

NUMBERS WITH FOUR CLOSE FACTORIZATIONS

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Abstract

In this paper, we study numbers n that can be factored in four different ways as $n = AB = (A + a_1)(B - b_1) = (A + a_2)(B - b_2) = (A + a_3)(B - b_3)$ with $B \le A$, $1 \le a_1 < a_2 < a_3 \le C$, and $1 \le b_1 < b_2 < b_3 \le C$. We obtain the optimal upper bound $A \le 0.047420\ldots \cdot C^3 + O(C)$. The key idea is to transform the original question into generalized Pell equations $ax^2 - by^2 = c$ and to study their solutions.

1. Introduction and Main Result

Integer factorization is a fundamental topic in number theory. In this paper, we continue the first author's work [1] and study close factorizations of integers. For example,

$$159600 = 399 \cdot 400 = 380 \cdot 420,$$

and

$$3950100 = 1900 \cdot 2079 = 1890 \cdot 2090 = 1881 \cdot 2100$$

are numbers with two and three close factorizations, respectively. One may ask whether there is any structure among such integers. Equivalently, one can interpret

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close factorizations of n as close lattice (integer) points on the hyperbola $xy = n$. Related earlier works include Cilleruelo and Jiménez-Urroz [3, 4] and Granville and Jiménez-Urroz [6]. Suppose a positive integer n can be factored in k different ways:

$$n = AB = (A + a_1)(B - b_1) = (A + a_2)(B - b_2) = \cdots = (A + a_{k-1})(B - b_{k-1}) \quad (1)$$

for some integers

$$1 \le B \le A, \ \ 1 \le a_1 < a_2 < \cdots < a_{k-1} \le C, \ \text{ and } \ 1 \le b_1 < b_2 < \cdots < b_{k-1} \le C \quad (2)$$

where C is a parameter measuring the "closeness" of the factors. A central question is to study the dependence of A and B on C and k. The first author [1] (and later in [2], which corrected an earlier mistake) proved the following upper bound for A and B when $k = 3$, and showed that no such upper bound exists when $k = 2$.

Theorem 1 ([1, 2])**.** *Let $C \ge 5$. Suppose n satisfies Equations (1) and (2) with $k = 3$. We have*

$$B \le A \le \frac{1}{4}C(C-1)^2.$$

Moreover, the upper bound $\frac{1}{4}C(C-1)^2$ can be attained if and only if $C = 2N + 1$ and

$$\begin{aligned} n =& [(2N+1)N^2 - (2N+1)] \cdot [(2N-1)(N-1)(N+1) + (2N-1)] \\ =& [(2N+1)N^2 - N] \cdot [(2N-1)(N-1)(N+1) + (N-1)] \\ =& [(2N+1)N^2] \cdot [(2N-1)(N-1)(N+1)]. \end{aligned}$$

In this paper, we continue the investigation when $k = 4$. Based on Lemma 1 from the next section, we know that a_{k-1} is the largest among all the a_i's and b_i's. Hence, one may simply set $C = a_{k-1}$ and we are interested in the ratio

$$R_k := \frac{A}{a_{k-1}^3} \quad \text{when} \ \ k = 2, 3, 4.$$

The above discussion shows that there is no uniform upper bound for R_2 while $R_3 \le 0.25$ with 0.25 being the best upper bound. Incidentally, Granville and Jiménez-Urroz [6, Theorem 1] proved (under our notation) that the least common multiple

$$\operatorname{lcm}(A, A + a_1, \ldots, A + a_{k-1}) \ge \frac{C_k A^k}{a_{k-1}^{\frac{k(k-1)}{2}}}$$

for some positive constant C_k (for example, $C_2 = 1$, $C_3 = 4$, and $C_4 = 50\sqrt{5}$). From this, one can deduce that

$$A^2 \ge \underbrace{AB}_{n} \ge \operatorname{lcm}(A, A + a_1, \ldots, A + a_{k-1}) \ge \frac{C_k A^k}{a_{k-1}^{k(k-1)/2}} \ \text{ and } \ \frac{A}{a_{k-1}^{\frac{k(k-1)}{2(k-2)}}} \le \frac{1}{C_k^{\frac{1}{k-2}}}. \quad (3)$$

When $k = 3$ or 4, the exponent $\frac{k(k-1)}{2(k-2)}$ in Equation (3) equals 3. This partially motivates our choice of a_{k-1}^3 in the ratio R_k when $k = 2, 3, 4$. In particular, Equation (3) gives $R_3 \le \frac{1}{4}$ (matching Theorem 1) and $R_4 \le \sqrt{\frac{1}{50\sqrt{5}}} = 0.094574\ldots$ respectively. Our main result is the following improvement.

Theorem 2. *Suppose n satisfies Equations (1) and (2) with $k = 4$. We have*

$$R_4 \le \frac{6+\sqrt{6}}{9(2+\sqrt{6})^2} + O\Big(\frac{1}{n^{1/3}}\Big) \approx 0.047420\ldots$$

when n is sufficiently large. Moreover, the above bound is best possible.

From the proof (with $x_2 = 49$ and $y_2 = 20$ in Equation (46)), we obtain the following numerical example from Equations (47) and (48):

$$\begin{aligned} &665165362680 \\ &\quad = \underbrace{902460}_{A} \cdot \underbrace{737058}_{B} = \underbrace{902520}_{A+a_1} \cdot \underbrace{737009}_{B-b_1} = \underbrace{902629}_{A+a_2} \cdot \underbrace{736920}_{B-b_2} = \underbrace{902727}_{A+a_3} \cdot \underbrace{736840}_{B-b_3} \end{aligned}$$

with

$$R_4 = \frac{902460}{(902727 - 902460)^3} = 0.0474126\ldots.$$

Thus, Theorem 2 shows an intimate connection between numbers with four close factorizations and solutions of generalized Pell equations $ax^2 - by^2 = c$.

Curiously, while studying divisors in arithmetic progressions, Cohen [5, Corollaire 1.4] proved similar but more general bounds (under our notation with $a_0 = 0$):

$$\operatorname{lcm}(A+a_0, \ldots, A+a_{k-1})^{\frac{t(t+1)}{2}} \prod_{0 \le i < j \le k-1} \gcd(A+a_i, A+a_j) \ge \prod_{0 \le i \le k-1} (A+a_i)^t$$

for all integers t. It follows that

$$(A^2)^{\frac{t(t+1)}{2}} a_{k-1}^{\frac{k(k-1)}{2}} \ge A^{kt} \quad \text{and} \quad \frac{A}{a_{k-1}^{\frac{k(k-1)}{2t(k-t-1)}}} \le 1, \tag{4}$$

which resembles Equation (3) when $t = 1$. When $k = 3$ or 4, Cohen's bound (4) yields a larger (and weaker) bound than Granville and Jiménez-Urroz's bound (3). However, when $k = 5$ and $t = 2$, the exponent of a_{k-1} in Equation (4) is $\frac{5}{2}$, which is smaller (and better) than that of Equation (3). This shows a change in behavior when $k = 5$. One may try to incorporate Proposition 1 of [6] into Cohen's argument in [5], but it seems to yield a ratio worse than 1. Thus, one future direction of research is to investigate numbers with $k = 5$ close factorizations and determine the optimal exponent λ_5 as well as the optimal ratio $R_5 = A/a_4^{\lambda_5}$.

This paper is organized as follows. First, we make some useful observations based on the four close factorizations. These include the special situation $a_3 b_1 - a_1 b_3 =$

$a_3b_2 - a_2b_3$ where we obtain the stronger bound $B \leq A \leq 0.25C^2$ (see Lemma 7). Next, we treat the special case when $a_2b_1 - a_1b_2 = 1$. The crux of the method is to transform our original question into generalized Pell equations (see Equations (20) or (33) for example). Based on this, we build up the general Pell-machinery. We also derive a formula for the ratio R_4 when solutions to Pell-type equations exist. Then we study various scenarios (as limited by Lemmas 3, 4, 5, 6, 7, and 12), and eliminate many of them through the use of divisibility and modular arithmetic. Finally, we combine everything together to deduce Theorem 2.

We adopt the following notation. When an integer a divides another integer b, we write $a \mid b$. Similarly, the symbol $a \nmid b$ means that a does not divide b. The big-O notation $f(n) = g(n) + O(h(n))$ means that, for some constant $c > 0$, we have $|f(n) - g(n)| \leq ch(n)$ for $n \geq 1$.

2. Some Initial Observations

First, we recall some basic tools from [1]. Expanding $AB = (A + a_i)(B - b_i)$ and simplifying, we obtain $a_iB - b_iA = a_ib_i$. Dividing both sides by b_iB, we obtain

$$\frac{a_i}{b_i} - \frac{A}{B} = \frac{a_i}{B}. \tag{5}$$

Similarly, if one divides both sides by a_iA instead, we obtain

$$\frac{B}{A} - \frac{b_i}{a_i} = \frac{b_i}{A}. \tag{6}$$

Applying Equation (5) with two different indices i, j and subtracting the two equations, we obtain

$$\frac{a_i}{b_i} - \frac{a_j}{b_j} = \frac{a_i - a_j}{B} \quad \text{or} \quad a_ib_j - a_jb_i = \frac{b_ib_j(a_i - a_j)}{B}. \tag{7}$$

Similarly, applying Equation (6) with two different indices i, j and subtracting, we obtain

$$\frac{b_j}{a_j} - \frac{b_i}{a_i} = \frac{b_i - b_j}{A} \quad \text{or} \quad a_ib_j - a_jb_i = \frac{a_ia_j(b_i - b_j)}{A}. \tag{8}$$

Now, we define the "*skews*":

$$D_{ij} := a_ib_j - a_jb_i \quad \text{for} \quad 1 \leq j \neq i \leq 3. \tag{9}$$

Note that $D_{ji} = -D_{ij}$. Since $0 < a_1 < a_2 < a_3$ and $0 < b_1 < b_2 < b_3$, we obtain

$$\begin{gathered} D_{ij} = \frac{b_ib_j(a_i - a_j)}{B} = \frac{a_ia_j(b_i - b_j)}{A} > 0, \\ B = \frac{b_ib_j(a_i - a_j)}{D_{ij}}, \quad \text{and} \quad A = \frac{a_ia_j(b_i - b_j)}{D_{ij}} \end{gathered} \tag{10}$$

for all $1 \leq j < i \leq 3$ from Equations (7) and (8).

Lemma 1. *For $1 \le i \le k-1$, we have $a_i > b_i$. We also have $a_i - a_j > b_i - b_j$ for all $1 \le j < i \le k-1$.*

Proof. Equation (5) implies

$$\frac{a_i}{b_i} = \frac{A + a_i}{B} > \frac{A}{B} \ge 1 \tag{11}$$

since $A \ge B$. This gives the first half of the lemma. Note: Using Equation (6) instead, one can obtain

$$\frac{b_i}{a_i} = \frac{B - b_i}{A} < \frac{B}{A} \le 1. \tag{12}$$

Equation (10) implies

$$\frac{B}{A} = \frac{b_i b_j (a_i - a_j)}{a_i a_j (b_i - b_j)} \quad \text{or} \quad \frac{a_i - a_j}{b_i - b_j} = \frac{B}{A} \cdot \frac{a_i}{b_i} \cdot \frac{a_j}{b_j}.$$

Applying Equation (11) to the above equation, we obtain

$$\frac{a_i - a_j}{b_i - b_j} > \frac{B}{A} \cdot \frac{A}{B} \cdot \frac{A}{B} = \frac{A}{B} \ge 1,$$

which gives the second half of the lemma since $b_i > b_j$. □

Next, we notice two simple relations among the skews D_{21}, D_{31}, and D_{32}.

Lemma 2. *We have the identities $D_{31}a_2 - D_{21}a_3 = D_{32}a_1$ and $D_{31}b_2 - D_{21}b_3 = D_{32}b_1$.*

Proof. From the definition of D_{ij}'s, we obtain

$$D_{31}a_2 - D_{21}a_3 = (a_3b_1 - a_1b_3)a_2 - (a_2b_1 - a_1b_2)a_3 = a_1a_3b_2 - a_1a_2b_3 = a_1D_{32}.$$

This gives the first identity. The second one follows in a similar manner. □

Lemma 3. *We have the inequality $D_{31} > D_{21}$.*

Proof. Since $D_{32} > 0$ and $a_1 > 0$, Lemma 2 yields $D_{31}a_2 - D_{21}a_3 > 0$ or $D_{31}a_2 > D_{21}a_3$. This gives $D_{31} > D_{21}a_3/a_2 > D_{21}$ since $a_3 > a_2$ by Lemma 1. □

Lemma 4. *We have the inequality $D_{32} + D_{21} > D_{31}$.*

Proof. From Equation (10), we obtain

$$\frac{b_2b_1(a_2 - a_1)}{D_{21}} = \frac{b_3b_1(a_3 - a_1)}{D_{31}} \quad \text{or} \quad D_{31}b_2(a_2 - a_1) = D_{21}b_3(a_3 - a_1).$$

Subtracting $D_{21}b_2(a_3 - a_1)$ from both sides, we obtain

$$b_2(D_{31}a_2 - D_{31}a_1 - D_{21}a_3 + D_{21}a_1) = D_{21}(b_3 - b_2)(a_3 - a_1) > 0.$$

Lemma 2 gives $-D_{21}a_3 = D_{32}a_1 - D_{31}a_2$. Substituting this into the above, we obtain

$$b_2(D_{31}a_2 - D_{31}a_1 + D_{32}a_1 - D_{31}a_2 + D_{21}a_1) = b_2a_1(D_{32} + D_{21} - D_{31}) > 0,$$

which proves the lemma since b_2 and a_1 are positive. □

Lemma 5. *Let $0 < a_1 < a_2 < a_3 \le C$ and $0 < b_1 < b_2 < b_3 \le C$. Suppose $a_i = (1+\lambda)a_j$ for some $1 \le j < i \le 3$ and $\lambda > 0$. We have*

$$B \le A < \frac{\lambda C^3}{D_{ij}(1+\lambda)^2}.$$

In particular, if one of D_{21}, D_{31}, D_{32} is greater than 5, then $B \le A \le \frac{C^3}{24} < 0.042C^3$.

Proof. Since $a_i = (1+\lambda)a_j \le C$, we obtain $a_j \le \frac{C}{(1+\lambda)}$. Also, $a_i - a_j = (1+\lambda)a_j - a_j = \lambda a_j$. Thus, applying Equation (10) and Lemma 1, we obtain

$$B \le A = \frac{a_ia_j(b_i - b_j)}{D_{ij}} < \frac{a_ia_j(a_i - a_j)}{D_{ij}} = \frac{a_ia_j^2\lambda}{D_{ij}} \le \frac{\lambda C^3}{D_{ij}(1+\lambda)^2},$$

which yields the first half of the lemma. By simple calculus, one can verify that the function $\frac{\lambda}{(1+\lambda)^2}$ is increasing on the interval $(0,1)$. Hence, if some $D_{ij} > 5$, the above inequality implies $B \le A \le \frac{C^3}{6\cdot 2^2}$, and the entire lemma follows. □

Another consequence of Lemma 5 is a bound on R_4 in certain cases.

Lemma 6. *Suppose $D_{31} = 5$. If $D_{21} \ge 3$ and $D_{32} \ge 4$, then $R_4 < 0.045$.*

Proof. Lemma 2 gives $D_{21}a_3 + D_{32}a_1 = 5a_2$. If $a_3 \ge 4.8a_1$, then the proof of Lemma 5 yields

$$B \le A < \frac{a_1a_3(a_3 - a_1)}{D_{31}} \le \frac{a_3^3}{24} \quad \text{and} \quad R_4 = \frac{A}{a_3^3} < \frac{1}{24} < 0.042.$$

If $a_3 < 4.8a_1$, then, since $D_{21} \ge 3$ and $D_{32} \ge 4$, we obtain $5a_2 \ge 3a_3 + 4a_1 > (3 + \frac{4}{4.8})a_3 = \frac{23}{6}a_3$. This implies $a_3 < \frac{30}{23}a_2$. Hence, if $a_3 = (1+\lambda)a_2$, then $0 < \lambda < \frac{7}{23}$. Applying Lemma 5 and recalling $C = a_3$, we obtain $A \le \frac{\lambda a_3^3}{D_{32}(1+\lambda)^2}$. Since $\frac{\lambda}{(1+\lambda)^2}$ is increasing on the interval, we can deduce that

$$R_4 = \frac{A}{a_3^3} \le \frac{7/23}{4(1+7/23)^2} < 0.045.$$

□

Finally, we consider the case of equal skews.

Lemma 7. *Suppose $D_{31} = D_{32}$ and suppose that*

$$n \;=\; AB \;=\; (A+a_1)(B-b_1) \;=\; (A+a_2)(B-b_2) \;=\; (A+a_3)(B-b_3)$$

admits four close factorizations. This gives the relations

$$A(A+a_3) \;=\; (A+a_1)(A+a_2) \quad \textit{and} \quad B(B-b_3) \;=\; (B-b_1)(B-b_2). \tag{13}$$

Moreover, we obtain the superior upper bound $B \le A \le C^2/4$.

Proof. First, Equation (9) and $D_{31} = D_{32}$ give

$$a_3b_1 - a_1b_3 = a_3b_2 - a_2b_3 \quad \text{or} \quad a_3(b_2-b_1) = b_3(a_2-a_1) \tag{14}$$

after some rearrangement. Next, Equation (10) implies

$$\frac{b_2b_1(a_2-a_1)}{B} = D_{21} = \frac{a_2a_1(b_2-b_1)}{A} \quad \text{or} \quad \frac{b_2b_1b_3(a_2-a_1)}{B} = \frac{a_2a_1b_3(b_2-b_1)}{A}.$$

Applying Equation (14) to the above equation, we obtain

$$\frac{b_2b_1a_3(b_2-b_1)}{B} = \frac{a_2a_1b_3(b_2-b_1)}{A} \quad \text{or} \quad \frac{b_2}{a_2}\cdot\frac{b_1}{a_1}\cdot\frac{a_3}{b_3} = \frac{B}{A}. \tag{15}$$

Furthermore, applying Equation (11) to Equation (15), we obtain

$$\frac{B}{A+a_2}\cdot\frac{B}{A+a_1}\cdot\frac{A+a_3}{B} = \frac{B}{A},$$

which gives the first half of Equation (13). Similarly, applying Equation (12) to Equation (15), we obtain the second half of Equation (13). Expanding the first equality of Equation (13) and simplifying, we obtain $A(a_3-a_2-a_1) = a_1a_2 > 0$ and, hence, $a_3 \ge a_1+a_2+1$. Finally, the last part of the lemma follows from

$$C \ge a_3 > a_1+a_2 \ge 2\sqrt{a_1a_2} = 2\sqrt{A(a_3-a_2-a_1)} \ge 2\sqrt{A}.$$

□

3. The Situation When $D_{21} = 1$

This special case $D_{21} = 1$ forms the prototype for subsequent arguments. One may restrict to $2 \le D_{31}, D_{32} \le 5$ by Lemmas 3 and 5. Also, Lemmas 4 and 7 allow us to focus on $D_{32} > D_{31} - 1$ with $D_{31} \ne D_{32}$. Equation (9) gives

$$\begin{cases} a_2b_1 - b_2a_1 = 1, \\ a_3b_1 - b_3a_1 = D_{31}. \end{cases} \tag{16}$$

The first equality in Equation (16) implies $\gcd(b_1, b_2) = 1$. From Equation (10), we obtain

$$D_{32}b_1(a_3 - a_1) = D_{31}b_2(a_3 - a_2). \tag{17}$$

Equation (17) implies $b_1 \mid D_{31}b_2(a_3-a_2)$ and $b_2 \mid D_{32}b_1(a_3-a_1)$. Since $\gcd(b_1, b_2) = 1$, we must have $b_1 \mid D_{31}(a_3 - a_2)$ and $b_2 \mid D_{32}(a_3 - a_1)$ by Euclid's lemma. Hence, we can write

$$b_1k_1 = D_{31}(a_3 - a_2) \quad \text{and} \quad b_2k_2 = D_{32}(a_3 - a_1)$$

for some integers k_1 and k_2. Substituting these into Equation (17), we can deduce that, if $k_1 = k_2 = k > 0$, then

$$b_1k = D_{31}(a_3 - a_2), \quad \text{and} \quad b_2k = D_{32}(a_3 - a_1). \tag{18}$$

By a similar argument, we can also obtain

$$a_1m = D_{31}(b_3 - b_2) \quad \text{and} \quad a_2m = D_{32}(b_3 - b_1) \tag{19}$$

for some integer $m > 0$. Lemma 2 and $D_{21} = 1$ give $a_3 = D_{31}a_2 - D_{32}a_1$ and $b_3 = D_{31}b_2 - D_{32}b_1$. Substituting these into the first equalities in Equations (18) and (19), we obtain

$$D_{31}\big((D_{31} - 1)a_2 - D_{32}a_1\big) = kb_1 \quad \text{and} \quad D_{31}\big((D_{31} - 1)b_2 - D_{32}b_1\big) = ma_1,$$

which yield

$$a_2 = \frac{kb_1 + D_{31}D_{32}a_1}{D_{31}(D_{31} - 1)} \quad \text{and} \quad b_2 = \frac{ma_1 + D_{31}D_{32}b_1}{D_{31}(D_{31} - 1)}.$$

Finally, substituting the above expressions into $a_2b_1 - a_1b_2 = 1$, we transform our original question into the following generalized Pell equation:

$$kb_1^2 - ma_1^2 = D_{31}(D_{31} - 1). \tag{20}$$

Next, by Lemma 1 and Equations (18) and (19), we obtain the following inequality:

$$k = \frac{D_{31}(a_3 - a_2)}{b_1} > \frac{D_{31}(a_3 - a_2)}{a_1} > \frac{D_{31}(b_3 - b_2)}{a_1} = m. \tag{21}$$

Multiplying the second equality in Equation (18) by the first equality in Equation (19), we obtain

$$b_2a_1km = D_{32}D_{31}(a_3 - a_1)(b_3 - b_2).$$

On the other hand, from the proof of Lemma 4, we obtain

$$b_2a_1(D_{32} - D_{31} + 1) = (b_3 - b_2)(a_3 - a_1).$$

Consequently, we have the relation:

$$km = D_{32}D_{31}(D_{32} - D_{31} + 1). \tag{22}$$

By Lemmas 3, 4, and 7, we can focus on $2 \le D_{31} < D_{32} \le 5$. Hence, $km = 12$, 24, 40, or 45. In many instances, Equation (20) has no integer solution by the following lemmas.

Lemma 8. *If $M \equiv 1 \pmod 4$ and $\tau \equiv 2 \pmod 4$, then the Pell-type equation $x^2 - My^2 = \tau$ has no integer solution.*

Proof. Since τ is even and M is odd, we obtain $x^2 - y^2 \equiv 0 \pmod 2$, which forces x and y to have the same parity. If both x and y are even, then $x^2 - My^2 \equiv 0 \pmod 4$, which cannot equal τ. If both x and y are odd, then $x^2 - My^2 \equiv 1 - 1 \cdot 1 \equiv 0 \pmod 4$, which also cannot equal τ. □

Lemma 9. *Suppose K, M, τ are positive integers and either (i) $-M\tau$ is not a square modulo d for some $d \mid K$ or (ii) $K\tau$ is not a square modulo d for some $d \mid M$. Then the Pell-type equation $Kx^2 - My^2 = \tau$ has no integer solution.*

Proof. We shall focus on condition (i) since the proof for condition (ii) is similar. Suppose there is an integer solution. Reducing the Pell-type equation modulo d, we obtain $-My^2 \equiv \tau \pmod d$ or $(My)^2 \equiv -M\tau \pmod d$. This contradicts $-M\tau$ being a non-square modulo d. □

Lemma 10. *Let K, M, and τ be integers. Suppose, for some prime p, the equation*

$$Kx^2 - My^2 = \tau \tag{23}$$

satisfies the conditions: (i) $p \mid K, \tau$ (say $K = pK'$ and $\tau = p\tau'$), (ii) $p \nmid M$, and (iii) $K'\tau'$ is not a square modulo p, then Equation (23) has no integer solution.

Proof. Since $p \mid K$, $p \mid \tau$, and $p \nmid M$, we must have $p \mid y$. Say $y = py'$ for some integer y'. Then Equation (23) can be rewritten as

$$pK'x^2 - Mp^2y'^2 = p\tau' \quad \text{or} \quad K'x^2 - Mpy'^2 = \tau',$$

and the lemma follows from Lemma 9 with $d = p$. □

Lemma 11. *Let K, M, and τ be integers. Suppose, for some prime p and some odd number $b > 0$, the equation*

$$Kx^2 - My^2 = \tau \tag{24}$$

satisfies the conditions: (i) $p^b \mid \tau$, (ii) $p^{b+1} \mid K$, (iii) $p \nmid M$, (iv) $p^{b+1} \nmid \tau$, then Equation (24) has no integer solution.

Proof. Suppose to the contrary that there exist integers x, y satisfying Equation (24). Since $p^b \mid \tau$ and $p^{b+1} \mid K$, there exist integers τ' and K' such that $\tau = \tau' p^b$ and $K = K' p^{b+1}$. So, Equation (24) becomes

$$K'p^{b+1}x^2 - My^2 = \tau' p^b. \tag{25}$$

Since $p \nmid M$, we obtain $p^b \mid y^2$. Suppose $y = p^r y'$ for some integer y' with $p \nmid y'$ and some positive integer r. This gives $p^b \mid p^{2r} y'^2$, which implies $2r \geq b$ or $r \geq b/2$. However, since b is odd, we must have $r \geq (b+1)/2$. Then p^{b+1} divides the left-hand side of Equation (25) but not the right-hand side, and the lemma follows. □

D_{31}	D_{32}	(k,m)	Pell Equation (20)	Solution
2	4	$(6,4)$	$6b_1^2-4a_1^2=2$	$b_1=1,\ a_1=1$
2	5	$(20,2)$	$20b_1^2-2a_1^2=2$	$b_1=1,\ a_1=3$
3	4	$(6,4)$	$6b_1^2-4a_1^2=6$	$b_1=5,\ a_1=6$
4	5	$(20,2)$	$20b_1^2-2a_1^2=12$	$b_1=1,\ a_1=2$
4	5	$(8,5)$	$8b_1^2-5a_1^2=12$	$b_1=2,\ a_1=2$

Table 1: Summary of underlined triples with solutions when $D_{21}=1$.

First, consider the case $km=45$. This can only arise from $D_{31}=3$ and $D_{32}=5$. Hence, both k and m are odd. Then Equation (20) gives $(kb_1)^2-45a_1^2=6k$, which has no integer solution by Lemma 8. The remaining cases give the Pell-type equation $kx^2-my^2=\tau$ with

$$\begin{aligned}(k,m,\tau)=&(12,1,2),(6,2,2),(4,3,2),(24,1,2),(12,2,2),(8,3,2),(6,4,2),\\&(40,1,2),(20,2,2),(10,4,2),(8,5,2),(24,1,6),(12,2,6),(8,3,6),\\&(6,4,6),(40,1,12),(20,2,12),(10,4,12),(8,5,12)\end{aligned}$$

by Equations (20), (21), and (22). If $4\mid k$, m is odd, and $4\nmid\tau$, then we can apply Lemma 9 with $d=4$ to show that the corresponding Pell-type equations have no integer solutions. For $5\mid k$ (apart from the triples $(20,2,2)$ and $(20,2,12)$), one can apply Lemma 9 with $d=5$ to show that the corresponding equations have no integer solutions. Thus, we are left with

$$\begin{aligned}(k,m,\tau)=&(6,2,2),(12,2,2),(6,4,2),(20,2,2),(12,2,6),(6,4,6),(20,2,12),(8,5,12)\\=&(3,1,1),(6,1,1),\underline{(3,2,1)},\underline{(10,1,1)},(6,1,3),\underline{(3,2,3)},\underline{(10,1,6)},\underline{(8,5,12)}\end{aligned}$$

after dividing the triples by $\gcd(k,m,\tau)$. The triples $(3,1,1)$ and $(6,1,1)$ give Pell-type equations with no integer solutions by Lemma 9 with $d=3$. The triple $(6,1,3)$ gives an equation with no integer solution by Lemma 10 with $p=3$.

We summarize the underlined triples in Table 1 together with their origins and the corresponding Pell-type equations.

4. General Pell-Machinery

In general, the skew D_{21} may not be 1. Then we cannot conclude $\gcd(b_1,b_2)=\gcd(a_1,a_2)=1$ and apply Euclid's lemma. To extend our previous argument to a general D_{21}, we introduce the following notation. Let

$$d_a:=\gcd(a_1,a_2),\ d_b:=\gcd(b_1,b_2),\ \begin{cases}a_1:=d_a\alpha_1\\a_2:=d_a\alpha_2,\end{cases}\ \begin{cases}b_1:=d_b\beta_1\\b_2:=d_b\beta_2,\end{cases}\tag{26}$$

It follows that $\gcd(\alpha_1, \alpha_2) = 1 = \gcd(\beta_1, \beta_2)$. Now, we are ready to generalize the argument in the previous section. From Equations (9) and (26), we obtain

$$\begin{cases} \alpha_2\beta_1 - \beta_2\alpha_1 = \frac{D_{21}}{d_a d_b}, \\ d_b a_3\beta_1 - d_a b_3\alpha_1 = D_{31}. \end{cases} \tag{27}$$

From Equation (10), we obtain

$$D_{32}\beta_1(a_3 - d_a\alpha_1) = D_{31}\beta_2(a_3 - d_a\alpha_2). \tag{28}$$

Equation (28) implies $\beta_1 \mid D_{31}\beta_2(a_3 - d_a\alpha_2)$ and $\beta_2 \mid D_{32}\beta_1(a_3 - d_a\alpha_1)$. Since $\gcd(\beta_1, \beta_2) = 1$, this gives $\beta_1 \mid D_{31}(a_3 - d_a\alpha_2)$ and $\beta_2 \mid D_{32}(a_3 - d_a\alpha_1)$ by Euclid's lemma. Say

$$\beta_1 k_1 = D_{31}(a_3 - d_a\alpha_2) \quad \text{and} \quad \beta_2 k_2 = D_{32}(a_3 - d_a\alpha_1).$$

Putting these equations into Equation (28), we obtain $k_1 = k_2 = k > 0$,

$$\beta_1 k = D_{31}(a_3 - d_a\alpha_2) \quad \text{and} \quad \beta_2 k = D_{32}(a_3 - d_a\alpha_1). \tag{29}$$

By a similar argument, we obtain

$$\alpha_1 m = D_{31}(b_3 - d_b\beta_2) \quad \text{and} \quad \alpha_2 m = D_{32}(b_3 - d_b\beta_1) \tag{30}$$

for some integer $m > 0$. Now, we apply Lemma 2 and obtain

$$a_3 = \frac{d_a(D_{31}\alpha_2 - D_{32}\alpha_1)}{D_{21}} \quad \text{and} \quad b_3 = \frac{d_b(D_{31}\beta_2 - D_{32}\beta_1)}{D_{21}}. \tag{31}$$

Putting Equation (31) into the first equalities in Equations (29) and (30), we obtain $d_a D_{31}[(D_{31} - D_{21})\alpha_2 - D_{32}\alpha_1] = D_{21}k\beta_1$ and $d_b D_{31}[(D_{31} - D_{21})\beta_2 - D_{32}\beta_1] = D_{21}m\alpha_1$, which yield

$$\alpha_2 = \frac{D_{21}k\beta_1 + d_a D_{31}D_{32}\alpha_1}{d_a D_{31}(D_{31} - D_{21})} \quad \text{and} \quad \beta_2 = \frac{D_{21}m\alpha_1 + d_b D_{31}D_{32}\beta_1}{d_b D_{31}(D_{31} - D_{21})}. \tag{32}$$

Finally, substituting Equation (32) into the first equality in Equation (27), we obtain the Pell-type equation:

$$d_b k\beta_1^2 - d_a m\alpha_1^2 = D_{31}(D_{31} - D_{21}). \tag{33}$$

Multiplying the second equality in Equation (29) with the first equality in Equation (30), we obtain

$$\beta_2\alpha_1 km = D_{32}D_{31}(a_3 - d_a\alpha_1)(b_3 - d_b\beta_2).$$

On the other hand, from the proof of Lemma 4, we can deduce that

$$d_a d_b\beta_2\alpha_1(D_{32} - D_{31} + D_{21}) = D_{21}(b_3 - d_b\beta_2)(a_3 - d_a\alpha_1).$$

Consequently, we obtain the relation:

$$km = \frac{d_a d_b}{D_{21}} D_{32}D_{31}(D_{32} + D_{21} - D_{31}). \tag{34}$$

By Lemma 1, we arrive at an inequality similar to Equation (21):

$$k = \frac{d_b D_{31}(a_3 - d_a\alpha_2)}{d_b\beta_1} > \frac{d_b D_{31}(a_3 - d_a\alpha_2)}{a_1} > \frac{d_b D_{31}(b_3 - d_b\beta_2)}{d_a\alpha_1} = \frac{d_b}{d_a} m. \tag{35}$$

5. A Formula for the Ratio R_4

Our ultimate goal is to estimate the ratio $R_4 = A/a_3^3$. To achieve this, we express everything in terms of α_1 (as it appears in Equation (33)).

Lemma 12. *For $1 \le D_{21}, D_{31}, D_{32} \le 5$, we have the formula*

$$R_4 = \frac{m(D_{31}-D_{21})\big(1+\frac{D_{21}}{D_{31}D_{32}}\sqrt{\frac{D_{32}D_{31}(D_{32}+D_{21}-D_{31})}{D_{21}}}\big)}{d_aD_{31}D_{32}^2\big(1+\frac{1}{D_{32}}\sqrt{\frac{D_{32}D_{31}(D_{32}+D_{21}-D_{31})}{D_{21}}}\big)^2}+O\Big(\frac{1}{\alpha_1^2}\Big).$$

Moreover, when $m = 1$, $2 \le D_{21} < D_{31} \le 5$, and $D_{32} + D_{21} > D_{31}$, one has $R_4 \le 0.04 + O(1/\alpha_1^2)$.

Proof. Since $d_ad_b = \gcd(a_1,a_2)\gcd(b_1,b_2) \mid a_2b_1 - b_2a_1 = D_{21} \le 5$, we obtain $1 \le d_ad_b \le 5$. Combining this with Equation (34) and Lemmas 3 and 4, it follows that $1 \le km \le 5\cdot 5\cdot 5\cdot 4 = 500$. From the Pell-type Equation (33), we obtain

$$\Big(\beta_1 - \sqrt{\frac{d_am}{d_bk}}\alpha_1\Big)\Big(\beta_1 + \sqrt{\frac{d_am}{d_bk}}\alpha_1\Big) = \beta_1^2 - \frac{d_am}{d_bk}\alpha_1^2 = O(1),$$

which implies

$$\frac{\beta_1}{\alpha_1} - \sqrt{\frac{d_am}{d_bk}} = O\Big(\frac{1}{\sqrt{(d_am)/(d_bk)}\alpha_1^2}\Big) = O\Big(\frac{1}{\alpha_1^2}\Big) \tag{36}$$

since $\frac{d_bk}{d_am} \le \frac{5\cdot 500}{1\cdot 1}$. Next, we find formulas for a_3, A, and R_4. Substituting Equation (32) into Equation (31), we obtain

$$a_3 = \frac{k\beta_1 + d_aD_{32}\alpha_1}{D_{31}-D_{21}}. \tag{37}$$

By Equations (10), (26), and (37), we obtain $A = \frac{a_2a_1(b_2-b_1)}{D_{21}} = \frac{d_a^2d_b\alpha_2\alpha_1(\beta_2-\beta_1)}{D_{21}}$. This gives

$$R_4 = \frac{A}{a_3^3} = \frac{(D_{31}-D_{21})^3d_a^2d_b\alpha_2\alpha_1(\beta_2-\beta_1)}{D_{21}(k\beta_1+d_aD_{32}\alpha_1)^3}. \tag{38}$$

From Equation (32), it follows that $\beta_2 - \beta_1 = \frac{D_{21}m\alpha_1 + d_bD_{31}(D_{32}+D_{21}-D_{31})\beta_1}{d_bD_{31}(D_{31}-D_{21})}$. Substituting this and Equation (32) into Equation (38), we obtain

$$R_4 = \frac{(D_{31}-D_{21})d_a\big(D_{21}k\frac{\beta_1}{\alpha_1} + d_aD_{31}D_{32}\big)\big(D_{21}m + d_bD_{31}(D_{32}+D_{21}-D_{31})\frac{\beta_1}{\alpha_1}\big)}{D_{21}D_{31}^2(k\frac{\beta_1}{\alpha_1}+d_aD_{32})^3}.$$

Finally, we apply Equation (36) to the above equation and obtain

$$\begin{aligned}R_4 =& \frac{(D_{31}-D_{21})d_a^2\big(D_{21}\sqrt{\frac{km}{d_ad_b}}+D_{31}D_{32}+O(\frac{1}{\alpha_1^2})\big)}{D_{21}D_{31}^2d_a^3\big(\sqrt{\frac{km}{d_ad_b}}+D_{32}+O(\frac{1}{\alpha_1^2})\big)^3}\\ &\cdot\Big(D_{21}m+D_{31}(D_{32}+D_{21}-D_{31})\sqrt{\frac{d_ad_bm}{k}}+O\Big(\frac{1}{\alpha_1^2}\Big)\Big)\\ =& \frac{(D_{31}-D_{21})D_{31}D_{32}\big(1+\frac{D_{21}}{D_{31}D_{32}}\sqrt{\frac{km}{d_ad_b}}\big)\big(1+O(\frac{1}{\alpha_1^2})\big)}{d_aD_{21}D_{31}^2D_{32}^3\Big(1+\frac{1}{D_{32}}\sqrt{\frac{km}{d_ad_b}}\Big)^3\big(1+O(\frac{1}{\alpha_1^2})\big)^3}\\ &\cdot D_{21}m\Big(1+\frac{D_{31}(D_{32}+D_{21}-D_{31})}{D_{21}}\sqrt{\frac{d_ad_b}{km}}\Big)\Big(1+O\Big(\frac{1}{\alpha_1^2}\Big)\Big).\end{aligned}$$

This gives the first half of the lemma since $\frac{1}{D_{32}}\sqrt{\frac{km}{d_ad_b}}=\frac{D_{31}(D_{32}+D_{21}-D_{31})}{D_{21}}\sqrt{\frac{d_ad_b}{km}}$ and $\frac{km}{d_ad_b}=\frac{D_{32}D_{31}(D_{32}+D_{21}-D_{31})}{D_{21}}$ by Equation (34).

A computer check shows that the ratio

$$\frac{(D_{31}-D_{21})\big(1+\frac{D_{21}}{D_{31}D_{32}}\sqrt{\frac{D_{32}D_{31}(D_{32}+D_{21}-D_{31})}{D_{21}}}\big)}{D_{31}D_{32}^2\big(1+\frac{1}{D_{32}}\sqrt{\frac{D_{32}D_{31}(D_{32}+D_{21}-D_{31})}{D_{21}}}\big)^2}$$

is at most 0.04 when $2\le D_{21}<D_{31}\le 5$ and $D_{32}+D_{21}>D_{31}$. The second half of the lemma follows since $m=1$ and $d_a\ge 1$. □

6. The Situation When $D_{21}=2$

Suppose $D_{21}=2$. We may restrict our attention to $3\le D_{31}\le 5$ by Lemmas 3 and 5. Also, Lemma 4 tells us that $D_{32}\ge D_{31}-1$. From Equations (9), (26), (34), and (33), we obtain

$$\begin{cases}\alpha_2\beta_1-\beta_2\alpha_1=\frac{2}{d_ad_b},\\ d_ba_3\beta_1-d_ab_3\alpha_1=D_{31},\end{cases}\tag{39}$$

$$km=\frac{d_ad_b}{2}D_{32}D_{31}(D_{32}-D_{31}+2),\tag{40}$$

and

$$d_bk\beta_1^2-d_am\alpha_1^2=D_{31}(D_{31}-2).\tag{41}$$

We focus on $3\le D_{31}\le 5$, $D_{32}\ge D_{31}-1$, and $D_{31}\ne D_{32}$ by Lemmas 3, 4, and 7. Note that $d_ad_b\mid 2$ since the left-hand side of Equation (39) is an integer. Moreover, we can drop the cases where d_bk and d_am are both even and D_{31} is odd, since Equation (41) cannot have an integer solution by parity considerations.

(d_a, d_b)	(k, m) since $k > \frac{d_b}{d_a} m$	Pell Equation (41)	Solution?
$(1,1)$	$(3,1)$	$3\beta_1^2 - \alpha_1^2 = 3$	$\beta_1 = 2, \alpha_1 = 3$
$(2,1)$	$(6,1)$	$6\beta_1^2 - 2\alpha_1^2 = 3$	No, by parity
	$\underline{\underline{(3,2)}}$	$3\beta_1^2 - 4\alpha_1^2 = 3$	$\beta_1 = 7$, $\alpha_1 = 6$
	$(2,3)$	$2\beta_1^2 - 6\alpha_1^2 = 3$	No, by parity
$(1,2)$	$(6,1)$	$12\beta_1^2 - \alpha_1^2 = 3$	$\beta_1 = 1$, $\alpha_1 = 3$

Table 2: All possibilities satisfy Equations (35) and (40) when $D_{31} = 3$, $D_{32} = 2$.

Also, we can restrict to $m > 1$ in view of Lemma 12. For example, when $D_{31} = 3$ and $D_{32} = 2$, Equation (40) gives $km = 3d_a d_b$. We include all possible Pell-type equations in Table 2 for illustrative purposes (but we shall omit entries involving $m = 1$ later).

First, we focus on the situation $D_{31} = 4$. If $D_{32} = 3$, we obtain $km = 6d_a d_b$ by Equation (40) and the following possibilities satisfying Equation (35) and $m > 1$:

$$(d_a, d_b, k, m) = (1,1,3,2), \underline{\underline{(2,1,6,2)}}, (2,1,4,3), (2,1,3,4), (1,2,6,2).$$

By Lemma 9 with $d = 3$, the above non-underlined tuples give Pell-type equations with no integer solutions. If $D_{32} = 5$, we obtain $km = 30d_a d_b$ by Equation (40) and the following possibilities satisfy Equation (35) and $m > 1$:

$$\begin{aligned}&(d_a, d_b, k, m)\\ &=(1,1,15,2), (1,1,10,3), (1,1,6,5), \underline{(2,1,30,2)}, \underline{(2,1,20,3)}, (2,1,15,4), \underline{\underline{(2,1,12,5)}},\\ &\quad (2,1,10,6), (2,1,6,10), (1,2,30,2), (1,2,20,3), \underline{(1,2,15,4)}, \underline{(1,2,12,5)}.\end{aligned}$$

By Lemma 9 with $d = 3$, the above non-underlined tuples give Pell-type equations with no integer solutions. As for the underlined tuples, all but $(2,1,12,5)$ yield equations with no integer solutions by Lemma 9 with $d = 5$.

Next, we treat the case $D_{31} = 5$. Then $D_{32} = 4$ and $km = 10d_a d_b$ by Equation (40). We have $(d_a, d_b, k, m) = (1,1,5,2), (2,1,5,4)$ as the only possibilities satisfying Equation (35) and $m > 1$. By Lemma 10 with $p = 5$, the corresponding Pell-type equations have no integer solutions.

Finally, we consider the case $D_{31} = 3$. The case $D_{32} = 2$ has already been treated in Table 2. If $D_{32} = 4$, then $km = 18d_a d_b$ by Equation (40). We have

$$(d_a, d_b, k, m) = (1,1,9,2), \underline{\underline{(1,1,6,3)}}, (2,1,9,4), \underline{(1,2,12,3)}$$

as the only possibilities satisfying Equation (35) and $m > 1$. The non-underlined tuples give Pell-type equations with no integer solutions by Lemma 11 with $p = 3$ and $b = 1$. The last underlined tuple yields the equation $24\beta_1^2 - 3\alpha_1^2 = 3$, which has no integer solution by Lemma 9 with $d = 4$.

If $D_{32} = 5$, then $km = 30d_a d_b$ by Equation (40). We have

$$(d_a, d_b, k, m) = (1,1,15,2), (1,1,10,3), (1,1,6,5), (2,1,15,4), (1,2,20,3), (1,2,12,5)$$

D_{31}	D_{32}	(d_a, d_b)	(k, m)	Pell Equation (41)	Solution
3	2	$(2,1)$	$(3,2)$	$3\beta_1^2 - 4\alpha_1^2 = 3$	$\beta_1 = 7,\ \alpha_1 = 6$
3	4	$(1,1)$	$(6,3)$	$6\beta_1^2 - 3\alpha_1^2 = 3$	$\beta_1 = 1,\ \alpha_1 = 1$
4	3	$(2,1)$	$(6,2)$	$6\beta_1^2 - 4\alpha_1^2 = 8$	$\beta_1 = 2,\ \alpha_1 = 2$
4	5	$(2,1)$	$(12,5)$	$12\beta_1^2 - 10\alpha_1^2 = 8$	$\beta_1 = 2,\ \alpha_1 = 2$

Table 3: Summary of double underlined tuples with solutions when $D_{21} = 2$.

as the only possibilities satisfying Equation (35) and $m > 1$. All the above tuples give Pell-type equations with no integer solutions by Lemma 10 with $p = 3$.

We summarize those equations with integer solutions in Table 3.

7. The Situation When $D_{21} = 3$

Suppose $D_{21} = 3$. We can restrict our attention to $4 \le D_{31} \le 5$ by Lemmas 3 and 5. Also, Lemmas 4 and 7 tell us that $D_{32} \ge D_{31} - 2$ and $D_{31} \ne D_{32}$. Then, Equations (34) and (33) give

$$km = \frac{d_a d_b}{3} D_{32} D_{31} (D_{32} - D_{31} + 3), \tag{42}$$

and

$$d_b k \beta_1^2 - d_a m \alpha_1^2 = D_{31}(D_{31} - 3). \tag{43}$$

Note that $d_a d_b \mid 3$ by Equation (27), and we obtain the following five cases. We can also restrict to $m > 1$ in view of Lemma 12.

First, we treat the case $D_{31} = 5$. If $D_{32} = 3$, then $km = 5 d_a d_b = 5$ or 15. Hence, k, m, d_a, and d_b are all odd. Then Equation (43) gives $(d_b k \beta_1)^2 - 5(d_a d_b \alpha_1)^2 = 6 d_b k$, which has no integer solution by Lemma 8. If $D_{32} = 4$, we obtain $R_4 < 0.045$ by Lemma 6.

It remains to consider $D_{31} = 4$. If $D_{32} = 2$, then $km = \frac{8 d_a d_b}{3}$ by Equation (42). Since $d_a = d_b = 1$ is impossible, we obtain $(d_a, d_b, k, m) = \underline{\underline{(3,1,4,2)}}, (3,1,2,4)$ as the only possibilities satisfying Equation (35) and $m > 1$. The non-underlined tuple yields the equation $2\beta_1^2 - 12\alpha_1^2 = 4$, which has no integer solution by Lemma 9 with $d = 3$.

If $D_{32} = 3$, then $km = 8 d_a d_b$. We have

$$(d_a, d_b, k, m) = \underline{\underline{(1,1,4,2)}}, (3,1,12,2), (3,1,8,3), (3,1,6,4), \underline{\underline{(3,1,4,6)}}, (3,1,3,8), \underline{\underline{(1,3,12,2)}}$$

as the only possibilities satisfying Equation (35) and $m > 1$. Considering the equations modulo 3, the non-underlined tuples yield Pell-type equations with no integer solutions since $D_{31}(D_{31} - 3) = 4 \not\equiv 0 \pmod 3$.

D_{31}	D_{32}	(d_a, d_b)	(k, m)	Pell Equation (43)	Solution
4	2	$(3,1)$	$(4,2)$	$4\beta_1^2 - 6\alpha_1^2 = 4$	$\beta_1 = 5,\ \alpha_1 = 4$
4	3	$(1,1)$	$(4,2)$	$4\beta_1^2 - 2\alpha_1^2 = 4$	$\beta_1 = 3,\ \alpha_1 = 4$
4	3	$(3,1)$	$(4,6)$	$4\beta_1^2 - 18\alpha_1^2 = 8$	$\beta_1 = 17,\ \alpha_1 = 8$
4	3	$(1,3)$	$(12,2)$	$36\beta_1^2 - 2\alpha_1^2 = 8$	$\beta_1 = 17,\ \alpha_1 = 4$
4	5	$(3,1)$	$(16,5)$	$16\beta_1^2 - 15\alpha_1^2 = 4$	$\beta_1 = 2,\ \alpha_1 = 2$

Table 4: Summary of double underlined tuples with solutions when $D_{21} = 3$.

If $D_{32} = 5$, then $km = \frac{80}{3} d_a d_b$ and $d_a = d_b = 1$ is impossible. We have

$$(d_a, d_b, k, m) = \underline{(3,1,40,2)}, (3,1,20,4), \underline{\underline{(3,1,16,5)}}, \underline{(3,1,10,8)}, \underline{(1,3,40,2)}, (1,3,20,4), (1,3,16,5)$$

as the only possibilities satisfying Equation (35) and $m > 1$. The first two non-underlined tuples yield Pell-type equations with no integer solutions by Lemma 9 with $d = 3$. The last non-underlined tuple yields an equation with no integer solution by Lemma 9 with $d = 5$. All the underlined tuples except $(3,1,16,5)$ yield equations with no integer solutions by considering them modulo 8, since squares are congruent to $0, 1$, or $4 \pmod 8$.

We summarize those equations with integer solutions in Table 4.

8. The Situation When $D_{21} = 4$

Suppose $D_{21} = 4$. We may restrict our attention to $D_{31} = 5$ by Lemmas 3 and 5. Also, Lemmas 4 and 7 tell us that $D_{32} \geq D_{31} - 3$ and $D_{31} \neq D_{32}$. Then, Equations (34) and (33) give

$$\begin{cases} \alpha_2\beta_1 - \beta_2\alpha_1 = \frac{4}{d_a d_b}, \\ d_b a_3 \beta_1 - d_a b_3 \alpha_1 = D_{31}, \end{cases} \tag{44}$$

$$km = \frac{d_a d_b}{4} D_{32} D_{31} (D_{32} - D_{31} + 4),$$

and

$$d_b k \beta_1^2 - d_a m \alpha_1^2 = D_{31}(D_{31} - 4). \tag{45}$$

Note that $d_a d_b \mid 4$ by Equation (44), and we obtain the following three cases. Again, we can drop those situations when $d_b k$ and $d_a m$ are both even and D_{31} is odd since Equation (45) has no integer solution by parity considerations. Again, we restrict to $m > 1$ in view of Lemma 12.

If $D_{32} = 2$, then $km = \frac{5}{2} d_a d_b$, and $d_a = d_b = 1$ is impossible. We have $(d_a, d_b, k, m) = \underline{(4,1,5,2)}$ as the only possibility satisfying Equation (35) and $m > 1$. It yields a Pell-type equation with integer solution.

D_{31}	D_{32}	(d_a, d_b)	(k, m)	Pell Equation (45)	Solution
5	2	$(4, 1)$	$(5, 2)$	$5\beta_1^2 - 8\alpha_1^2 = 5$	$\beta_1 = 19$, $\alpha_1 = 15$
5	3	$(2, 1)$	$(5, 3)$	$5\beta_1^2 - 6\alpha_1^2 = 5$	$\beta_1 = 11$, $\alpha_1 = 10$
5	3	$(4, 1)$	$(5, 6)$	$5\beta_1^2 - 24\alpha_1^2 = 5$	$\beta_1 = 11$, $\alpha_1 = 5$

Table 5: Summary of underlined tuples with solutions when $D_{21} = 4$.

If $D_{32} = 3$, then $km = \frac{15}{2} d_a d_b$, and $d_a = d_b = 1$ is impossible. We have

$$(d_a, d_b, k, m) = \underline{(2, 1, 5, 3)}, (2, 1, 3, 5), (4, 1, 15, 2), \underline{(4, 1, 5, 6)}$$

as the only possibilities satisfying Equation (35) and $m > 1$. By Lemma 10 with $p = 5$, the non-underlined tuples yield Pell-type equations with no integer solutions.

If $D_{32} = 4$, then $R_4 < 0.045$ by Lemma 6.

We summarize those equations with integer solutions in Table 5.

9. Proof of Theorem 2

Proof of Theorem 2. If one of D_{21}, D_{31}, D_{32} is greater than 5, then Lemma 5 with $C = a_3$ implies $R_4 < 0.042$, which is smaller than the upper bound in the theorem when n is sufficiently large. If $D_{31} = D_{32}$, then Lemma 7 implies $R_4 \leq 0.25/a_3$, which is much smaller than 0.042 when n (and, hence, a_3) is large. By Lemma 6, we obtain $R_4 < 0.045$ when $D_{31} = 5$, $D_{21} \geq 3$, and $D_{32} \geq 4$. Therefore, we can narrow our attention to $1 \leq D_{21}, D_{31}, D_{32} \leq 5$ with $D_{31} \neq D_{32}$ and omit the cases with $D_{31} = 5$, $D_{21} \geq 3$, and $D_{32} \geq 4$. Based on the previous sections, we see that the four close factorizations of n imply an integer solution to a certain Pell-type Equation (33). However, as shown in the tables in the previous sections, many selections of the parameters D_{21}, D_{31}, D_{32}, d_a, d_b, k, and m yield no integer solutions. Thus, we can focus on those Pell-type equations with integer solutions, and we summarize them in Table 6 below, using Lemma 12 to compute R_4 (ignoring the error term and the entries involving $m = 1$ when $D_{21} \geq 2$).

One can see that the largest ratio (shown in red and the only one exceeding 0.045) comes from the case when $D_{21} = 1$, $D_{31} = 3$, $D_{32} = 4$, $d_a = d_b = 1$, and $(k, m) = (6, 4)$. Putting these parameters into Lemma 12, we arrive at

$$R_4 = \frac{6 + \sqrt{6}}{9(2 + \sqrt{6})^2} + O\Big(\frac{1}{\alpha_1^2}\Big).$$

From Equation (37) and Lemma 12, we know that $a_3 = O(\alpha_1)$ since $d_a \leq 5$ and $k \leq 500$. It follows that $\sqrt{n} \leq A \leq a_2 a_1 (b_2 - b_1) < a_3^3 = O(\alpha_1^3)$ by Equation (10) and Lemma 1. This implies $\frac{1}{\alpha_1^3} = O(\frac{1}{\sqrt{n}})$ or $\frac{1}{\alpha_1^2} = O(\frac{1}{n^{1/3}})$, and Theorem 2 follows. □

D_{ij} Skews			Divisors		Parameters		Ratio $R_4 = A/a_3^3$
D_{21}	D_{31}	D_{32}	d_a	d_b	k	m	
1	2	4	1	1	6	4	0.04072067323
1	2	5	1	1	20	2	0.01272913946
1	3	4	1	1	6	4	0.04742065558
1	4	5	1	1	20	2	0.01539501058
1	4	5	1	1	8	5	0.03848752646
2	3	2	2	1	3	2	0.03774955135
2	3	4	1	1	6	3	0.02512626585
2	4	3	2	1	6	2	0.02371032779
2	4	5	2	1	12	5	0.01762424561
3	4	2	3	1	4	2	0.02036033661
3	4	3	3	1	4	6	0.02512626585
3	4	3	1	3	12	2	0.02512626585
3	4	3	1	1	4	2	0.02512626585
3	4	5	3	1	16	5	0.00715749421
4	5	2	4	1	5	2	0.01272913946
4	5	3	2	1	5	3	0.01576260533
4	5	3	4	1	5	6	0.01576260533

Table 6: All cases from Tables 1, 3, 4, 5 with solutions and their R_4 ratios.

The generalized Pell Equation (33) corresponding to the largest ratio is

$$6\beta_1^2 - 4\alpha_1^2 = 6 \quad \text{which is equivalent to} \quad \beta_1^2 - 6\Big(\frac{\alpha_1}{3}\Big)^2 = 1$$

since 3 must divide α_1 and, hence, $\frac{\alpha_1}{3}$ is an integer. By the theory of Pell equations, all integer solutions are generated from the fundamental solution: for each integer $i \geq 1$,

$$\beta_1 + \frac{\alpha_1}{3}\sqrt{6} = (5 + 2\sqrt{6})^i =: x_i + y_i\sqrt{6}. \tag{46}$$

With $b_1 = \beta_1 = x_i$ and $a_1 = \alpha_1 = 3y_i$, we can construct the following example:

$$a_1 = 3y_i,\ b_1 = x_i,\ a_2 = x_i{+}6y_i,\ b_2 = 2x_i{+}2y_i,\ a_3 = 3x_i{+}6y_i,\ b_3 = 2x_i{+}6y_i, \tag{47}$$

$$A = 3y_i(x_i + 2y_i)(x_i + 6y_i),\quad B = 2x_i(x_i + y_i)(x_i + 3y_i),\quad \text{and}\quad n = AB \tag{48}$$

using Equations (10), (26), (31), and (32). One can verify that

$$n = AB = (A + a_1)(B - b_1) = (A + a_2)(B - b_2) = (A + a_3)(B - b_3)$$

and

$$\frac{A}{a_3^3} = \frac{3y_i(x_i+2y_i)(x_i+6y_i)}{(3x_i+6y_i)^3} = \frac{y_i(x_i+6y_i)}{9(x_i+2y_i)^2} = \frac{y_i\big((6+\sqrt{6})y_i+O(\frac{1}{y_i})\big)}{9\big((2+\sqrt{6})y_i+O(\frac{1}{y_i})\big)^2}$$
$$= \frac{6+\sqrt{6}+O(\frac{1}{y_i^2})}{9\big(2+\sqrt{6}+O(\frac{1}{y_i^2})\big)^2} = \frac{6+\sqrt{6}}{9(2+\sqrt{6})^2}+O\Big(\frac{1}{\alpha_1^2}\Big) = \frac{6+\sqrt{6}}{9(2+\sqrt{6})^2}+O\Big(\frac{1}{n^{1/3}}\Big)$$

since $x_i - y_i\sqrt{6} = \frac{1}{x_i+y_i\sqrt{6}}$; hence $x_i = \sqrt{6}y_i + O(\frac{1}{y_i})$. This shows that the upper bound in Theorem 2 is best possible.

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