

FLIPPED MMPS, NEW INVARIANT MEASURES, AND MAPS WITH TWO LINEAR BRANCHES

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Abstract

The first part deals with piecewise fractional linear maps with three branches. Given a map T, a map R is called a related map, if some branches of T are replaced by a "flipped" branch, namely a branch of $1-T$. The main question is if T and R have a common invariant measure. The short second part presents invariant measures of a new type. The third part is devoted to piecewise linear maps with one non-linear branch and two linear branches

1. Introduction

Since Rényi's foundational work [4], ergodic theory connected to number theory has seen a lot of publications (see [3], [5] [8]). In this paper, Moebius maps (i.e., piecewise fractional linear maps) are discussed ([5], [6], [7] [9], [10], [11]). There is a given partition p_j, $j \in J$, of the unit interval $B = [0,1]$, where J is a finite or countably infinite set such that the map $T : [0,1] \to [0,1]$ is bijective from $]p_{j-1}, p_j[$, $j \in J$, onto $]0,1[$. The inverse map V_j is called an *(inverse) branch* and is given as

$$V_j(x) = \frac{c_j + d_j x}{a_j + b_j x},\ j \in J.$$

The Jacobians of these maps are given by

$$\omega_j = \omega_j(x) = \frac{|a_j d_j - b_j c_j|}{(a_j + b_l x)^2},\ j \in J.$$

We will use the same notation for the associated matrices

$$V_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix},\ j \in J.$$

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Let $\epsilon = 1$ stand for an increasing map and $\epsilon = -1$ for a decreasing map. We say that the map T is of *type* $(\epsilon_1, \epsilon_2, ...)$, $\epsilon_j \in \{1, -1\}$, if V_j is increasing whenever $\varepsilon_j = 1$ and decreasing whenever $\varepsilon_j = -1$ If the parameters satisfy some conditions the map T is ergodic and admits a σ-finite invariant measure. These conditions are different for increasing and decreasing branches and will be given later.

In several investigations the question has been to find an explicit shape of the density of the invariant measure.Three possibilities are used. A map $T^* : B^* \to B^*$ is called a *dual map* if B^* is an interval and T^* is a piecewise fractional linear map whose branches are given by the adjoint matrices

$$V_j^* = \begin{pmatrix} a_j & c_j \\ b_j & d_j \end{pmatrix}, j \in J.$$

Then the function

$$h(x) = \int_{B^*} \frac{dy}{(1+xy)^2}$$

is (up to a multiplicative factor) the density of the invariant measure for T. If there is a map $\Psi_T(t) = \frac{B+Dt}{A+Bt}$ such that $\Psi_T \circ T = T^* \circ \Psi_T$ (or equivalently $\Psi \circ V_j = V_j^* \circ \Psi$) then the dual is called a *natural dual*. This implies the following relation for the associated matrices

$$(M) \quad \begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} = \begin{pmatrix} a_j & c_j \\ b_j & d_j \end{pmatrix} \begin{pmatrix} A & B \\ B & D \end{pmatrix}, j \in J.$$

For natural duals we then put $B^* = \Psi_T[0, 1]$ which is an interval with endpoints $\frac{B}{A}$ and $\frac{B+D}{A+B}$.

In the other case it is called an *exceptional dual*. If $B^* = [\eta, \theta]$, $\eta < \theta$, then the invariant density is in both cases

$$h(x) = \frac{\theta}{1+\theta x} - \frac{\eta}{1+\eta x}.$$

If the set B^* shrinks to a point τ one can see this case as a natural dual as well as an exceptional one. We call it a *singular dual*. In this case the density of the invariant measure is $h(x) = \frac{1}{(1+\tau x)^2}$.

However, there are other examples where we can write down an explicit form of the density of the invariant measure. This method has been called a 1-*step extension* (see [13]). An example is the following map. We start with

$$Tx = \frac{x}{1-x}, 0 \le x \le \frac{1}{2}, Tx = \frac{1}{x} - 1, \frac{1}{2} \le x \le 1$$

and define the *jump transformation* $S = T^2$ on $[0, \frac{1}{2}]$ and $S = T$ on $[\frac{1}{2}, 1]$. Then

$$Sx = \frac{x}{1-2x},\ 0 \le x \le \frac{1}{3}, Sx = \frac{1}{x} - 2,\ \frac{1}{3} < x \le \frac{1}{2}, Sx = \frac{1}{x} - 1,\ \frac{1}{2} < x \le 1.$$

The density of the invariant measure is given by

$$h(x) = \frac{1}{x} \sum_{j=0}^{\infty} \left(\frac{1}{1+2jx} - \frac{1}{1+(2j+1)x} \right).$$

This is a function with infinitely many poles, and hence not rational. The set $B^* = \bigcup_{j=0}^{\infty} [2j, 2j+1]$ is associated with this density.

The first part of the paper deals with maps with three branches. Its inverse branches are

$$V_\lambda = \begin{pmatrix} 3 & 3\lambda - 3 \\ 0 & \lambda \end{pmatrix}, V_\mu = \begin{pmatrix} 9 & 3\mu - 9 \\ 3 & 2\mu - 3 \end{pmatrix}, V_\nu = \begin{pmatrix} 3 & \nu - 3 \\ 2 & \nu - 2 \end{pmatrix}$$

with $\lambda, \mu, \nu > 0$. The equation (M) leads to three homogeneous linear equations for A, B, D. If the determinant of this system vanishes, then a dual map exists. This condition is abbreviated by (C).The map T has a natural dual, if and only if $\lambda\mu + \mu = \lambda\nu + 3\lambda$. The triple $(\lambda, \mu, \nu) = (1, 3, 3)$ belongs to the map with three linear branches. One or more branches of the map T are *flipped*, this means, they are substituted by the branch of $1-T$. This map will be called a *related map* R. We investigate the question if the map T and a related map R have a common invariant measure.The maps T and R have the same invariant measure, if $\psi_T[0, 1] = \psi_R[0, 1]$.

The second part deals with the discovery of new explicitly given densities for some special cases. Example 1 in Section 2 presents a map with four branches. The density of its invariant measure is

$$h(x) = \frac{1}{1+x} - \frac{1}{2+x} + \frac{1}{3-x}.$$

The third part is illustrated by an example. The invariant measure for the map $Tx = 3x \bmod 1$ is Lebesgue measure with density $h(x) = 1$. The map with $Tx = 3x$, $0 < x < \frac{1}{3}$, $Tx = \frac{-3+9x}{5-3x}$, $\frac{1}{3} < x < \frac{2}{3}$, and $Tx = \frac{-2+3x}{3-2x}$, $\frac{2}{3} < x < 1$ has an invariant measure with density $h(x) = \frac{1}{1+x}$. However, the map $Tx = 3x \bmod 1$, $0 < x < \frac{2}{3}$ and $Tx = \frac{-2+3x}{1+\gamma-\gamma x}$, $\frac{2}{3} < x < 1$, $0 < 3\gamma \leq 1$ has an invariant measure, but no explicit form of the density is known (except when $\gamma = 0$).

2. Related Maps

2.1 Related Maps with One Flip

Here we consider maps R where one branch is substituted by the branch related to $1 - T$. Note that in this case the entries A, B, and D of a possible map Ψ_R are determined by the two unchanged branches. We first look on the map R_1 where

the branch V_λ is replaced by

$$W_\lambda = \begin{pmatrix} 3\lambda & -3\lambda+3 \\ \lambda & -\lambda \end{pmatrix}.$$

Then we consider the map R_2 where the branch V_μ is replaced by

$$W_\mu = \begin{pmatrix} 3\mu & -3\mu+9 \\ 2\mu & -2\mu+3 \end{pmatrix},$$

and the map R_3 where the branch V_ν is replaced by

$$W_\nu = \begin{pmatrix} \nu & -\nu+3 \\ \nu & -\nu+2 \end{pmatrix}.$$

Theorem 1. *(1) The maps T and R_1 both have a natural dual and the same invariant measure, if the conditions*

$$(CT)\ \ \lambda\mu+\mu = \lambda\nu+3\lambda$$

and

$$(CR_1)\ \ \lambda\mu\nu+3\lambda\mu+12\mu = 9\nu+27$$

are satisfied.
(2) The maps T and R_2 both have a natural dual and the same invariant measure, if the conditions

$$(CT)\ \ \lambda\mu+\mu = \lambda\nu+3\lambda$$

and

$$(CR_2)\ \ \lambda\mu\nu = 9$$

are satisfied.
(3) The maps T and R_3 both have a natural dual and the same invariant measure, if the conditions

$$(CT)\ \ \lambda\mu+\mu = \lambda\nu+3\lambda$$

and

$$(CR_3)\ \ 4\lambda\nu-\mu\nu-\lambda\mu\nu+3\lambda+3 = 0$$

are satisfied.

Proof. (1) We start with $A = 4\mu-3\nu-9$, $B = -6\mu+3\nu+9$, and $D = -\mu\nu+9\mu-3\nu-9$. The condition $A(3-3\lambda)-4\lambda B = \lambda D$ gives (CR_1). We insert $\nu = \frac{\lambda\mu+\mu-3\lambda}{\lambda}$ into (CR_1) and obtain

$$\lambda^2\mu+\lambda(\mu+3)-9 = 0.$$

$\lambda = \frac{3}{4}$, $\mu = \frac{36}{7}$, and $\nu = 9$ are solutions.
(2) We take $A = 3-\lambda$, $B = 3\lambda-3$, and $D = 3+\lambda\nu-6\lambda$. The condition

$A(-3\mu+9)+B(-5\mu+3)=2\mu D$ gives (CR_2). We insert $\lambda=\frac{9}{\mu\nu}$ into (CT) and obtain

$$\mu^2\nu+9\mu-9\nu-27=0.$$

Then $\lambda=\mu=3$ and $\nu=1$ are solutions.
(3) We now take $A=3-\lambda$, $B=3\lambda-3$, and $D=\lambda\mu-9\lambda+\mu+3$. Then $A(3-\nu)+B(2-2\nu)=\nu D$ leads to (CR_3). If we multiply (CT) by ν and add it to (CR_3), we obtain

$$\lambda\nu^2-\lambda\nu-3\lambda-3=0.$$

Here, $\lambda=\frac{27}{13}$, $\mu=\frac{153}{40}$, and $\nu=\frac{8}{3}$ are solutions. □

2.2 Related Maps with Two Flips

Again, we have to look at three cases. R_{12} has the branches W_λ and W_μ, R_{23} the branches W_μ and W_ν, and R_{13} the branches W_λ and W_ν.

Theorem 2. *(1) If the conditions*

$$(CT)\ \ \lambda\mu+\mu=\lambda\nu+3\lambda$$

and

$$(CR_{12})\ \ \mu\nu-\lambda\nu-3\lambda-3=0$$

are satisfied then the maps T and R_{12} both have a natural dual but they have the same invariant measure only in the linear case.
(2) If the conditions

$$(CT)\ \ \lambda\mu+\mu=\lambda\nu+3\lambda$$

and

$$(CR_{23})\ \ \lambda\mu\nu+3\mu-9\nu+3\lambda\mu=0$$

are satisfied then the maps T and R_{23} both have a natural dual but they have the same invariant measure only in the linear case.
(3) If the conditions

$$(CT)\ \ \lambda\mu+\mu=\lambda\nu+3\lambda$$

and

$$(CR_{13})\ \ \lambda\mu\nu+3\mu\nu-9\nu-9=0$$

are satisfied then the maps T and R_{13} both have a natural dual. They have the same invariant measure only in the linear case.

Proof. (1) Using the branches W_μ and V_ν, we obtain $A' = \mu\nu - 3$, $B' = 9 - \mu\nu$, and $D' = -18 + 3\nu + \mu\nu$. The condition $A'(-3\lambda+3) - 4\lambda B' = \lambda D'$ gives condition (CR_{12}). If we insert $\nu = \frac{\lambda\mu+\mu-3\lambda}{\lambda}$ into this condition we obtain the quadratic equation

$$\mu^2(1+\lambda) - \mu(4\lambda + \lambda^2) - 3\lambda = 0.$$

Then $\lambda = \frac{1}{2}$, $\mu = 2$, and $\nu = 3$ are solutions.

Clearly,

$$\frac{B+D}{A+B} = \frac{\nu-3}{2} = \frac{B'+D'}{A'+B'},$$

but

$$\frac{B}{A} = \frac{3\lambda-3}{3-\lambda} = \frac{B'}{A'} = \frac{9-\mu\nu}{\mu\nu-3}$$

leads to $\lambda\mu\nu = 9$. We insert $\nu = \frac{9}{\lambda\mu}$ into equation (CT) and obtain a second quadratic equation

$$(1+\lambda)\mu^2 - 3\lambda\mu - 9 = 0.$$

Then we find $\mu = 3$. The first quadratic equation shows $\lambda = 1$ and condition (CT) gives $\nu = 3$, the linear case.

(2) In this case we find $A = A' = 3 - \lambda$, $B = B' = 3\lambda - 3$, and $D' = \frac{9-6\lambda\mu+3\mu}{\mu} = \frac{3+3\lambda+3\nu-5\lambda\nu}{\nu}$ and obtain in a similar way condition (CR_{23}). A solution for conditions (CT) and (CR_{23}) is given by $\lambda = 1$, $\mu = \frac{9}{2}$, and $\nu = 6$. From

$$\frac{B+D}{A+B} = \frac{\nu-3}{2} = \frac{B+D'}{A+B} = \frac{9-3\lambda\mu}{2\lambda\mu}$$

we deduce $D = D'$ and $\lambda\mu\nu = 9$. Then we obtain the quadratic equation

$$(\lambda+1)\mu^2 - 3\lambda\mu - 9 = 0$$

and hence $\mu = 3$. If we insert $\nu = \frac{\lambda\mu+\mu-3\lambda}{\lambda}$ into (CR_{23}) then we obtain a second quadratic equation

$$(\lambda^2+\lambda)\mu^2 - (6\lambda+9)\mu + 27\lambda = 0$$

and $\lambda = 1$. Again we find the linear case.

(3) In this case we take $A' = 2\lambda\nu + 2\lambda$, $B' = -2\lambda\nu + 3\nu - 3\lambda$, and $D' = 2\lambda\nu + 6\lambda - 6\nu + 6$. Then we obtain (CR_{13}). A solution for conditions (CT) and (CR_{13}) is given by $\lambda = 2$, $\mu = 3$, and $\nu = \frac{3}{2}$. From

$$\frac{\nu-3}{2} = \frac{B+D}{A+B} = \frac{B'+D'}{A'+B'} = \frac{3\lambda-3\nu+6}{3\nu-\lambda}$$

we find $\lambda = \frac{3\nu^2-3\nu-12}{\nu+3}$. Since

$$\frac{3\lambda-3}{3-\lambda} = \frac{B}{A} = \frac{B'}{A'} = \frac{-2\lambda\nu+3\nu-3\lambda}{2\lambda\nu+2\lambda},$$

we deduce the equation

$$\nu^5 - \nu^4 - 8\nu^3 + 15\nu + 9 = (\nu - 3)(\nu^2 - 3)(\nu + 1)^2 = 0.$$

However, $\nu = 3$ leads to the linear case and $\nu = \sqrt{3}$ gives $\lambda < 0$.

If $\frac{B}{A} = \frac{B'+D'}{A'+B'}$, namely

$$\frac{3\lambda - 3}{3 - \lambda} = \frac{3\lambda - 3\nu + 6}{3\nu - \lambda},$$

we obtain $\lambda\nu = 3$. If we insert this value in (CR_{13}) then we we get $\mu + \mu\nu = 3\nu + 3$ and eventually $\mu = 3$. □

2.3 Related Maps with Three Flips

Here, R_{123} is the only case. However, the results are a little bit surprising.

Theorem 3. *The conditions*

$$(CT)\ \ \lambda\mu + \mu = \lambda\nu + 3\lambda$$

and

$$(CR_{123})\ \ \lambda\mu - 4\lambda\nu + \mu\nu - 3\lambda + 4\mu - 3\nu = 0$$

characterize a natural dual of of T and R_{123}. These maps have the same invariant measure if and only if $\mu = 3$ and $\lambda\nu = 3$.

Proof. We obtain $A' = \lambda + \lambda\mu$, $B' = -3\lambda + 2\mu - \lambda\mu$, and $D' = 9\lambda - 5\mu + \lambda\mu + 3$. The condition

$$(CR_{123})\ \ \lambda\mu - 4\lambda\nu + \mu\nu - 3\lambda + 4\mu - 3\nu = 0$$

follows in the usual way. Examples are easy to find, e.g., $\lambda = 2$ and $\mu = \nu = 6$.

If $\mu = 3$ and $\lambda\nu = 3$, then a natural dual exists for both maps. We can take $A' = 2\lambda$, $B' = 3 - 3\lambda$, and $D' = 6\lambda - 6$ and furthermore $A = 3 - \lambda$, $B = 3\lambda - 3$, and $D = 6 - 6\lambda$. Then we calculate $\frac{B}{A} = \frac{B'+D'}{A'+B'}$ and $\frac{B+D}{A+B} = \frac{B'}{A'}$. A nice example is given by $\lambda = 3$, $\mu = 3$, and $\nu = 1$.

If $\frac{B}{A} = \frac{B'+D'}{A'+B'}$ and $\frac{B+D}{A+B} = \frac{B'}{A'}$ then we deduce $\mu(\lambda + 1) = 3(\lambda + 1)$. This gives $\mu = 3$. If $\frac{B}{A} = \frac{B'}{A'}$ and $\frac{B+D}{A+B} = \frac{B'+D'}{A'+B'}$ then we deduce two equations, namely

$$\lambda^2\mu + \lambda(-\mu^2 + 4\mu + 3) - \mu^2 = 0$$

and

$$\lambda^2\mu + \lambda(\mu + 3) - 3\mu = 0.$$

Hence $\lambda(\mu^2 - 3\mu) = 3\mu - \mu^2$. Since $\lambda \neq -1$, we get $\mu = 3$. □

The condition $\mu = \nu = 3\lambda$ give solutions of (CT) and (CR_{123}). However, they provide different invariant measures (with one exception, the linear case).

Remark 1. This paper covers only a small fraction of possible combinations but it tries to show what can be expected. A last example will be given. We construct an exceptional dual T^* with order of branches $\lambda\nu\mu$ or $\mu\nu\lambda$. We look for parameters λ, μ, and ν such that the map R_{123} has a natural dual by condition (CR_{123}). Clearly, the linear case with $\lambda = 1$, $\mu = 3$, and $\nu = 3$ is a solution but other solutions seem to be unusually complicated. An example is given by λ, the positive root of $448\lambda^2 + 283\lambda - 1113 = 0$, $\mu = \frac{63}{16}$, and $\nu = \frac{7\lambda+7}{4}$. For the exceptional dual T^* we find the interval B^* with endpoints $\frac{1}{2}$ and $\frac{3\lambda-3}{3-\lambda}$.

3. New Densities through the Back Door

The starting point for this paper was a second look at a generalization of continued fractions (see [1] and [12]). The proof needs only the following well known theorem.

Theorem 4. *Let $U : [0.1] \to [0, 1]$ be a piecewise fractional map with invariant measure ν and $Z : [0.1] \to [0, 1]$ be a piecewise fractional map with invariant measure μ. If a piecewise fractional linear map ϕ with $\phi[0, 1] = [0, 1]$ satisfies $\phi \circ U = Z \circ \phi$, then $\nu(E) = \mu(\phi^{-1}E)$ is an invariant measure for U.*

Proof. We use the following calculation:

$$\nu(T^{-1}E) = \mu(\phi^{-1}T^{-1}E) = \mu(S^{-1}\phi^{-1}E) = \mu(\phi^{-1}E) = \nu(E).$$

□

Let ζ_k denote the inverse branches of ϕ. If h is the density of μ and g the density of ν then

$$g(x) = \sum_k h(\zeta_k(x)) \left|\zeta_k'(x)\right|.$$

If $Ux = T(Sx)$ and $Zx = S(Tx)$ then we take $\phi = S$. If h is the density of the invariant measure for U and g the density of the invariant measure for Z then

$$g(x) = \sum_j h(V_j x)\omega_j(x).$$

We now give some examples and remark that the densities given by this device have no obvious relation to a possible dual map of Z^*. There are some exceptions (see Remark 1 and Example 6). Clearly, if S and T have the same invariant measure, this measure is also invariant for U and Z.

Example 1. Let S be the map on $[0,1]$ with the inverse branches $V_\alpha x = \frac{1-x}{2+x}$ and $V_\beta x = \frac{3-2x}{3-x}$, and let T be the map on $[0,1]$ with the inverse branches $V_\gamma x = \frac{3-3x}{6-x}$ and $V_\delta x = \frac{3-2x}{3-x}$. Then $U = T \circ S$ has the inverse branches $V_{\alpha\gamma} x = \frac{3+2x}{15-5x}$, $V_{\alpha\delta} x = \frac{x}{9-4x}$, $V_{\beta\gamma} x = \frac{4+x}{5}$, and $V_{\beta\delta} x = \frac{3+x}{6-x}$. This map has a natural dual on $B^* = [-\frac{1}{2}, 0]$. Therefore the density of the invariant measure is $h(x) = \frac{1}{2-x}$. The density of the invariant measure of $Z = S \circ T$ is

$$g(x) = h(V_\alpha x)\omega_\alpha(x) + h(V_\beta x)\omega_\beta(x) = \frac{1}{1+x} - \frac{1}{2+x} + \frac{1}{3-x}.$$

If one uses the inverse branches of Z, namely $V_{\gamma\alpha} x = \frac{3+6x}{11+7x}$, $V_{\beta\gamma} x = \frac{3x}{15-4x}$, $V_{\delta\alpha} x = \frac{4+5x}{5+4x}$, and $V_{\delta\beta} x = \frac{3+x}{6-x}$ one can also verify Kuzmin's equation for g by direct calculation.

The next examples are given in shorter form!

Example 2. Let $V_\alpha x = \frac{x}{1+x}$ and $V_\beta x = \frac{1}{2-x}$ be the inverse branches of S and $V_\gamma x = \frac{x}{3-x}$, and let $V_\delta x = \frac{1}{2-x}$ be the inverse branches of T. The map $U = T \circ S$ has density $h(x) = \frac{1}{1-x}$, and $Z = S \circ T$ has density $g(x) = \frac{1}{1+x} + \frac{1}{1-x} - \frac{1}{2-x}$.

Example 3. If $V_\alpha x = \frac{1-x}{2}$, $V_\beta x = \frac{1+x}{2}$, $V_\gamma x = \frac{x}{1+x}$, and $V_\delta x = \frac{1+x}{1+3x}$, then U has the invariant density $h(x) = \frac{1}{x}$ and Z has the invariant density $g(x) = \frac{1}{1-x} + \frac{1}{1+x}$. This density looks familiar, but the branches of the dual map of Z give no clear explanation.

Example 4. Let $Sx = \frac{1}{x} - k$, $k = \lfloor \frac{1}{x} \rfloor$, and $Tx = ax - j$, where $a \geq 2$ a fixed natural number and $j = \lfloor ax \rfloor$. Then

$$Ux = T(Sx) = \frac{a}{x} - (ak + j).$$

The density of its invariant measure is well known, namely $h(x) = \frac{1}{a+x}$.

Now we look at

$$Zx = S(Tx) = \frac{1 + kj - akx}{-j + ax}.$$

Our method gives the density of the invariant measure as

$$g(x) = \sum_{m=1}^{\infty} h\Big(\frac{1}{m+x}\Big) \frac{1}{(m+x)^2} = \sum_{m=1}^{\infty} \frac{1}{(am + 1 + ax)(m+x)}.$$

Example 5. If $Sx = \frac{x}{1-x}$, $k = \lfloor \frac{x}{1-x} \rfloor$, and $Tx = ax - \lfloor ax \rfloor$, then the density for $U = T \circ S$ is given by $h(x) = \frac{1}{a-1+x}$ and the density for $Z = S \circ T$ is

$$g(x) = \sum_{m=0}^{\infty} \frac{1}{(am + a - 1 + ax)(m + 1 + x)}.$$

Remark 2. If all branches of the map U have a common fixed point κ then $\xi = -\frac{1}{\kappa}$ is a common fixed point of the branches of U^*. Then U has a singular dual with density $\frac{1}{(1+\xi x)^2}$, and Z has the singular dual with density $\frac{1}{(1+\eta x)^2}$ where $\eta = S\xi$.

Example 6. Let $Sx = \frac{2-4x}{2+x}, 0 \le x < \frac{1}{2}$, and $Sx = \frac{-3+6x}{2+x}, \frac{1}{2} \le x \le 1$. The second map T is given as $Tx = \frac{4x}{3-2x}, 0 \le x < \frac{1}{2}$ and $Tx = \frac{4-4x}{1+2x}, \frac{1}{2} \le x \le 1$. The map $U = T \circ S$ has the density $h(x) = \frac{1}{(2+x)^2}$, and $Z = S \circ T$ has the density $g(x) = 1$ which corresponds to $\eta = S\frac{1}{2} = 0$.

Remark 3. This method was used to give a derivation for the density of the invariant measure of an f-expansion on $[0, 1]$, namely the map $f^{-1}x = Ax+(2-A)x^2$ mod 1 (see [14]), using the invariant measure of $Tx = Ax + (4-2A)x^2$ if $0 \le x < \frac{1}{2}$, and $Tx = 1 - A + (-4 + 3A)x + (4 - 2A)x^2$ if $\frac{1}{2} \le x \le 1$ (see [10]).

4. The Problem with Two Linear Branches

If $Vx = \frac{c+dx}{a}$, which means that V is linear, then $V^*y = \frac{dy}{a+cy}$ and $V^*0 = 0$. This observation leads to an interesting problem. If just one of the three branches is linear then there are good chances of finding the density of the invariant measure. If all three branches are linear then the density is $h(x) = 1$. If two branches are linear and just one branch is non-linear, then the shape of the density of the invariant measure (if it exists) is unknown!

This paper gives a partial solution to this problem, namely to consider a class of piecewise fractional linear maps with three branches but two branches which are not full.

4.1 The First Construction

We use the partition $0 < \frac{1}{3} < \frac{2}{3} < 1$. Let T be a fractional linear map such that T is linear on $[0, \frac{1}{3}]$ and on $[\frac{2}{3}, 1]$ with $T[0, \frac{1}{3}] = T[\frac{2}{3}, 1] = [\frac{1}{3}, \frac{2}{3}]$ and non-linear on $[\frac{1}{3}, \frac{2}{3}]$ and $T[\frac{1}{3}, \frac{2}{3}] = [0, 1]$.

We consider a jump transformation S, namely the Moebius map defined by $S = T^2$ on $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $S = T$ on $[\frac{1}{3}, \frac{2}{3}]$. If ν is an invariant measure for S with density h then μ (with density g), defined by $\mu(E) = \nu(E) + \nu(T^{-1}E \cap [0, \frac{1}{3}]) + \nu(T^{-1}E \cap [\frac{2}{3}, 1])$, is invariant for T.

Since S is a piecewise fractional linear map with three branches, we try to find a natural dual. If $V_\alpha, V_\beta, V_\gamma$ are the three inverse branches of T then $V_{\alpha\beta} = V_\alpha V_\beta, V_\beta, V_{\gamma\beta} = V_\gamma V_\beta$ are the three inverse branches of S. Note that the density g is given as

$$g(x) = h(x),\ 0 < x < \frac{1}{3}, \frac{2}{3} < x < 1$$

$$g(x) = h(x) + h(V_\alpha x)\omega_\alpha(x) + h(V_\gamma x)\omega_\gamma(x),\ \frac{1}{3} < x < \frac{2}{3}.$$

Here ω_α and ω_γ denote the Jacobians of V_α and V_γ. The middle branch is

$$V_\beta x = \frac{1 + (1+2\beta)x}{3 + 3\beta x}$$

in the increasing case and

$$V_\beta x = \frac{2 + (-1+\beta)x}{3 + 3\beta x}$$

in the decreasing case.

The result coincides with the result for $\beta = 0$, namely $h(x) = 1$ and $g(x) = 3$ on $[\frac{1}{3}, \frac{2}{3}]$.

As mentioned in the introduction, we write $\varepsilon = 1$ to indicate an increasing branch and $\varepsilon = -1$ to indicate a decreasing branch. The *type* of T is denoted $[\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_\gamma]$. The type of S is $[\varepsilon_\alpha\varepsilon_\beta, \varepsilon_\beta, \varepsilon_\gamma\varepsilon_\beta]$ We assume $-1 < \beta$ and for the increasing case $-1 < \beta \leq 2$. The value $\beta = 0$ belongs to the linear map.

Theorem 5. *(1) If the map T is of type $[1,1,1]$, then S has a natural dual and*

$$h(x) = \frac{1}{(2 - \beta + 3\beta x)(2 + 3\beta x)}.$$

(2) If the map T is of type $[1,-1,1]$, then S has a natural dual and

$$h(x) = \frac{1}{(4 + \beta + 3\beta x)(4 + 3\beta x)}.$$

Proof. We list the relevant (inverse) branches of T and S:

$$\varepsilon_\alpha = 1 : V_\alpha x = \frac{-1+3x}{3}; \qquad \varepsilon_\alpha = -1 : V_\alpha x = \frac{2-3x}{3};$$
$$\varepsilon_\beta = 1 : V_\beta x = \frac{1+(1+2\beta)x}{3+3\beta x}; \qquad \varepsilon_\beta = -1 : V_\beta x = \frac{2+(-1+\beta)x}{3+3\beta x};$$
$$\varepsilon_\gamma = 1 : V_\gamma x = \frac{1+3x}{3}; \qquad \varepsilon_\gamma = -1 : V_\gamma x = \frac{4-3x}{3}.$$

To determine the map S, one calculates $V_{\alpha\beta}x = V_\alpha(V_\beta x)$ and $V_{\gamma\beta}x = V_\gamma(V_\beta x)$. We obtain

$$\varepsilon_{\alpha\beta} = 1 : V_{\alpha\beta}x = \frac{(1+\beta)x}{3+3\beta x}; \qquad \varepsilon_{\alpha\beta} = -1 : V_{\alpha\beta}x = \frac{1-x}{3+3\beta x};$$
$$\varepsilon_{\gamma\beta} = 1 : V_{\gamma\beta}x = \frac{2+(1+3\beta)x}{3+3\beta x}; \qquad \varepsilon_{\gamma\beta} = -1 : V_{\gamma\beta}x = \frac{3+(-1+2\beta)x}{3+3\beta x}.$$

We look for a map $\Psi x = \frac{B+Dx}{A+Bx}$ such that $\Psi(V_i x) = V_i^*(\Psi x)$, $i \in \{\alpha\beta, \beta, \gamma\beta\}$. Then we obtain three homogeneous equations for the values A, B, D. A non-trivial

solution requires that the determinant vanishes, $DET = 0$.
(1) For Type $[1, 1, 1]$, we find the homogeneous equations

$$A(3\beta) + B(-2 + \beta) = 0$$

$$A3\beta + B(-2 + 2\beta) - D = 0$$

$$A3\beta + B(-2 + 3\beta) - 2D = 0.$$

Then $DET = 0$ and we find $A = 2 - \beta$, $B = 3\beta$, and $D = 3\beta^2$. As an example we take $\beta = 2$. Then we have

$$V_{\alpha\beta}x = \frac{x}{1 + 2x}, V_\beta x = \frac{1 + 5x}{3 + 6x}, V_{\gamma\beta}x = \frac{2 + 7x}{3 + 6x}$$

and

$$h(x) = \frac{1}{x(1 + 3x)}.$$

(2) For Type $[1, -1, 1]$, we again we find that $DET = 0$ and the values $A = 4 + \beta$, $B = 3\beta$, and $D = 3\beta^2$. As an example we take $\beta = 1$. Then we have

$$V_{\alpha\beta}x = \frac{1 - x}{3 + 3x}, V_\beta x = \frac{2}{3 + 3x}, V_{\gamma\beta}x = \frac{3 + x}{3 + 3x}$$

and

$$h(x) = \frac{1}{(4 + 3x)(5 + 3x)}.$$

□

Theorem 6. *If $\beta \neq 0$, then the map S has no natural dual for the other 6 types of T.*

Proof. Since the map $\psi x = 1 - x$ is an isomorphism for the pairs $[1, -1, -1]$, $[-1, -1, 1]$ and $[1, 1, -1]$, $[-1, 1, 1]$, just 4 cases remain. In all these cases we find that DET is a constant times $\beta(1 + \beta)$. Therefore, $DET = 0$ implies $\beta = 0$, the linear case, or $\beta = -1$, not an allowed value. □

4.2 A General Result

We started with $p_1 = \frac{1}{3}$ and $p_2 = \frac{2}{3}$. That choice was a very fortuitous one. The next theorem is a little bit surprising.

Theorem 7. *Let $0 < p_1 < p_2 < 1$. Then, for type $[1, 1, 1]$ and for type $[1, -1, 1]$ of T, the map S has a natural dual if and only if $p_1^2 + p_2^2 - p_1p_2 - p_1 = 0$.*

Proof. The rrelevant maps are as follows:

$$\varepsilon_\alpha = 1 : V_\alpha x = \frac{-p_1^2 + p_1 x}{p_2 - p_1}$$

$$\varepsilon_\beta = 1 : V_\beta x = \frac{p_1 + (p_2 - p_1 + p_2\beta)x}{1 + \beta x}, \; \varepsilon_\beta = -1 : V_\beta x = \frac{p_2 + (-p_2 + p_1 + p_1\beta)x}{1 + \beta x}$$

$$\varepsilon_\gamma = 1 : V_\gamma x = \frac{p_2^2 - p_1 + (1 - p_2)x}{p_2 - p_1}.$$

For type $[1, 1, 1]$ we calculate

$$V_{\alpha\beta} x = \frac{(p_1 + p_1\beta)x}{1 + \beta x}, \; V_{\gamma\beta} x = \frac{p_2 + (1 - p_2 + \beta)x}{1 + \beta x}$$

and find $DET = (p_1^2 + p_2^2 - p_1 p_2 - p_1)(\beta^2 + \beta)$. For type $[1, -1, 1]$ we calculate

$$V_{\alpha\beta} x = \frac{p_1 - p_1 x}{1 + \beta x}, \; V_{\gamma\beta} x = \frac{1 + (-1 + p_2 + p_2\beta)x}{1 + \beta x}$$

and find $DET = (p_1^2 + p_2^2 - p_1 p_2 - p_1)(\beta^2 + \beta)$. In both cases, $DET = 0$ only if $\beta = 0$, $\beta = -1$ (not an allowed value) or $p_1^2 + p_2^2 - p_1 p_2 - p_1 = 0$. □

If $p_1 + p_2 = 1$ then we find $p_1 = \frac{1}{3}$ and $p_2 = \frac{2}{3}$. Another rational solution is $p_1 = \frac{1}{7}$ and $p_2 = \frac{3}{7}$.

Theorem 8. *For $\beta \neq 0$ the map S has no natural dual for the other 6 types of T.*

Proof. Again we find DET is a multiple of $\beta(1 + \beta)$. □

Remark 4. In this paper we restricted our investigation to natural duals of the map S. The search for exceptional duals is confronted with a great nmber of configurations (see [6]). We give one example of a singular dual. Let $p = \frac{1+\sqrt{8}}{7}$. Then we choose $p_1 = \frac{1-p}{2}$, $p_2 = p$, and $\beta = 1$. Then $\tau = \frac{1}{p}$ is a common fixed point for all three branches and $h(x) = \frac{1}{(p+x)^2}$ is the invariant density of S.

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