

SUMS OF DISTINCT DIVISORS OF FACTORIALS

Scott D. Hughes
Independent Researcher
v@deltagray.com

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Abstract

An integer $N \geq 1$ is *practical* if every integer $1 \leq m \leq N$ is a sum of distinct divisors of N. For practical N, let $h(N)$ be the least k such that every integer $1 \leq m \leq N$ is a sum of at most k distinct divisors of N. We prove that $h(n!) \leq (2\log 2 + o(1))\, n/\log n$. This improves the bounds of order $n/(\log n)^{1/2-\varepsilon}$ established by Tenenbaum and Yokota. We combine their decreasing greedy construction with the sharper factorial divisor-gap estimate of Berend and Harmse. Counting the steps separately below and above $\sqrt{n!}$, with the upper range handled through reciprocal divisors, retains the leading coefficient in the gap exponent and yields the explicit constant $2\log 2$.

1. Introduction

Throughout, log denotes the natural logarithm and $\lg t = \log_2 t$. An integer $N \geq 1$ is *practical* if every $1 \leq m \leq N$ is a sum of distinct divisors of N. For practical N, let

$$h(N) = \min\bigl\{k : \text{ every } 1 \leq m \leq N \text{ is a sum of at most } k \text{ distinct divisors of } N\bigr\}.$$

Erdős proved that $h(n!) \leq n$ and asked for the true order of $h(n!)$ [2, pp. 37–38]. Lemma 4 of Tenenbaum and Yokota [5] is stated with N_k denoting either $k!$ or $\prod_{j\leq k} p_j$. In particular, for every $\eta > 0$, it gives

$$h(n!) \ll_\eta \frac{n}{(\log n)^{1/2-\eta}}.$$

Yokota's 1995 note [6] reaches the same exponent by a decreasing greedy construction. For every fixed $0 < \varepsilon < \frac{1}{2}$ and all sufficiently large k, its proof represents each integer $1 \leq S < 2k!$ as a sum of distinct divisors of $k!$, using

$$m \ll_\varepsilon \frac{k}{(\log k)^{1/2-\varepsilon}}$$

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terms. Its Lemma 3 uses the older gap input $\varepsilon_k = \exp\{-(\log k)^{3/2-\varepsilon}\}$, furnished by Tenenbaum's factorial small-interval estimates. See the deduction on pp. 6–7 and Theorem 4 of [3], with the corrections in [4]. Neither [5] nor [6] states a bound of order $n/\log n$. Theorem 1 gives this order with an explicit leading constant.

Theorem 1. *We have*

$$h(n!) \;\le\; (2\log 2 + o(1))\,\frac{n}{\log n}$$

as $n \to \infty$.

Our proof of Theorem 1 retains the greedy construction of [5, 6] and uses the 1993 factorial gap estimate of Berend and Harmse directly. We recompute the step counts in both ranges, retaining the coefficient $1/(2\log 2)$ in the gap exponent to obtain the constant $2\log 2$. Sections 2 and 3 give the details.

2. The Two Ingredients

The first ingredient in the proof of Theorem 1 is the following divisor-gap estimate for factorials.

Theorem 2 ([1, Theorem 2])**.** *For* $n \ge 2^{16}$ *and* $\sqrt{(n-1)!} \le D \le \sqrt{n!}$, *there is a divisor* x *of* $n!$ *such that*

$$\left|\frac{x}{D} - 1\right| \;\le\; 5\cdot 10^7 \Big(\frac{\lg n}{n}\Big)^{\frac{\lg n - \lg(\lg n)+1}{2} + \lg e} \;\le\; \Big(\frac{1}{n}\Big)^{\frac{\lg n}{2} - \lg(\lg n)}.$$

Put

$$\varepsilon_j \;:=\; \Big(\frac{1}{j}\Big)^{\frac{\lg j}{2} - \lg(\lg j)},$$

the rightmost bound above at $n = j$, so that

$$\log\frac{1}{\varepsilon_j} \;=\; \Big(\frac{\lg j}{2} - \lg(\lg j)\Big)\log j \;=\; \frac{(\log j)^2}{2\log 2}\Big(1 - \frac{2\log(\log j/\log 2)}{\log j}\Big). \tag{1}$$

Since $\big(\frac{\lg x}{2} - \lg(\lg x)\big)\log x$ is increasing for $x \ge 2^{16}$, the sequence $(\varepsilon_j)_{j\ge 2^{16}}$ is decreasing.

Corollary 1. *Let* $j \ge 2^{16}$, *and let* $a < b$ *be consecutive divisors of* $n!$ *(for any* $n \ge j$*) with* $\sqrt{(j-1)!} \le \sqrt{ab} \le \sqrt{j!}$. *Then* $\log(b/a) \le 3\varepsilon_j$.

Proof. Apply Theorem 2 at $D = \sqrt{ab}$. The resulting divisor x of $j!$ divides $n!$. Since a and b are consecutive divisors of $n!$, either $x \le a$ or $x \ge b$. If $x \le a$, then $1 - \varepsilon_j \le x/D \le a/D = \sqrt{a/b}$, so $b/a \le (1-\varepsilon_j)^{-2}$. If $x \ge b$, then $\sqrt{b/a} = b/D \le x/D \le 1+\varepsilon_j$, so $b/a \le (1+\varepsilon_j)^2$. Since $-\log(1-u) \le u/(1-u) \le \frac{4}{3}u$ for $0 \le u \le \frac{1}{4}$, both cases give $\log(b/a) \le 3\varepsilon_j$. Here $\varepsilon_j \le \varepsilon_{2^{16}} = (2^{-16})^{8-4} = 2^{-64} < \frac{1}{4}$. □

The second ingredient is the greedy mechanism used in the proof of [5, Lemma 4]. We use the standard fact, recalled there, that the ratio of two consecutive divisors of $n!$ does not exceed 2.

Lemma 1 (Greedy step)**.** *Let N be an integer whose consecutive divisors have ratio at most 2, and let $1 \le R \le N$. If R divides N (in particular, if $R = N$), the greedy expansion terminates at this step. Otherwise, let $d < R < b$ be the consecutive divisors of N bracketing R. Then*

$$R - d \;<\; d, \qquad R - d \;\le\; 2R \log \frac{b}{d}.$$

Consequently, successive divisors chosen by the greedy expansion are strictly decreasing, hence distinct.

Proof. Since $R < b \le 2d$, we have $R - d < b - d \le d$, so the next chosen divisor is smaller than d. Moreover, $R - d < b - d = d(b/d - 1) \le R\,(b/d - 1) \le 2R \log(b/d)$, because $b/d \in [1, 2]$ and $x - 1 \le 2 \log x$ on that interval. □

3. Proof of Theorem 1

Proof of Theorem 1. Set $j_0 = 2^{16}$ and $T_0 = 2\sqrt{(j_0 + 1)!}$, an absolute constant, and write $N = n!$. We may assume that $n \ge j_0^2$, since the bound of the theorem is asymptotic. Let $1 \le m \le N$ and run the greedy expansion. Write $R_0 = m > R_1 > \cdots$ for the successive remainders (the expansion terminates whenever a remainder divides N) and $\ell(R) = \log R$. All step counts below concern nonterminal subtractions. If the process terminates at a positive divisor of N, that final divisor contributes one additional term, absorbed in the $O(1)$ count of the remaining steps at the end of the proof.

Two symmetric facts about the bracketing divisors $d < R < b$ locate the Berend–Harmse window. If $R \le \sqrt{N}$, then $db \le N$: otherwise $N/d < b$, while $N/d \ge N/R \ge \sqrt{N} \ge R > d$, so N/d would be a divisor strictly between d and b. Symmetrically, if $R > \sqrt{N}$, then $db \ge N$.

`Case 1.` We bound the number of steps in the lower range $T_0 \le R \le \sqrt{N}$. Let $j(R) \in [j_0 + 1, n]$ be minimal with $R \le \sqrt{j(R)!}$. For a nonterminal step at R with bracketing divisors $d < R < b$, the geometric mean $\sqrt{db}$ lies in $[R/2, 2R]$ (as $d > R/2$ and $b < 2R$) and, by the preceding paragraph, $\sqrt{db} \le \sqrt{N}$. Each window $[\sqrt{(j-1)!}, \sqrt{j!}]$ has logarithmic width $\frac{1}{2} \log j \ge \log 2$, so $\sqrt{db}$ lies in the window of index $j(R) - 1$, $j(R)$, or $j(R) + 1$. Moreover, $j(R) - 1 \ge j_0$ by the choice of T_0. When $j(R) = n$, the shared endpoint $\sqrt{N}$ is assigned to the window of index n. Corollary 1 and the monotonicity of ε_j give $\log(b/d) \le 3\varepsilon_{j(R)-1} =: \frac{1}{2}\delta_{j(R)}$, and by Equation (1), applied at $j - 1$ (the index shift and the constant $\log 6$ are absorbed

in the error term),

$$s_j \;:=\; \log\frac{1}{\delta_j} \;=\; \frac{(\log j)^2}{2\log 2}\Big(1+O\Big(\frac{\log\log j}{\log j}\Big)\Big).$$

Lemma 1 gives $R_{i+1} \le 2R_i\log(b/d) \le R_i\,\delta_{j(R_i)}$, that is,

$$\ell(R_{i+1}) \;\le\; \ell(R_i) - s_{j(R_i)}.$$

Each step descends by at least $s_{j(R_i)}$. Moreover, s_j is increasing in j and $j(R)$ is nondecreasing in R, so $s_{j(e^\ell)} \le s_{j(R_i)}$ throughout the interval $[\ell(R_{i+1}),\ell(R_i)]$ of the ith step, whence $\int_{\ell(R_{i+1})}^{\ell(R_i)} d\ell/s_{j(e^\ell)} \ge 1$. The step intervals are disjoint, and all but possibly the final one (which may cross $\log T_0$) lie inside $[\log T_0, \frac12\log N]$. Hence,

$$\begin{aligned}\#\{\text{steps with } T_0 \le R \le \sqrt{N}\} &\le 1 + \int_{\log T_0}^{\frac12\log N} \frac{d\ell}{s_{j(e^\ell)}} \;\le\; O(1) + \sum_{j=j_0+1}^{n} \frac{\frac12\log j}{s_j} \\ &= \log 2 \sum_{j=j_0+1}^{n} \frac{1}{\log j} + O\Big(\sum_{j=j_0+1}^{n} \frac{\log\log j}{(\log j)^2}\Big) + O(1) \\ &= \Big(\log 2 + O\Big(\frac{\log\log n}{\log n}\Big)\Big)\frac{n}{\log n},\end{aligned}$$

since the integral over the window of index j, of logarithmic width $\frac12\log j$, contributes at most $\frac12\log j/s_j$, and $\sum_{j=j_0+1}^{n} 1/\log j \sim n/\log n$.

`Case 2.` We bound the number of steps in the upper range $\sqrt{N} < R \le N/T_0$. Put $U = N/R$. If $d < R < b$ are the bracketing divisors of R, then N/b and N/d are consecutive divisors of N with $N/b < U < N/d$ and with the same ratio b/d. Their geometric mean $N/\sqrt{db}$ lies in $[U/2, 2U]$ and, since $db \ge N$, is at most $\sqrt{N}$. Thus, the window argument of the lower range applies verbatim *at the mirror position*, giving

$$R_{i+1} \;\le\; \delta_{j(U_i)}\,R_i, \qquad \text{that is,} \qquad \log U_{i+1} - \log U_i \;\ge\; s_{j(U_i)},$$

where now $U_i = N/R_i$ *increases* along the expansion. The charging integral of the lower range cannot simply be mirrored (the monotonicity of $s_{j(U)}$ runs the wrong way), so we count these steps dyadically instead. Let

$$R_n := \Big\lfloor \log_2\frac{n}{2j_0} \Big\rfloor, \qquad J_r := \frac{n}{2^{r+1}} \quad (0 \le r \le R_n),$$

and let

$$\mathcal{B}_r := \{j \in \mathbb{Z} : J_r < j \le 2J_r\}.$$

Then $J_r \ge j_0$ for $0 \le r \le R_n$, and these blocks cover all indices $j > J_{R_n}$ up to n. The remaining indices satisfy $j \le J_{R_n} < 2j_0$. Consider the steps whose start

satisfies $j(U_i) \in \mathcal{B}_r$. Their values of $\log U_i$ lie in an interval of length at most

$$L_r \;\le\; \tfrac{1}{2} \sum_{J_r < j \le 2J_r} \log j \;=\; \tfrac{1}{2} J_r \log J_r + O(J_r),$$

where the sum is over integers. Successive starts in this block are separated in $\log U$ by at least $s_{\lfloor J_r \rfloor + 1}$, so their number is at most

$$1 + \frac{L_r}{s_{\lfloor J_r \rfloor + 1}} \;=\; \Big(\log 2 + O\Big(\frac{\log\log J_r}{\log J_r}\Big)\Big) \frac{J_r}{\log J_r} + O(1).$$

The relative-error terms are summable at the required scale, in the sense that

$$\sum_{r=0}^{R_n} \frac{J_r \log\log J_r}{(\log J_r)^2} = O\left(\frac{n \log\log n}{(\log n)^2}\right) = o\left(\frac{n}{\log n}\right).$$

Indeed, put $g(t) = \log\log t/(\log t)^2$. The function g is decreasing for $t \ge j_0$. For the blocks with $J_r \ge \sqrt{n}$, we have $g(J_r) \le g(\sqrt{n})$ and $\sum J_r = O(n)$, giving $O(n \log\log n/(\log n)^2)$. For the remaining blocks, we have $g(J_r) = O(1)$ and $\sum J_r = O(\sqrt{n})$, which is $o(n/\log n)$. The remaining indices have $j(U_i) < 2j_0$, hence $T_0 \le U_i \le \sqrt{(2j_0)!}$. This is a fixed logarithmic interval, and the positive lower bound for the step size gives $O(1)$ further steps. The accumulated $O(1)$ terms from the $O(\log n)$ dyadic blocks total $O(\log n) = o(n/\log n)$. Summing over the blocks with $J_r \ge j_0$ and using $\sum_{r=0}^{R_n} J_r/\log J_r = (1 + o(1))\, n/\log n$, we obtain

$$\#\{\text{steps with } \sqrt{N} < R \le N/T_0\} \;\le\; (\log 2 + o(1))\, \frac{n}{\log n}.$$

It remains to count the steps not covered by Cases 1 and 2, namely those with $N/T_0 < R \le N$ and those with $1 \le R < T_0$. For $N/T_0 < R \le N$, the mirror $U = N/R$ lies below T_0. Every nonterminal step still has $d > R/2$, hence $R_{i+1} < R_i/2$, so this region contributes at most $\log T_0/\log 2 + O(1) = O(1)$ steps. The same halving argument disposes of $1 \le R < T_0$ at the bottom. Adding the counts for the lower range, the upper range, and these $O(1)$ remaining steps proves the theorem. □

4. Remarks

Remark 1. The proof gives the more explicit relative error $O(\log\log n/\log n)$ in the constant, inherited from the $\lg(\lg n)$ term in Theorem 2. Berend and Harmse note (their Remark 2) that, up to a bounded power of $\lg n$, their bound is the best their method produces.

Remark 2. In the other direction, write $T = \tau(n!)$ and $k = h(n!)$. If $k \le T/2$, then

$$n! \;\le\; \sum_{i=0}^{k} \binom{T}{i} \;\le\; (eT/k)^k.$$

We only need the estimate $\log T \ll n/\log n$, which follows from Chebyshev's bound $\pi(x) \ll x/\log x$. Indeed, in the sum $\log T = \sum_{p \le n} \log(v_p(n!)+1)$, where v_p denotes the p-adic valuation, the primes $p \le \sqrt{n}$ contribute $O(\sqrt{n}\log n)$. For $p > \sqrt{n}$ in $(n/2^{r+1}, n/2^r]$, we have $v_p(n!) = \lfloor n/p \rfloor < 2^{r+1}$, and there are $O(n/(2^r \log n))$ such primes. Their total contribution is therefore

$$\ll \frac{n}{\log n} \sum_{r \ge 0} \frac{r+1}{2^r} \ll \frac{n}{\log n}.$$

Taking logarithms in the counting inequality and using $\log(n!) \asymp n \log n$ now gives $k \gg (\log n)^2$. If $k > T/2$, the same conclusion follows from $T \ge n$, since every integer from 1 to n divides $n!$. Erdős asked whether $h(n!) < n^{o(1)}$, or even $h(n!) < (\log n)^{O(1)}$ [2, pp. 37–38].

Remark 3. Yokota's note [6] uses the same two greedy ranges separated at $\sqrt{k!}$, with the upper range mirrored through $k!/S$. Its Lemma 2 supplies the ratio-at-most-2 fact used in Lemma 1. The sharper estimate of Berend and Harmse [1, Theorem 2] gives

$$\log(1/\varepsilon_j) = \frac{(\log j)^2}{2\log 2} + O(\log j \log\log j).$$

The present proof applies this estimate directly and counts the steps in both ranges to retain its leading coefficient. The contribution of Theorem 1 is the resulting bound with constant $2\log 2$. The greedy construction is due to Tenenbaum and Yokota and to Yokota, and the sharper gap estimate is due to Berend and Harmse. The paper of Berend and Harmse appeared in 1993. Yokota's paper was received in September 1994 and cites Tenenbaum's 1987 paper and Hardy and Wright.

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