preliminaries

In this section, we provide some fundamental theta-function identities and existing lemmas, which will later be used to prove the conjecture. The following identities, due to Ramanujan and recorded in [3, Entry 22, pp. 36-37], and also appearing in [7, Equations (1.5.5) and (1.5.8)], will be used in the proof of our result:

$$\begin{aligned}
\varphi(q) &= 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{f_2^5}{f_1^2 f_4^2},\\
\varphi(-q) &= 1 + 2\sum_{k=1}^{\infty} (-1)^k q^{k^2} = \frac{f_1^2}{f_2}.
\end{aligned}$$

We repeatedly use the following congruence, which can be easily obtained by employing the binomial theorem.

Lemma 1. *For any positive integers k and m, and any prime p, we have*

$$f_m^{p^k} \equiv f_{mp}^{p^{k-1}} \pmod{p^k}.$$

Lemma 2 ([3, Entry 25, p. 40])**.** *We have*

$$\varphi(q)\varphi(-q) = \varphi(-q^2)^2.$$

Lemma 3 ([6, Lemma 2.2])**.** *Let m be a positive integer. Then for all $1 \leq n \leq 2m$, we have*

$$\binom{2^m}{n} 2^n \equiv 0 \pmod{2^{m+\lfloor (n-1)/2 \rfloor + 1}}.$$

3. Main Results

We now state the main result of this paper.

Theorem 1. *For $k \geq 1$ and $j, i \geq 0$, we have*

$$\begin{aligned}
\bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(3n+2) &\equiv 0 \pmod{2^{k+1}},\\
\bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(9n+3) &\equiv 0 \pmod{2^{k+2}},\\
\bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(9n+6) &\equiv 0 \pmod{2^{k+2}}.
\end{aligned}$$

Proof. We have

$$\bar{A}_{r,s}(q) = \sum_{n=0}^{\infty} \bar{a}_{r,s}(n)q^n = \frac{f_2^{3s-2r}}{f_1^{2s} f_4^{s-r}}.$$

Put $r = 2^{k+1}j + 2^k - 1$ and $s = 2^k i + 1$ to obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(n)q^n &= \frac{f_2^{3\cdot 2^k i+3-2^{k+2}j-2^{k+1}+2}}{f_1^{2^{k+1}i+2} f_4^{2^k i+1-2^{k+1}j-2^k+1}}\\
&= \frac{f_2^{3\cdot 2^k i+3}}{f_1^{2^{k+1}i+2}} \cdot \left(\frac{f_4^{2^{k+1}}}{f_2^{2^{k+2}}}\right)^j \cdot \frac{f_4^{2^k}}{f_2^{2^{k+1}}} \cdot \frac{f_2^2}{f_4^{2^k i+1} \cdot f_4}. \qquad (2)
\end{aligned}$$

We use the congruence from Lemma 1 with $m = 2$ and $p = 2$ to obtain

$$f_2^{2^{k+2}} \equiv f_4^{2^{k+1}} \pmod{2^{k+2}}.$$

Therefore, from Equation (2), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(n)q^n &\equiv \frac{f_2^{3\cdot 2^k i+3}}{f_1^{2^{k+1}i+2}} \cdot \frac{f_4^{2^k}}{f_2^{2^{k+1}}} \cdot \frac{f_2^2}{f_4^{2^k i+1} \cdot f_4} \pmod{2^{k+2}}\\
&\equiv \left(\frac{f_2^3}{f_1^2 f_4}\right)^{2^k i+1} \cdot \left(\frac{f_4}{f_2^2}\right)^{2^k} \cdot \frac{f_2^2}{f_4} \pmod{2^{k+2}}\\
&\equiv \left(\frac{f_2^5}{f_1^2 f_4^2}\right)^{2^k i+1} \cdot \left(\frac{f_4}{f_2^2}\right)^{2^k i} \cdot \left(\frac{f_4}{f_2^2}\right)^{2^k} \pmod{2^{k+2}}.
\end{aligned}$$

Using the identities $\varphi(q) = \dfrac{f_2^5}{f_1^2 f_4^2}$ and $\varphi(-q) = \dfrac{f_1^2}{f_2}$, we obtain the following congruence:

$$\sum_{n=0}^{\infty} \bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(n)q^n \equiv \varphi(q)^{2^k i+1} \left(\frac{1}{\varphi(-q^2)}\right)^{2^k i} \left(\frac{1}{\varphi(-q^2)}\right)^{2^k} \pmod{2^{k+2}}.$$

By using the identity $\varphi(q)\varphi(-q)=\varphi(-q^2)^2$ from Lemma 2, we can further simplify this as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{a}_{2^{k+1}j+2^k-1,2^ki+1}(n)q^n &\equiv \varphi(q)\varphi(q)^{2^{k+1}i}\left(\frac{1}{\varphi(-q^2)}\right)^{2^{k+1}i}\left(\frac{\varphi(-q^2)}{\varphi(q)}\right)^{2^ki}\\ &\quad\times\left(\frac{1}{\varphi(-q^2)}\right)^{2^k}\pmod{2^{k+2}}\\ &\equiv \varphi(q)\varphi(q)^{2^ki}\frac{1}{\varphi(-q)^{2^ki}}\left(\frac{\varphi(-q)}{\varphi(q)}\right)^{2^{k-1}i}\\ &\quad\times\left(\frac{1}{\varphi(q)\varphi(-q)}\right)^{2^{k-1}}\pmod{2^{k+2}}.\end{aligned}$$

Moreover, using the identity [7, Equation (1.5.16)]

$$\frac{1}{\varphi(-q)}=\varphi(q)\varphi(q^2)^2\varphi(q^4)^4\cdots$$

and recalling the fact that $\varphi(q)=1+2\sum\limits_{k=1}^{\infty}q^{k^2}$, we obtain

$$\begin{aligned}\varphi(q)^{2^{k+1}i}&=\left(1+2\sum_{k=1}^{\infty}q^{k^2}\right)^{2^{k+1}i}\\ &=1+2^{k+1}i\cdot 2\sum_{k=1}^{\infty}q^{k^2}+\sum_{j=2}^{2^{k+1}i}\binom{2^{k+1}i}{j}\left(2\sum_{k=1}^{\infty}q^{k^2}\right)^j\\ &\equiv 1\pmod{2^{k+2}}.\end{aligned}$$

Therefore, using this congruence property, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{a}_{2^{k+1}j+2^k-1,2^ki+1}(n)q^n &\equiv \varphi(q)\varphi(q)^{2^ki}(\varphi(q)\varphi(q^2)^2)^{2^ki}\left(\frac{\varphi(-q)}{\varphi(q)}\right)^{2^{k-1}i}\\ &\quad\times(\varphi(q^2)^2)^{2^{k-1}}\pmod{2^{k+2}}\\ &\equiv \varphi(q)\varphi(q)^{2^{k+1}i}\varphi(q^2)^{2^{k+1}i}\left(\frac{\varphi(-q)}{\varphi(q)}\right)^{2^{k-1}i}\\ &\quad\times\varphi(q^2)^{2^k}\pmod{2^{k+2}}\\ &\equiv \varphi(q)\left(\frac{\varphi(-q)}{\varphi(q)}\right)^{2^{k-1}i}\varphi(q^2)^{2^k}\pmod{2^{k+2}}.\end{aligned}\tag{3}$$

Notice that

$$\begin{aligned}\frac{\varphi(-q)}{\varphi(q)} &= \frac{1-2q+2q^4-2q^9+\cdots}{1+2q+2q^4+2q^9+\cdots}\\ &= (1-2q+2q^4-2q^9+\cdots)(1-2q+4q^2-8q^3+\cdots)\\ &= (1-4q+8q^2-16q^3+32q^4-\cdots)\\ &= 1+4(-q+2q^2-4q^3+8q^4+\cdots)\\ &= 1+4H(q), \end{aligned} \tag{4}$$

where $H(q) = -q+2q^2-4q^3+8q^4+\cdots$. Also,

$$\begin{aligned}\varphi(q^2)^{2^k} &= (1+2\sum_{n\geq 1} q^{2n^2})^{2^k}\\ &= 1+2^{k+1}\sum_{n\geq 1} q^{2n^2} + \binom{2^k}{2}\left(2\sum_{n\geq 1} q^{2n^2}\right)^2 + \sum_{m=3}^{2^k}\binom{2^k}{m}\left(2\sum_{n\geq 1} q^{2n^2}\right)^m\\ &= 1+2^{k+1}\sum_{n\geq 1} q^{2n^2} + 2^{k+1}(2^k-1)\left(\sum_{\alpha,\beta\geq 1} q^{2(\alpha^2+\beta^2)}\right)\\ &\quad + \sum_{m=3}^{2^k}\binom{2^k}{m}\left(2\sum_{n\geq 1} q^{2n^2}\right)^m. \end{aligned}$$

From Lemma 3, we get $2^m\binom{2^k}{m} \equiv 0 \pmod{2^{k+2}}$ for $m \geq 3$. This gives,

$$\begin{aligned}\varphi(q^2)^{2^k} &\equiv 1+2^{k+1}\sum_{n\geq 1} q^{2n^2} + 2^{k+1}\left(\sum_{\alpha,\beta\geq 1} q^{2(\alpha^2+\beta^2)}\right) \pmod{2^{k+2}}\\ &\equiv 1+2^{k+1}F(q) \pmod{2^{k+2}}, \end{aligned} \tag{5}$$

where $F(q) = \sum_{n\geq 1} q^{2n^2} + \left(\sum_{\alpha,\beta\geq 1} q^{2(\alpha^2+\beta^2)}\right)$. Combining Equations (3), (4), and (5), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(n)q^n &\equiv \varphi(q)\left(\frac{\varphi(-q)}{\varphi(q)}\right)^{2^{k-1}i}\varphi(q^2)^{2^k} \pmod{2^{k+2}}\\ &\equiv \varphi(q)\left[1+4H(q)\right]^{2^{k-1}i}\left[1+2^{k+1}F(q)\right] \pmod{2^{k+2}}\\ &\equiv \varphi(q)[1+2^{k+1}iH(q)][1+2^{k+1}F(q)] \pmod{2^{k+2}}\\ &\equiv (1+2\sum_{n\geq 1} q^{n^2})[1-2^{k+1}iq][1+2^{k+1}F(q)] \pmod{2^{k+2}} \end{aligned}$$

$$\begin{aligned}
&\equiv (1+2\sum_{n\geq 1} q^{n^2})[1-2^{k+1}iq+2^{k+1}F(q)] \pmod{2^{k+2}}\\
&\equiv 1+2\sum_{n\geq 1} q^{n^2}-2^{k+1}iq+2^{k+1}F(q) \pmod{2^{k+2}}\\
&\equiv 1+2\sum_{n\geq 1} q^{n^2}-2^{k+1}iq\\
&\qquad +2^{k+1}\left[\sum_{n\geq 1} q^{2n^2}-\sum_{\alpha,\beta\geq 1} q^{2(\alpha^2+\beta^2)}\right] \pmod{2^{k+2}}.
\end{aligned} \tag{6}$$

The fourth congruence holds because of the fact that $H(q) \equiv -q \pmod 2$. The possible residues of $n^2 \pmod 9$ are $0, 1, 4$, and 7. Hence, the coefficients of q^{9n+3} and q^{9n+6} are 0 in $\sum_{n\geq 1} q^{n^2}$. The possible residues of $2n^2 \pmod 9$ are $0, 2, 5$, and 8. So, the coefficient of q^{9n+3} and q^{9n+6} is 0 in $\sum_{n\geq 1} q^{2n^2}$. The possible residues of $2(\alpha^2+\beta^2) \pmod 9$ are $0, 1, 2, 4, 5, 7$, and 8. So, the coefficient of q^{9n+3} and q^{9n+6} is 0 in $\sum_{\alpha,\beta\geq 1} q^{2(\alpha^2+\beta^2)}$.
As a result,

$$\begin{aligned}
\bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(9n+3) &\equiv 0 \pmod{2^{k+2}},\\
\bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(9n+6) &\equiv 0 \pmod{2^{k+2}}.
\end{aligned}$$

Also, taking Equation (6) modulo 2^{k+1} instead gives

$$\sum_{n=0}^{\infty} \bar{a}_{2^{k+1}j+2^k-1,2^k i+1}(n)q^n \equiv 1+2\sum_{n\geq 1} q^{n^2} \pmod{2^{k+1}}.$$

The possible residues of $n^2 \pmod 3$ are 0 and 1. So, the coefficient of q^{3n+2} is 0 in $\sum_{n\geq 1} q^{n^2}$. Therefore,

$$\bar{a}_{2^{k+1}j+2^k-1,\,2^k i+1}(3n+2) \equiv 0 \pmod{2^{k+1}}.$$

This completes the proof. □

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