

AN APPROXIMATE FUNCTIONAL EQUATION FOR ELEMENTS OF THE SELBERG CLASS

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Abstract

We prove a smoothed approximate functional equation for functions in the Selberg class, in the spirit of the approximate functional equation of Iwaniec and Kowalski for L-functions. As an application, we show that for the Riemann zeta-function our formula contains Lavrik's approximate functional equation in the strip $0 \leq \sigma \leq 1$, with a sharper error term, and we discuss its relation with the classical approximate functional equation of Hardy and Littlewood.

1. Introduction

Let $F(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, where $s = \sigma + it$, be an L-function whose Dirichlet series converges in the half-plane $\sigma > \sigma_0$. In many situations it is desirable to express $F(s)$ in the region $\sigma \leq \sigma_0$ by finite Dirichlet series with a good error term. Such expressions are known as *approximate formulas* or *approximate functional equations* for $F(s)$. Recall that the Riemann zeta-function is given by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\sigma > 1$. A first approximation of $\zeta(s)$ is

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}),$$

uniformly for $\sigma \geq \sigma_0 > 0$ and $|t| < 2\pi x/C$, where $C > 1$ is a given constant (see [15, Theorem 4.11]). A much better approximation in the critical strip is the classical approximate functional equation of Hardy and Littlewood [3].

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Theorem 1 ([3]). *Let $0 \le \sigma \le 1$, let $x, y, t > C > 0$, and let $2\pi xy = t$. Then, uniformly in σ,*

$$\zeta(s) = \sum_{n \le x} n^{-s} + \chi(s) \sum_{n \le y} n^{s-1} + O(x^{-\sigma}) + O(t^{1/2-\sigma} y^{\sigma-1}), \tag{1}$$

where $\chi(s)$ is defined by the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$.

As an application of Equation (1), Ingham [5] proved the mean value formulas

$$\int_0^T \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^2 dt = T \log T + (2\gamma_0 - 1 - \log 2\pi)T + O(T^{1/2+\varepsilon})$$

and

$$\int_0^T \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^4 dt = \frac{1}{2\pi^2} T \log^4 T + O(T \log^3 T), \tag{2}$$

where γ_0 denotes Euler's constant. Mean value estimates of this kind, in particular Equation (2), are key ingredients in zero-density estimates for $\zeta(s)$. Let $N(\sigma, T)$ denote the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $\beta > \sigma$ and $0 < \gamma \le T$. Ingham [6] proved that

$$N(\sigma, T) \ll T^{3(1-\sigma)/(2-\sigma)} \log^5 T$$

uniformly for $1/2 \le \sigma \le 1$, and Huxley [4] proved that

$$N(\sigma, T) \ll T^{3(1-\sigma)/(3\sigma-1)} \log^{44} T$$

uniformly for $3/4 \le \sigma \le 1$.

Approximate functional equations have been established for many of the classical L-functions of number theory. Titchmarsh [14] proved that if $x, y, t > C > 0$ and $4\pi^2 xy = t^2$, then, uniformly for $0 \le \sigma \le 1$,

$$\zeta^2(s) = \sum_{n \le x} d(n) n^{-s} + \chi^2(s) \sum_{n \le y} d(n) n^{s-1} + O(x^{1/2-\sigma} \log t).$$

For $\zeta^k(s)$ the corresponding formulas are discussed by Ivić [7, Chapter 4]. Lavrik [10] proved approximate functional equations for Dirichlet L-functions which played a crucial role in the subsequent development of the subject; see also Suetuna [12] and Tatuzawa [13]. Potter [11] derived an approximate functional equation for Epstein zeta-functions, Wilton [16] for the product of two zeta-functions, and Apostol and Sklar [1] for Hecke's Dirichlet series. Harcos [2] established a uniform approximate functional equation for principal L-functions on GL_m.

A general smoothed approximate functional equation is given by Iwaniec and Kowalski [8, Theorem 5.3]. In their notation, let $L(f, s) = \sum_n \lambda_f(n) n^{-s}$ be an L-function of conductor q, with gamma factor $\gamma(f, s)$, completed L-function $\Lambda(f, s) = q^{s/2}\gamma(f, s)L(f, s)$, and root number $\epsilon(f)$.

Theorem 2 ([8, Theorem 5.3]). *Let $G(u)$ be any function which is holomorphic and bounded in the strip $-4 < \Re u < 4$, even, and normalized by $G(0) = 1$. Let $X > 0$. Then for s in the strip $0 \le \sigma \le 1$ we have*

$$L(f,s) = \sum_n \frac{\lambda_f(n)}{n^s} V_s\Big(\frac{n}{X\sqrt{q}}\Big) + \epsilon(f,s) \sum_n \frac{\overline{\lambda_f(n)}}{n^{1-s}} V_{1-s}\Big(\frac{nX}{\sqrt{q}}\Big) + R,$$

where

$$V_s(y) = \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \frac{\gamma(f,s+u)}{\gamma(f,s)} \frac{du}{u}, \qquad \epsilon(f,s) = \epsilon(f) q^{\frac{1}{2}-s} \frac{\gamma(f,1-s)}{\gamma(f,s)},$$

and $R = 0$ if $\Lambda(f,s)$ is entire. Otherwise, R is given by the residues of $X^u \Lambda(f,s+u) G(u) u^{-1} / q^{s/2} \gamma(f,s)$ at $u = 1-s$ and $u = -s$.

In this paper we extend Theorem 2 of Iwaniec and Kowalski to all functions in the Selberg class (defined in Section 2). Our proof follows the method that Iwaniec and Kowalski use to prove Theorem 2. The main result is Theorem 3 in Section 2. In Section 3 we collect some preliminary lemmas, and in Section 4 we prove Theorem 3. In Section 5 we specialize to the Riemann zeta-function; we show that Lavrik's approximate functional equation for $\zeta(s)$ follows from Theorem 3 in the strip $0 \le \sigma \le 1$, with a sharper error term, and we discuss the relation with Theorem 1.

2. The Selberg Class and the Main Result

We give a brief description of the Selberg class; we refer the reader to [9] for an excellent survey. The Selberg class $\mathbb{S}$ consists of functions F satisfying the following axioms.

(i) (Dirichlet series) The Dirichlet series

$$F(s) = \sum_{n \ge 1} \frac{a(n)}{n^s}$$

converges absolutely for $\sigma > 1$.

(ii) (Analytic continuation) There is an integer $m \ge 0$ such that $(s-1)^m F(s)$ is an entire function of finite order.

(iii) (Functional equation) The function F satisfies a functional equation of the form

$$\Phi(s) = \omega \overline{\Phi(1-\overline{s})},$$

where ω is a complex number with $|\omega| = 1$ and

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s).$$

Here, we assume $Q > 0$, $r \geq 0$, $\lambda_j > 0$, and $\Re(\mu_j) \geq 0$.

(iv) (Ramanujan hypothesis) For every $\varepsilon > 0$ we have $a(n) \ll_\varepsilon n^\varepsilon$.

(v) (Euler product) For $\sigma > 1$ we have

$$\log F(s) = \sum_{n \geq 1} \frac{b(n)}{n^s},$$

where $b(n) = 0$ unless $n = p^\ell$ for some prime p and some integer $\ell \geq 1$, and $b(n) \ll n^\theta$ for some $\theta < 1/2$.

Examples of elements of $\mathbb{S}$ are the Riemann zeta-function, Dirichlet L-functions attached to primitive characters, Dedekind zeta-functions and Hecke L-functions of number fields, and the normalized L-functions of holomorphic newforms. Artin L-functions belong to $\mathbb{S}$ provided Artin's conjecture holds for them, and more generally the L-functions of cuspidal automorphic representations are expected to belong to $\mathbb{S}$.

Throughout the paper we write

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j), \qquad \Phi(s) = \gamma(s) F(s),$$

and $\mu_j = \alpha_j + i\beta_j$ with α_j, β_j real. For a function H we write $\bar{H}(w) = \overline{H(\overline{w})}$. Thus,

$$\bar{F}(w) = \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^w}, \qquad \bar{\gamma}(w) = Q^w \prod_{j=1}^{r} \Gamma(\lambda_j w + \bar{\mu}_j),$$

and the functional equation (iii) reads $\Phi(w) = \omega \bar{\Phi}(1 - w)$. We denote by m the order of the pole of F at $w = 1$, so that $m = 0$ if F is entire. All implied constants may depend on F and G.

The main result of this paper is the following.

Theorem 3. *Let $F(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ be a function in $\mathbb{S}$. Let $G(u)$ be any function which is holomorphic and bounded in the strip $-4 < \Re u < 4$, even, and normalized by $G(0) = 1$. Let $X > 0$. Let s be in the strip $0 \leq \sigma \leq 1$, with $s \neq 0, 1$ and s not a pole of γ (the last condition is automatic if $\sigma > 0$). Then*

$$F(s) = \frac{1}{\gamma(s)} \sum_{n=1}^{\infty} \frac{a(n)}{n^s} V_s\Big(\gamma, \frac{n}{X}\Big) + \frac{\omega}{\gamma(s)} \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^{1-s}} V_{1-s}(\bar{\gamma}, nX) - R(s), \qquad (3)$$

where, for $y > 0$,

$$V_s(\gamma, y) = \frac{1}{2\pi i} \int_{(3)} y^{-u} \gamma(s+u) G(u) \frac{du}{u},$$

and

$$R(s) = \frac{1}{\gamma(s)}\Big(\mathrm{Res}_{u=1-s} + \mathrm{Res}_{u=-s}\Big) X^u \Phi(s+u)\frac{G(u)}{u}. \tag{4}$$

In particular, $R(s) = 0$ *if* F *is entire. If* $m = 1$ *and* $\kappa = \mathrm{Res}_{w=1}F(w)$*, then*

$$R(s) = \frac{1}{\gamma(s)}\Big(\kappa\gamma(1)\frac{X^{1-s}G(1-s)}{1-s} + \omega\,\overline{\kappa\gamma(1)}\,\frac{X^{-s}G(s)}{s}\Big). \tag{5}$$

Remark 1. The residual term enters Equation (3) with a minus sign because $R(s)$ is defined in Equation (4) as the sum of the residues. Both residues, at $u = 1-s$ and at $u = -s$, have to be taken into account: by Lemma 1 below, Φ has a pole at $w = 0$ whenever F has a pole at $w = 1$. For $m \geq 2$ the residues can be computed from the Laurent expansions of Φ at $w = 1$ and at $w = 0$; they involve the derivatives of $X^u G(u)u^{-1}$ up to order $m-1$.

3. Preliminary Lemmas

We first determine the poles of the completed function Φ.

Lemma 1. *Let* $F \in \mathbb{S}$*. Then* $\Phi(w) = \gamma(w)F(w)$ *is meromorphic in* $\mathbb{C}$*, and it is holomorphic except possibly for poles at* $w = 1$ *and* $w = 0$*, each of order at most* m*. If* $m = 1$ *and* $\kappa = \mathrm{Res}_{w=1}F(w)$*, then*

$$\mathrm{Res}_{w=1}\Phi(w) = \kappa\gamma(1), \qquad \mathrm{Res}_{w=0}\Phi(w) = -\omega\,\overline{\kappa\gamma(1)}. \tag{6}$$

Proof. If $\Re w > 0$, then $\Re(\lambda_j w + \mu_j) > 0$ for every j, since $\lambda_j > 0$ and $\Re\mu_j \geq 0$. Hence, γ is holomorphic in $\Re w > 0$, and by axiom (ii) Φ is holomorphic there except possibly at $w = 1$, where it has a pole of order at most m. If $\Re w < 1$, then $\Re(1-\bar{w}) > 0$, and the functional equation $\Phi(w) = \omega\overline{\Phi(1-\bar{w})}$ shows that Φ is holomorphic there except possibly at $w = 0$, where it has a pole of order at most m. The two half-planes cover $\mathbb{C}$. If $m = 1$, then $\Phi(w) = \kappa\gamma(1)(w-1)^{-1} + O(1)$ near $w = 1$, and therefore, near $w = 0$,

$$\Phi(w) = \omega\overline{\Phi(1-\bar{w})} = \omega\,\overline{\Big(\frac{\kappa\gamma(1)}{-\bar{w}} + O(1)\Big)} = -\frac{\omega\,\overline{\kappa\gamma(1)}}{w} + O(1).$$

□

Next we need bounds for γ and F in vertical strips. We use Stirling's formula in the form

$$|\Gamma(x+iy)| = \sqrt{2\pi}\,|y|^{x-1/2}e^{-\pi|y|/2}\big(1 + O(|y|^{-1})\big), \tag{7}$$

which holds uniformly for x in a compact set and $|y| \geq 1/4$; in particular, $|\Gamma(x+iy)| \asymp |y|^{x-1/2}e^{-\pi|y|/2}$ in this range.

Lemma 2. *Let $F \in \mathbb{S}$ and let $a < b$ be fixed. Put $\Lambda = \lambda_1 + \cdots + \lambda_r$ and $T_0 = 1 + \max_j (1 + |\beta_j|)/\lambda_j$. There is a constant $c = c(F, a, b)$ such that, for $a \le \Re w \le b$ and $|\Im w| \ge T_0$,*

$$\gamma(w) \ll |\Im w|^c \exp\Big(-\frac{\pi}{2}\Lambda |\Im w| \Big) \tag{8}$$

and

$$F(w) \ll |\Im w|^c. \tag{9}$$

Consequently, $\Phi(w) \ll |\Im w|^{2c} \exp(-\frac{\pi}{2}\Lambda|\Im w|)$ in the same range.

Proof. Write $w = x + iy$. For $|y| \ge T_0$ we have $|\Im(\lambda_j w + \mu_j)| = |\lambda_j y + \beta_j| \ge \lambda_j |y| - |\beta_j| \ge 1$, and $\Re(\lambda_j w + \mu_j) = \lambda_j x + \alpha_j$ lies in a compact set. Since $\big||\lambda_j y + \beta_j| - \lambda_j |y|\big| \le |\beta_j|$, Equation (7) gives

$$|\Gamma(\lambda_j w + \mu_j)| \ll |y|^{c_j} e^{-\pi \lambda_j |y|/2}$$

for some constant c_j, and Equation (8) follows on taking the product over j, since $|Q^w| = Q^x$ is bounded.

For the proof of Equation (9), let $\delta = 1/2$. For $\Re w \ge 1 + \delta$, the Dirichlet series converges absolutely, so $F(w) \ll 1$. For $\Re w \le -\delta$, the functional equation gives

$$F(w) = \omega \, \frac{\bar\gamma(1 - w)}{\gamma(w)} \, \bar F(1 - w),$$

where $\bar F(1 - w) \ll 1$ since $\Re(1 - w) \ge 1 + \delta$. By Equation (7) applied to the numerator and the denominator, the exponential factors cancel up to a bounded factor, and the quotient $\bar\gamma(1-w)/\gamma(w)$ is bounded by a power of $|y|$ for $a \le \Re w \le -\delta$ and $|y| \ge T_0$. Hence, $F(w)$ is bounded by a power of $|y|$ on the lines $\Re w = -\delta$ and $\Re w = 1+\delta$. By axiom (ii), $(w-1)^m F(w)$ is entire of finite order, so the Phragmén–Lindelöf principle applied to $(w - 1)^m F(w)$ in the strip $-\delta \le \Re w \le 1 + \delta$ shows that $F(w)$ is bounded by a power of $|y|$ in this strip for $|y| \ge T_0$. Combining the three ranges gives Equation (9); see also [9]. □

4. Proof of Theorem 3

We are now ready to prove our main theorem.

Proof of Theorem 3. Consider the integral

$$I(X, s) := \frac{1}{2\pi i} \int_{(3)} X^u \Phi(s + u) G(u) \frac{du}{u} = \frac{1}{2\pi i} \int_{(3)} X^u \gamma(s + u) F(s + u) G(u) \frac{du}{u}.$$

On the line $\Re u = 3$ we have $\Re(s + u) \ge 3$, so the Dirichlet series of $F(s + u)$ converges absolutely and uniformly. Since $\sum_n |a(n)| n^{-\sigma-3} < \infty$ and, by Lemma 2,

$\int_{(3)} |\gamma(s+u)G(u)u^{-1}|\,|du| < \infty$, we may interchange summation and integration. This gives

$$I(X,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \cdot \frac{1}{2\pi i} \int_{(3)} \Big(\frac{n}{X}\Big)^{-u} \gamma(s+u)G(u)\frac{du}{u} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} V_s\Big(\gamma, \frac{n}{X}\Big). \quad (10)$$

Now we move the line of integration to $\Re u = -3$, applying Cauchy's theorem to the rectangle with vertices $\pm 3 \pm iT$, where T is large. By Lemma 1, the poles of the integrand inside this rectangle can only be at $u = 0$, $u = 1-s$, and $u = -s$; these three points are distinct because $s \neq 0, 1$, and they lie in $-1 \leq \Re u \leq 1$. No poles lie on the vertical sides, since there $\Re(s+u) \in [-3,-2] \cup [3,4]$. Since $G(0) = 1$, the residue at $u = 0$ is $\Phi(s)$, and by Equation (4) the residues at $u = 1-s$ and $u = -s$ add up to $\gamma(s)R(s)$. On the horizontal sides $\Im u = \pm T$ the integrand is bounded by

$$\max(X^3, X^{-3})\Big(\sup_{|\Re u|<4} |G(u)|\Big)\, T^{-1}\, |\Phi(s+u)|,$$

which tends to 0 as $T \to \infty$ by Lemma 2. Hence,

$$I(X,s) = \Phi(s) + \gamma(s)R(s) + \frac{1}{2\pi i} \int_{(-3)} X^u \Phi(s+u)G(u)\frac{du}{u}. \quad (11)$$

In the last integral we replace u by $-u$. Since G is even and the orientation of the line of integration is reversed, we obtain

$$\frac{1}{2\pi i} \int_{(-3)} X^u \Phi(s+u)G(u)\frac{du}{u} = -\frac{1}{2\pi i} \int_{(3)} X^{-u} \Phi(s-u)G(u)\frac{du}{u}.$$

By the functional equation (iii),

$$\Phi(s-u) = \omega\, \bar{\Phi}(1-s+u) = \omega\, \bar{\gamma}(1-s+u)\bar{F}(1-s+u),$$

and on the line $\Re u = 3$ we have $\Re(1-s+u) \geq 3$. Expanding $\bar{F}$ into its absolutely convergent Dirichlet series and interchanging summation and integration as before, we get

$$\begin{aligned}\frac{1}{2\pi i} \int_{(-3)} X^u \Phi(s+u)G(u)\frac{du}{u} &= -\omega \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^{1-s}} \\ &\quad \times \frac{1}{2\pi i} \int_{(3)} (nX)^{-u} \bar{\gamma}(1-s+u)G(u)\frac{du}{u} \\ &= -\omega \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^{1-s}} V_{1-s}(\bar{\gamma}, nX).\end{aligned}$$

Inserting this and Equation (10) into Equation (11), we obtain

$$\gamma(s)F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} V_s\Big(\gamma, \frac{n}{X}\Big) + \omega \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^{1-s}} V_{1-s}(\bar{\gamma}, nX) - \gamma(s)R(s).$$

Since Γ has no zeros and s is not a pole of γ, we have $0 < |\gamma(s)| < \infty$, and dividing by $\gamma(s)$ gives Equation (3). Clearly, $R(s) = 0$ if F is entire, since then Φ is entire by Lemma 1. Finally, if $m = 1$, then $X^u\Phi(s+u)G(u)/u$ has simple poles at $u = 1-s$ and $u = -s$, and Equation (5) follows from Equation (6) and $G(-s) = G(s)$. This completes the proof of Theorem 3. □

5. Applications to the Riemann Zeta-Function

In this section we specialize Theorem 3 to the Riemann zeta-function. We first write down the resulting approximate functional equation for $\zeta(s)$ together with an explicit bound for its residual term. We then show that, in the strip $0 \le \sigma \le 1$, Lavrik's approximate functional equation for $\zeta(s)$ is a special case of our formula, with a sharper error term. Finally, we estimate the weights that occur in it and compare the result with the classical approximate functional equation of Hardy and Littlewood (Theorem 1).

5.1. The Approximate Functional Equation for $\zeta(s)$

Let $F(s) = \zeta(s)$. Here, $Q = \pi^{-1/2}$, $r = 1$, $\lambda_1 = 1/2$, $\mu_1 = 0$, and $\omega = 1$, so that

$$\gamma(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big) = \bar{\gamma}(s), \qquad \gamma(s)\zeta(s) = \gamma(1-s)\zeta(1-s).$$

Moreover, $m = 1$, $\kappa = 1$, and $\gamma(1) = \pi^{-1/2}\Gamma(1/2) = 1$. We have

$$\chi(s) = \frac{\gamma(1-s)}{\gamma(s)} = \pi^{s-1/2}\frac{\Gamma\big(\frac{1-s}{2}\big)}{\Gamma\big(\frac{s}{2}\big)}.$$

Theorem 3 gives, for any $X > 0$, $0 \le \sigma \le 1$, and $s \ne 0, 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\frac{V_s(\gamma, n/X)}{\gamma(s)} + \chi(s)\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\frac{V_{1-s}(\gamma, nX)}{\gamma(1-s)} - R(s), \tag{12}$$

where

$$R(s) = \frac{1}{\gamma(s)}\Big(\frac{X^{1-s}G(1-s)}{1-s} + \frac{X^{-s}G(s)}{s}\Big).$$

Now take $G(u) = e^{u^2}$, which is entire, even, bounded in $|\Re u| < 4$, and satisfies $G(0) = 1$. We have $|G(1-s)| = e^{(1-\sigma)^2 - t^2}$ and $|G(s)| = e^{\sigma^2 - t^2}$, and by Equation (7)

$$\frac{1}{|\gamma(s)|} = \frac{\pi^{\sigma/2}}{|\Gamma(s/2)|} \ll |t|^{(1-\sigma)/2}e^{\pi|t|/4}, \qquad |t| \ge 1.$$

Since $|s|, |1-s| \ge |t|$, we obtain

$$R(s) \ll \big(X^{1-\sigma} + X^{-\sigma}\big)e^{-t^2/2}, \qquad 0 \le \sigma \le 1,\ |t| \ge 1. \tag{13}$$

Thus, for $G(u) = e^{u^2}$, the residual term $R(s)$ is extremely small as soon as $|t|$ is large. Note, however, that the weights $V_s(\gamma, n/X)/\gamma(s)$ are not compactly supported in n; they decay rapidly for large n, as we show in Lemma 3 below.

5.2. Lavrik's Approximate Functional Equation

The approximate functional equation obtained by Lavrik (see [15, Section 4.20] and [10]) is as follows:

$$\begin{aligned}\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} w_s\Big(\frac{n}{x}\Big) + \chi(s)^k \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{1-s}} w_{1-s}\Big(\frac{n}{y}\Big) \\ + O\Big(x^{1-\sigma} \log^k(2+x)\, e^{-t^{1/4}}\Big),\end{aligned} \tag{14}$$

uniformly for $t \geq 1$, $|\sigma| \leq \frac{1}{2}t$, $xy = (t/2\pi)^k$, and $x, y \gg 1$, for any fixed positive integer k. Here, for $s' = s$ or $s' = 1 - s$,

$$w_{s'}(\xi) = \frac{1}{2\pi i} \int_{(c)} \Big(\Big(\frac{t}{2}\Big)^{-z/2} \frac{\Gamma\big(\frac{1}{2}(s'+z)\big)}{\Gamma\big(\frac{1}{2}s'\big)}\Big)^k \xi^{-z} e^{z^2} \frac{dz}{z}, \tag{15}$$

where $\xi > 0$ and $c > \max(0, -\Re s')$; note that $t = \Im s$ appears in both kernels.

We show that for $k = 1$ and $0 \leq \sigma \leq 1$, Equation (14) is a consequence of Theorem 3, in fact with a sharper error term. Let $t \geq 1$, $G(z) = e^{z^2}$, and $s' \in \{s, 1-s\}$. Since

$$\frac{\gamma(s'+z)}{\gamma(s')} = \pi^{-z/2} \frac{\Gamma(\frac{1}{2}(s'+z))}{\Gamma(\frac{1}{2}s')},$$

we have

$$\frac{V_{s'}(\gamma, y)}{\gamma(s')} = \frac{1}{2\pi i} \int_{(3)} y^{-z} \pi^{-z/2} \frac{\Gamma\big(\frac{1}{2}(s'+z)\big)}{\Gamma\big(\frac{1}{2}s'\big)} e^{z^2} \frac{dz}{z}.$$

The integrand is holomorphic in $\Re z > \max(0, -\Re s')$ and decays rapidly as $|\Im z| \to \infty$, so the line $\Re z = 3$ may be replaced by $\Re z = c$ for any $c > \max(0, -\Re s')$. Since

$$\Big(\xi\sqrt{\frac{t}{2\pi}}\Big)^{-z} \pi^{-z/2} = \Big(\frac{t}{2}\Big)^{-z/2} \xi^{-z},$$

comparison with Equation (15) for $k = 1$ gives

$$\frac{1}{\gamma(s')} V_{s'}\Big(\gamma, \xi\sqrt{\frac{t}{2\pi}}\Big) = w_{s'}(\xi), \qquad \xi > 0. \tag{16}$$

Corollary 1. *Let $t \geq 1$, $0 \leq \sigma \leq 1$, and let $x, y > 0$ with $xy = t/2\pi$. Then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} w_s\Big(\frac{n}{x}\Big) + \chi(s) \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} w_{1-s}\Big(\frac{n}{y}\Big) + O\Big(\big(x^{1-\sigma} + x^{-\sigma}\big) e^{-t^2/4}\Big).$$

In particular, for $x \gg 1$ the error term is $O(x^{1-\sigma} e^{-t^2/4})$.

Proof. Apply Equation (12) with $G(u) = e^{u^2}$ and $X = x\sqrt{2\pi/t}$. Then

$$\frac{n}{X} = \frac{n}{x}\sqrt{\frac{t}{2\pi}}, \qquad nX = nx\sqrt{\frac{2\pi}{t}} = \frac{n}{y}\sqrt{\frac{t}{2\pi}},$$

because $x = (t/2\pi)/y$. By Equation (16), the two weights in Equation (12) are $w_s(n/x)$ and $w_{1-s}(n/y)$. By Equation (13),

$$R(s) \ll \Big(x^{1-\sigma}\Big(\frac{2\pi}{t}\Big)^{\frac{1-\sigma}{2}} + x^{-\sigma}\Big(\frac{t}{2\pi}\Big)^{\frac{\sigma}{2}}\Big)e^{-t^2/2} \ll \big(x^{1-\sigma} + x^{-\sigma}\big)t^{1/2}e^{-t^2/2},$$

and $t^{1/2}e^{-t^2/2} \ll e^{-t^2/4}$. If $x \geq c_0 > 0$, then $x^{-\sigma} \leq c_0^{-1}x^{1-\sigma}$. □

Thus, in the range $0 \leq \sigma \leq 1$, Lavrik's formula for $k = 1$ is a reparametrization of Equation (12) with $G(u) = e^{u^2}$, through the change of variables

$$x = X\sqrt{\frac{t}{2\pi}}, \qquad y = \frac{1}{X}\sqrt{\frac{t}{2\pi}}.$$

The factor $\chi(s)$ in front of the second sum arises from the identity $1/\gamma(s) = \chi(s)/\gamma(1-s)$, and the free parameter X of Theorem 3 corresponds to the ratio $x/y = X^2$.

Remark 2. For $k \geq 2$, the function $\zeta^k(s)$ satisfies axioms (i)–(iii) and (v), with gamma factor $\gamma(s)^k$, $\omega = 1$, and a pole of order k at $s = 1$, and it satisfies axiom (iv) because $d_k(n) \ll_\varepsilon n^\varepsilon$; hence $\zeta^k \in \mathbb{S}$. The computation leading to Equation (16) applies verbatim and identifies the weights of Theorem 3 with Lavrik's kernels in Equation (15), now with $x = X(t/2\pi)^{k/2}$ and $y = X^{-1}(t/2\pi)^{k/2}$, so that $xy = (t/2\pi)^k$. In this case $R(s)$ comes from poles of order k at $u = 1-s$ and $u = -s$, and we do not estimate it here. We also note that Lavrik's result holds in the wider range $|\sigma| \leq \frac{1}{2}t$, which is not covered by Theorem 3.

5.3. The Weights and the Classical Approximate Functional Equation

We now describe the size of the weights $w_{s'}(\xi)$, where $s' \in \{s, 1-s\}$, for $k = 1$. We write $s' = \sigma' + it'$, so that $0 \leq \sigma' \leq 1$ and $|t'| = t$.

Lemma 3. *Let $J \geq 0$ be an integer and $A = 2J + 3/2$. Uniformly for $t \geq 1$, $0 \leq \sigma \leq 1$, $s' \in \{s, 1-s\}$, and $\xi > 0$ we have*

$$w_{s'}(\xi) \ll_A \xi^{-A}, \tag{17}$$

and, for $0 < \xi \leq 1$,

$$w_{s'}(\xi) = 1 + O_A\big(\xi^A\big) + O_A\big(\xi^{\sigma'}e^{-t^2/2}\big). \tag{18}$$

Proof. Write $z = a + ib$ and

$$K(z) = \Big(\frac{t}{2}\Big)^{-z/2} \frac{\Gamma\big(\frac{1}{2}(s'+z)\big)}{\Gamma\big(\frac{1}{2}s'\big)}.$$

We first show that, for $a = \pm A$,

$$K(a+ib) \ll_A (1+|b|)^{2A+1} e^{\pi|b|}. \tag{19}$$

Since $|t'/2| = t/2 \geq 1/2$, Equation (7) gives $|\Gamma(\frac{1}{2}s')|^{-1} \asymp (t/2)^{(1-\sigma')/2} e^{\pi t/4}$. If $|b| \leq t/2$, then $|t'+b|$ lies between $t/2$ and $3t/2$, and Equation (7) gives

$$\Big|\Gamma\big(\tfrac{1}{2}(s'+z)\big)\Big| \asymp_A \Big|\frac{t'+b}{2}\Big|^{\frac{\sigma'+a-1}{2}} e^{-\pi|t'+b|/4} \ll_A \Big(\frac{t}{2}\Big)^{\frac{\sigma'+a-1}{2}} e^{-\pi t/4+\pi|b|/4},$$

so that $K(z) \ll_A e^{\pi|b|/4}$. If $|b| > t/2$, we use $|\Gamma(\frac{1}{2}(s'+z))| \ll_A (1+|b|)^A$. This follows from Equation (7) when $|t'+b| \geq 1$, and from the fact that $\frac{1}{2}(\sigma'+a)$ has distance at least $1/4$ from the poles $0, -1, -2, \ldots$ of Γ when $|t'+b| < 1$ (here, we use $a = \pm(2J+3/2)$ and $0 \leq \sigma' \leq 1$). Together with $|\Gamma(\frac{1}{2}s')|^{-1} \ll (1+|b|)^{1/2} e^{\pi t/4}$, $(t/2)^{\mp A/2} \ll (1+|b|)^{A/2}$, and $e^{\pi t/4} \leq e^{\pi|b|/2}$, this gives Equation (19).

To prove Equation (17) we move the line of integration in Equation (15) to $\Re z = A$. No poles are crossed, since the integrand is holomorphic in $\Re z > \max(0, -\sigma')$, and the horizontal integrals tend to 0 because $|e^{z^2}| = e^{a^2-b^2}$. By Equation (19),

$$w_{s'}(\xi) \ll_A \xi^{-A} \int_{-\infty}^{\infty} (1+|b|)^{2A+1} e^{\pi|b|+A^2-b^2}\, db \ll_A \xi^{-A}.$$

To prove Equation (18) we move the line of integration to $\Re z = -A$. We cross the simple pole at $z = 0$, with residue 1, and the simple poles of $\Gamma(\frac{1}{2}(s'+z))$ at $z = -s' - 2j$ with $\sigma' + 2j < A$, that is, $0 \leq j \leq J$; the line $\Re z = -A$ itself contains no pole. Since $\mathrm{Res}_{w=-j}\Gamma(w) = (-1)^j/j!$, the residue at $z = -s'-2j$ equals

$$\rho_j = \frac{2(-1)^j}{j!}\, \xi^{s'+2j} \Big(\frac{t}{2}\Big)^{\frac{s'+2j}{2}} \frac{1}{\Gamma\big(\frac{1}{2}s'\big)} \frac{e^{(s'+2j)^2}}{-(s'+2j)}.$$

Using $|e^{(s'+2j)^2}| = e^{(\sigma'+2j)^2-t^2}$, $|s'+2j| \geq t \geq 1$, and the bound for $|\Gamma(\frac{1}{2}s')|^{-1}$ above, we get

$$\rho_j \ll_A \xi^{\sigma'+2j}(1+t)^{A+1} e^{\pi t/4-t^2} \ll_A \xi^{\sigma'+2j} e^{-t^2/2} \leq \xi^{\sigma'} e^{-t^2/2}, \qquad 0 < \xi \leq 1.$$

Finally, by Equation (19) the integral over $\Re z = -A$ is

$$\ll_A \xi^A \int_{-\infty}^{\infty} (1+|b|)^{2A+1} e^{\pi|b|+A^2-b^2}\, db \ll_A \xi^A.$$

This proves Equation (18). □

By Lemma 3, for any L tending to infinity with t, the weight $w_s(n/x)$ is close to 1 for $n \le x/L$ and close to 0 for $n \ge xL$; similarly for $w_{1-s}(n/y)$. Thus, Corollary 1 expresses $\zeta(s)$ by two smoothed Dirichlet polynomials of effective lengths x and y with $xy = t/2\pi$, which is the same balance of lengths as in the Hardy–Littlewood approximate functional equation (Theorem 1).

Remark 3. Theorem 1 does not follow from Corollary 1 merely by replacing the weights with the sharp cut-offs $\mathbf{1}_{\{n\le x\}}$ and $\mathbf{1}_{\{n\le y\}}$. Indeed, in the transition range $x/L < n < xL$ the weight $w_s(n/x)$ is neither close to 0 nor close to 1 (it varies smoothly as a function of $\log(n/x)$). By Lemma 3 with $L = t^{\varepsilon}$, the trivial estimate for

$$\sum_{n=1}^{\infty} \frac{1}{n^s}\Big(w_s\Big(\frac{n}{x}\Big) - \mathbf{1}_{\{n\le x\}}\Big)$$

is only $O(x^{1-\sigma}t^{\varepsilon})$, while the trivial estimate for the corresponding term for the second sum is $O(t^{1/2-\sigma}y^{\sigma}t^{\varepsilon})$, since $|\chi(s)| \asymp t^{1/2-\sigma}$. For $x = y = \sqrt{t/2\pi}$ both are of order $t^{(1-\sigma)/2+\varepsilon}$, which is not smaller than the trivial bound for the sums themselves. To obtain the error terms $O(x^{-\sigma}) + O(t^{1/2-\sigma}y^{\sigma-1})$ of Theorem 1, one must exploit the oscillation of n^{-it} in the transition range, for instance by the methods of [15, Chapter IV]. For $x = y = \sqrt{t/2\pi}$, Theorem 1 gives the error term $O(t^{-\sigma/2})$; at $\sigma = 1/2$ this is $O(t^{-1/4})$, which is of the correct order of magnitude by the Riemann–Siegel formula. The smoothed formula of Theorem 3 and the sharp formula of Theorem 1 are therefore complementary: the former is an exact identity with a very small and explicit residual term, which is convenient in the study of moments, whereas the latter involves Dirichlet polynomials with sharp cut-offs.

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