CYCLIC-PATTERN-AVOIDING STACKS

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Received: 8/10/25, Accepted: 12/22/25, Published: 1/5/26

Abstract
In 2020, Cerbai, Claesson, and Ferrari generalized West stack-sorting maps to the stack-sorting maps s_σ which avoid a specified pattern σ. Our paper introduces cyclic-pattern-avoiding maps $s_{[\sigma]}$ and consecutive-cyclic-pattern-avoiding maps $s_{[\underline{\sigma}]}$ which are natural analogues of s_σ. In particular, we study the case of length 3 patterns and prove that our stack-sorting machine $\mathrm{SC}_{[123]}$ sorts any permutation of length n within $n-2$ iterations when $n \geq 3$, where $\mathrm{SC}_{[123]}$ is defined as $\mathrm{SC}_{[\sigma]} = s \circ s_{[\sigma]}$, with s being West's deterministic stack-sorting map. Additionally, we characterize the graphs generated from the stack-sorting machine $\mathrm{SC}_{[321]}$. Lastly, we identify the permutations in $|\mathrm{Sort}_n(s_{[\underline{123}]})|$ and $|\mathrm{Sort}_n(s_{[\underline{321}]})|$.

1. Introduction

In 1990, West [10] introduced a deterministic stack-sorting map $s : S_n \to S_n$. At each step of the algorithm, before adding the first remaining element of the input permutation onto the top of the stack, elements are removed from the top of the stack until the stack is empty or its top element exceeds the current first input element (see, for example, Figure 1). West [10] proved that $s(\pi) = \mathrm{id}$ if and only if π avoids the 231 pattern [10]. Since West's introduction of the deterministic stack-sorting map, researchers have studied many variations of s [5, 6, 7, 10, 2, 3]. The stack-sorting machine [4] has also been studied extensively from a sorting point of view.

In 2020, Cerbai, Claesson, and Ferrari [4] generalized West's sorting stack to include pattern avoidance of arbitrary length. For each pattern σ, they defined the map $s_\sigma : S_n \to S_n$, which processes permutations through a stack under the new

DOI: 10.5281/zenodo.18154216

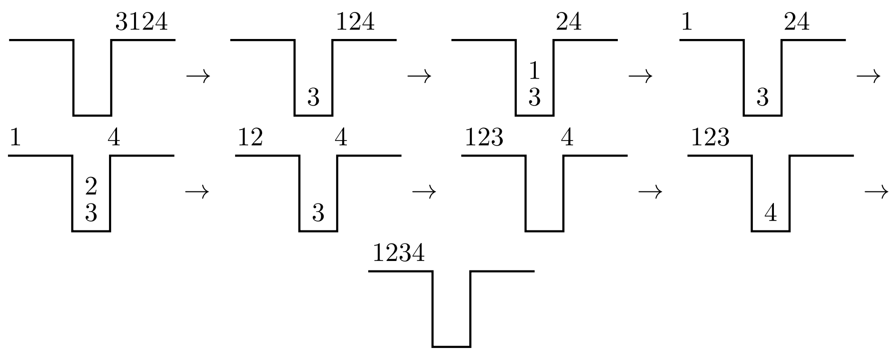

Figure 1: West stack-sorting map s on $\pi = 3124$.

condition that elements in the stack must avoid the pattern σ when read from top to bottom. West's stack-sorting map is a special case of s_σ in which $\sigma = 21$.

In 2021, Berlow [2] introduced a generalized map s_T that avoids a set T of patterns. The s_T map operates by removing the minimum number of elements from the top of the stack necessary to ensure that appending the next element in the permutation to the stack will not induce any pattern in T. The cyclic map $s_{[\sigma]}$, which is the focus of our paper, is a special case of s_T in which $T = [\sigma]$, representing the set of all rotations of the pattern σ.

Babson and Steingrímsson [1] first introduced vincular pattern avoidance, where vincular patterns can additionally require some elements to be adjacent when considering whether a permutation contains the pattern; see Steingrímsson [1] for a survey of the study of vincular patterns, which he refers to as generalized patterns.

Our paper introduces cyclic-pattern-avoiding stacks $s_{[\sigma]}$ and their corresponding cyclic-pattern-avoiding machines $\mathrm{SC}_{[\sigma]}$, which are analogues of the classical-pattern-avoiding stack-sorting maps s_σ and stack-sorting machines SC_σ[2]. The maps $s_{[\sigma]}$ and $\mathrm{SC}_{[\sigma]}$ operate on the same principle as s_σ and SC_σ with the added condition that the stack must *cyclically avoid* the given permutation pattern when read from top to bottom. Formally, $s_{[\sigma]}(\pi)$ is the output permutation produced by processing π through a stack that avoids all patterns in $[\sigma]$. The map removes elements from the top of the stack when necessary to avoid the formation of any pattern in $[\sigma]$ by the incoming element. Once all input elements have been processed, the elements of the stack are removed from the top of the stack one by one and appended to the output permutation in order. Figure 2 illustrates the process using the example $s_{[123]}(3124) = 2143$. The cyclic avoidance machine $\mathrm{SC}_{[\sigma]}$ is defined as $s \circ s_{[\sigma]}$. For example, $\mathrm{SC}_{[123]}(3124) = s(2143) = 1234$.

Stacks avoiding cyclic patterns of length 2 simply return the input permutation, hence, our paper will focus on cyclic patterns of length 3. The only two distinct

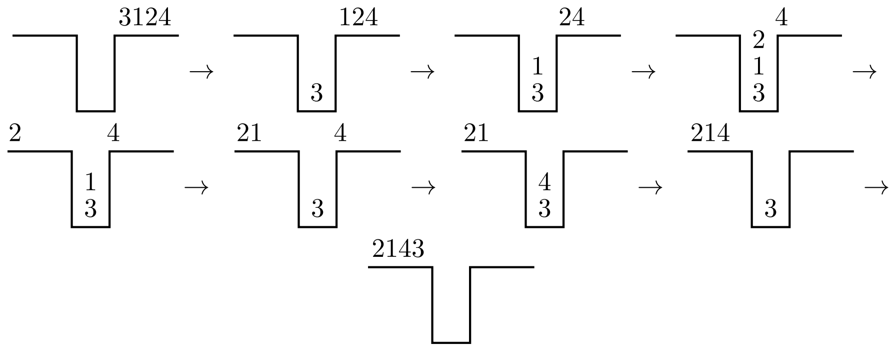

Figure 2: Cyclic stack-sorting map $s_{[123]}$ on $\pi = 3124$.

length 3 patterns up to rotation are 123 and 321, thus, the only two distinct length 3 cyclic avoidance maps and machines are the [123] and [321]-avoiding maps and machines.

West's stack-sorting machine gets its name from its ability to sort a permutation of length n using at most $n-1$ applications of the mapping s_{21} [10]. A natural question that arises is whether our cyclic maps also satisfy such a property. Our first main result is that $\mathrm{SC}_{[123]}$ sorts any permutation π of length n into $\mathrm{id_n}$, the length n identity permutation, through $n-2$ iterations when $n \geq 3$ and that this bound is tight. To show the tightness of the bound, we define a permutation ξ_n where $n \geq 1$ as follows. If n is even, let $\xi_n = 1, 3, \cdots, n-3, n-1, 2, 4, \cdots, n-2, n$. If n is odd, let $\xi_n = 2, 4, \cdots, n-3, n-1, 1, 3, \cdots, n-2, n$. For example, $\xi_6 = 135246$ and $\xi_7 = 2461357$.

Theorem 1. *For any permutation $\pi \in S_n$ where $n \geq 3$, we have $\mathrm{SC}_{[123]}^{\mathrm{n-2}}(\pi) = \mathrm{id_n}$ and $\mathrm{SC}_{[123]}^{\mathrm{n-3}}(\xi_n) \neq \mathrm{id_n}$.*

As an example of the above result, consider the length 3 permutation 231. It requires $3-1=2$ iterations of applying the West stack to map it to the identity. In other words, $s(s(231)) = s(213) = 123$. Meanwhile, the [123]-avoiding stack-sorting machine, $\mathrm{SC}_{[123]}$, maps 231 to 123 in one iteration. To illustrate the tightness of the bound, consider $\mathrm{SC}_{[123]}(\xi_4) = \mathrm{SC}_{[123]}(1324) = 3124 \neq \mathrm{id}_4$.

In general, consider the directed graph formed by the mapping $\mathrm{SC}_{[\sigma]}$ on S_n, where each permutation $\pi \in S_n$ is a vertex and there is an edge from π to $\mathrm{SC}_{[\sigma]}(\pi)$ when $\mathrm{SC}_{[\sigma]}(\pi) \neq \pi$. In the graph formed by $\mathrm{SC}_{[123]}$, Theorem 1 implies that every vertex has a directed path ending at $\mathrm{id_n}$, and that the graph has no cycles. In contrast, in the directed graph formed by $\mathrm{SC}_{[321]}$, when $n \geq 4$, the identity has a directed path to a vertex that is part of a cycle. Our second main result concerns the length of this cycle and where it begins.

Theorem 2. *For all* $n \geq 4$ *and* $m \geq \lceil \frac{n-1-\lceil \frac{2n-1}{3} \rceil}{2} \rceil$*, we have that*

$$\mathrm{SC}^{\mathrm{m}}_{[321]}(\mathrm{id_n}) = \mathrm{SC}^{\mathrm{m}+\lceil \frac{2n-1}{3} \rceil}_{[321]}(\mathrm{id_n}).$$

The next few theorems focus on consecutive cyclic avoidance machines. The consecutive cyclic avoidance stack, denoted $s_{[\underline{\sigma}]}$, is motivated by Defant and Zheng [6], who introduced consecutive-pattern-avoiding stack-sorting maps in 2021. Consecutive cyclic avoidance maps $s_{[\underline{\sigma}]}$ and consecutive avoidance machines $\mathrm{SC}_{[\underline{\sigma}]}$ are natural analogues of consecutive stack-sorting maps. Formally, $s_{[\underline{\sigma}]}(\pi)$ is the output permutation produced by processing π through a stack that *consecutively* avoids all patterns in $[\sigma]$. Elements are removed from the top of the stack when necessary to avoid the formation of any consecutive pattern in $[\sigma]$ by the incoming element. Once all input elements have been processed, the elements in the stack are removed from the top of the stack and appended to the output permutation in the order they were removed from the stack. Figure 3 illustrates the process through the example $s_{[\underline{123}]}(3124) = 4213$. Meanwhile, the consecutive cyclic avoidance machine $\mathrm{SC}_{[\underline{\sigma}]}$ is defined as $s \circ s_{[\underline{\sigma}]}$. For example, $\mathrm{SC}_{[\underline{123}]}(3124) = s(4213) = 1234$.

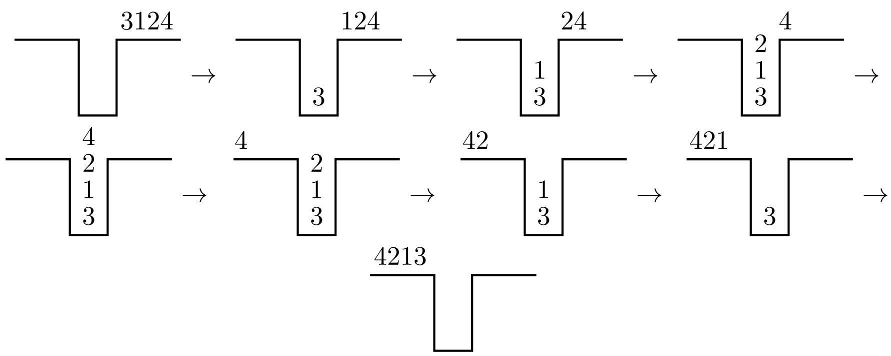

Figure 3: Consecutive cyclic stack-sorting map $s_{[\underline{123}]}$ on $\pi = 3124$.

The final main results concern the number of permutations sorted by $s_{[\underline{123}]}$ and $s_{[\underline{321}]}$. Theorem 3 states that no permutations of length at least 4 are sorted by $s_{[\underline{123}]}$.

Theorem 3. *For all* $n \geq 4$*, the equation* $|\mathrm{Sort_n}(s_{[\underline{123}]})| = 0$ *holds.*

Meanwhile, for all $n \geq 2$ the number of permutations of length n sorted by $s_{[\underline{321}]}$ is 2^{n-2}, and all such permutations are enumerated in our proof.

Theorem 4. *For all* $n \geq 2$*, the equation* $|\mathrm{Sort_n}(s_{[\underline{321}]})| = 2^{n-2}$ *holds.*

In Section 2, we establish the preliminaries. In Section 3, we prove Theorems 1 to 4. In Section 4, we suggest future directions.

2. Preliminaries

A *permutation* $\pi \in S_n$ is a sequence of length n such that $\pi = \pi_1\pi_2 \ldots \pi_n$, where π_i are distinct natural numbers from 1 to n. A *stack* is a structure that can store an ordered list of elements. The only two types of allowed modifications to a stack are adding an element to the top of the stack and removing an element from the top of the stack. A *pattern* σ is a permutation $\sigma_1\sigma_2 \ldots \sigma_{|\sigma|}$ which will be used specifically in the context of pattern avoidance. The *reverse* of a permutation is defined by $\text{Rev}(\pi_1\pi_2 \cdots \pi_n) = \pi_n \cdots \pi_2\pi_1$. For instance, $\text{Rev}(2314) = 4132$. The *standardization* of a sequence τ of distinct numbers, denoted by $\text{st}(\tau)$, is the permutation in S_n obtained by replacing the i^{th}-smallest entry in the sequence with i for all $1 \leq i \leq n$. For example, $\text{st}(315) = 213$, since 3 is the second largest element, 1 is the smallest element, and 5 is the largest element. Two sequences of distinct numbers, τ and τ', have the same *relative order* if $\text{st}(\tau) = \text{st}(\tau')$. For instance, 213 and 315 have the same relative order. A permutation π *contains* a pattern σ if there exists a sequence of indices $i_1 < i_2 \ldots < i_k$ such that $\text{st}(\pi_{\text{i}_1} \ldots \pi_{\text{i}_\text{k}}) = \sigma$. For example, 52413 contains the pattern 132 since $\text{st}(243) = 132$. A permutation π *contains* a sequence $a_1, a_2, \ldots, a_k$ *without gaps* if there exists an index i where $1 \leq i \leq n + 1 - k$ such that $\pi_i, \pi_{i+1}, \cdots, \pi_{i+k-1} = a_1, a_2, \cdots, a_k$. For instance, the permutation 42351 contains the sequence $2, 3, 5$ without gaps. A stack *avoids* a pattern if at all times, the stack does not contain the pattern when read from top to bottom. A stack *cyclically avoids* a pattern if the stack avoids all rotations of the pattern when read from top to bottom. Additionally, the *reduction* of a sequence π, denoted by $\text{red}(\pi)$, is defined to be the permutation obtained from replacing every maximal consecutive subsequence of contiguous numbers with its minimum element and then standardizing the resulting permutation. For example, the maximal consecutive subsequences of contiguous numbers in the permutation 16783425 are $1, 678, 34, 2, 5$. Hence, $\text{red}(16783425) = \text{st}(16325) = 15324$. Finally, for a given map $f : S_n \to S_n$, define $\text{Sort}_\text{n}(f)$ to be the pre-image of $\{\text{id}_\text{n}\}$ under f. For instance, $\text{Sort}_3(\text{s}_{\underline{[321]}}) = \{312, 321\}$ since these are all the permutations in S_3 sorted by $s_{\underline{[321]}}$.

3. Proofs of the Main Results

3.1. Proof of Theorem 1

We first prove that reducing a permutation π does not change the number of iterations of $\text{SC}_{[123]}$ required to sort π.

Lemma 1. *For $\pi \in S_n$, we have that $\text{red}(\text{SC}_{[123]}(\text{red}(\pi))) = \text{red}(\text{SC}_{[123]}(\pi))$. Also, any sequence of the form $a, a+1, \cdots, a+k$ that π contains without gaps is added onto and removed from the stack together.*

Proof. Suppose that a permutation π contains the sequence $a, a+1, \cdots, a+k$ without gaps. Before a is added onto the $s_{[123]}$ stack, all elements on the stack must be less than a or greater than $a+k$. Thus, if adding a does not induce the patterns [123] in the stack, then adding any element in $a, a+1, \cdots, a+k$ will not induce the patterns [123] with earlier elements in the stack. Also note that no two or three elements from the above sequence will induce a [123] pattern with earlier elements on the stack either. Hence, the elements in $a, a+1, \cdots, a+k$ will be added onto the $s_{[123]}$ stack together, without inducing the patterns [123]. Similarly, the sequence $a, a+1, \cdots, a+k$ will also be removed from the stack consecutively. Thus, $s_{[123]}(\pi)$ contains the sequence $\mathrm{Rev}(a, a+1, \cdots, a+k)$ without gaps. Applying s will then reverse $\mathrm{Rev}(a, a+1, \cdots, a+k)$. Hence, $\mathrm{SC}_{[123]}(\pi) = s \circ s_{[123]}(\pi)$ contains the sequence $a, a+1, \cdots, a+k$ without gaps. Thus, any contiguous subsequence of consecutive numbers in π also appears in $\mathrm{SC}_{[123]}(\pi)$, and they are always added onto and removed from the stacks together.

Hence, each maximal contiguous subsequence of consecutive numbers in π can be treated as a single element with respect to the reduction operation and $\mathrm{SC}_{[123]}$. Thus, $\mathrm{red}(\mathrm{SC}_{[123]}(\mathrm{red}(\pi))) = \mathrm{red}(\mathrm{SC}_{[123]}(\pi))$. □

Consequently, the number of iterations required to sort a permutation π equals the number of permutations required to sort $\mathrm{red}(\pi)$.

Corollary 1. *For $\pi \in S_n$, we have that $\mathrm{SC}^{\mathrm{k}}_{[123]}(\pi) = \mathrm{id_n}$ if and only if $\mathrm{SC}^{\mathrm{k}}_{[123]}(\mathrm{red}(\pi))) = \mathrm{id}_{|\mathrm{red}(\pi)|}$.*

Next, we prove that $\mathrm{red} \circ \mathrm{SC}_{[123]}$ reduces the length of any permutation.

Lemma 2. *For any $\pi \in S_n$ with $n \geq 2$, the equation $|\mathrm{red}(\mathrm{SC}_{[123]}(\pi))| < |\pi|$ holds.*

Proof. It suffices to show that $\mathrm{SC}_{[123]}(\pi)$ contains a sequence of the form $a, a+1$ without gaps. First, suppose that $\pi_1 \neq n$. Note that right before the element $\pi_1 + 1$ is added onto the $s_{[123]}$ stack, the stack should only contain π_1, otherwise it would induce the pattern 231 or 312. Thus, π_1 and $\pi_1 + 1$ are consecutive elements at the bottom of the stack and are the last two elements removed from the $s_{[123]}$ stack so $s_{[123]}(\pi)$ ends with the sequence $\pi_1 + 1, \pi_1$. Clearly, after applying s to $s_{[123]}(\pi)$, the element π_1 appears right before $\pi_1 + 1$ in $\mathrm{SC}_{[123]}(\pi)$.

Now, suppose $\pi_1 = n$. Then $s_{[123]}(\pi)$ ends with n, and thus, $s(s_{[123]}(\pi))$ ends with the sequence $(n-1, n)$. Hence, $\mathrm{SC}_{[123]}(\pi)$ contains a sequence of the form $(a, a+1)$ without gaps. Therefore, $|\mathrm{red}(\mathrm{SC}_{[123]}(\pi))| < |\pi|$. □

We finish by using Corollary 1 and Lemma 2 to prove Theorem 1.

Proof of Theorem 1. We first use induction to prove that $\mathrm{SC}^{n-2}_{[123]}(\pi) = \mathrm{id_n}$ for $\pi \in S_n$ with $n \geq 3$. The base case $n = 3$ is clear by straightforward verification. Now assume that for all n where $3 \leq n \leq m$, the equation $\mathrm{SC}^{\mathrm{n-2}}_{[123]}(\pi) = \mathrm{id_n}$ holds.

For any $\pi \in S_{m+1}$, Lemma 2 implies that $|\mathrm{red}(\mathrm{SC}_{[123]}(\pi))| < |\pi| = m+1$. Thus, by our inductive hypothesis, $\mathrm{SC}_{[123]}^{\mathrm{m-2}}(\mathrm{red}(\mathrm{SC}_{[123]}(\pi))) = \mathrm{id}_{|\mathrm{red}(\mathrm{SC}_{[123]}(\pi))|}$. Then by Corollary 1, $\mathrm{SC}_{[123]}^{\mathrm{m-1}}(\pi) = \mathrm{id_{m+1}}$. Hence, $\mathrm{SC}_{[123]}^{\mathrm{n-2}}(\pi) = \mathrm{id_n}$ for all $\pi \in S_n$, which proves that any length n permutation π can be sorted through at most $n-2$ iterations of $\mathrm{SC}_{[123]}$.

We now show the tightness of the bound by proving $\mathrm{SC}_{[123]}^{\mathrm{n-3}}(\xi_n) \neq \mathrm{id_n}$ for $n \geq 3$ by induction on n. The base case $n=3$ is clear. Now assume that for $n=m$, the equation $\mathrm{SC}_{[123]}^{\mathrm{n-3}}(\xi_n) \neq \mathrm{id_n}$ holds.

Suppose $n = m+1$. If $m+1$ is even, then

$$\mathrm{SC}_{[123]}(\xi_{\mathrm{m+1}}) = 3, 5, \cdots, m, 1, 2, 4, \cdots, m-1, m+1$$

which implies that $\mathrm{red}(\mathrm{SC}_{[123]}(\xi_{\mathrm{m+1}})) = \xi_m$. In the case where $m+1$ is odd, we have that

$$\mathrm{SC}_{[123]}(\xi_{\mathrm{m+1}}) = 1, 4, 6, \cdots, m, 2, 3, 5, \cdots, m+1$$

which implies that $\mathrm{red}(\mathrm{SC}_{[123]}(\xi_{\mathrm{m+1}})) = \xi_m$. Thus, $\mathrm{red}(\mathrm{SC}_{[123]}(\xi_{\mathrm{m+1}})) = \xi_m$ always holds. By the inductive hypothesis, $\mathrm{SC}_{[123]}^{\mathrm{m-3}}(\xi_m) \neq \mathrm{id}_{|\xi_{\mathrm{m}}|}$.

Then by Corollary 1, $\mathrm{SC}_{[123]}^{\mathrm{m-2}}(\xi_{m+1}) \neq \mathrm{id}_{|\xi_{\mathrm{m+1}}|}$. Hence, $\mathrm{SC}_{[123]}^{\mathrm{n-3}}(\xi_n) \neq \mathrm{id_n}$ always holds for $n \geq 3$. □

3.2. Proof of Theorem 2

Define the *superimpose* operation $\mathrm{si}(\tau_1, \tau_2)$ which operates on two disjoint sequences of distinct numbers, τ_1 and τ_2, as follows. The last element of τ_2 is added onto the top of a stack followed by the last two elements of τ_1 in reverse order, or followed by the last element of τ_1 if τ_1 has only one element left. The process is repeated until either τ_1 or τ_2 is empty, in which case all remaining elements are added onto the top of the stack in reverse order. The output sequence is then obtained by reading the stack from top to bottom. For example, $\mathrm{si}((1,2,3,4,5),(6,7,8,9,10)) = 6,7,1,8,2,3,9,4,5,10$ since the elements are added onto the stack in the order $\underline{10}, 5, 4, \underline{9}, 3, 2, \underline{8}, 1, \underline{7}, \underline{6}$ where the underlined elements are from the sequence $6, 7, 8, 9, 19$ and all other elements are from $1, 2, 3, 4, 5$.

Lemma 3. *For all n, m where $n \geq 4$ and $1 \leq m < \lceil \frac{2n-1}{3} \rceil$, we have that*

$$\mathrm{SC}_{[321]}(\mathrm{si}((1,2,\cdots,m),(m+1,m+2,\cdots,n))) = \mathrm{si}((1,2,\cdots,m+1),(m+2,m+3,\cdots,n)).$$

Proof. Let $\pi = \mathrm{si}((1,2,\cdots,m),(m+1,m+2,\cdots,n))$. Since $m < \lceil \frac{2n-1}{3} \rceil$, the first element of π is $m+1$. Hence, when applying $s_{[321]}$ to π, the element $m+1$ remains at the bottom of the stack and is the last element to be removed from the stack. Then whenever π contains a sequence (i,j) satisfying $i < m+1 < j$ without gaps, all elements less than i are removed from the stack before i is added onto the stack, and the remaining elements on the stack are in an increasing order from top to bottom. Thus, j is added onto the stack immediately after i. Since $\pi = \mathrm{si}((1, 2, \cdots, m), (m+1, m+2, \cdots, n))$, either j is the last element of π or the element right after j in π is $i+1$. If j is the last element of π, then j is removed from the stack, and i is removed from the stack immediately after. If $i+1$ is right after j in π, then $i+1, i, m+1$ forms the 213 pattern, hence j and i are consecutively removed from the stack in that order before $i+1$ is added. In either case, the order of the elements i, j is swapped. Thus, $s_{[321]}(\pi)$ is equivalent to the permutation obtained by swapping all adjacent pairs of elements i, j in π with i preceding j and $i < m+1 < j$, and then placing the element $m+1$ at the end of π since $m+1$ is removed from the stack last.

Next, when applying s to $s_{[321]}(\pi)$, each element $j > m+1$ remains on the stack until right before the element $j+1$ is added or the final operation in which every element in the stack is removed and appended to the output permutation in the same order of removal. Thus, $\mathrm{SC}_{[321]}(\pi) = s(s_{[321]}(\pi)) = \mathrm{si}((1, 2, \cdots, m+1), (m+2, m+3, \cdots, n))$. □

Example 1. To illustrate the process described in the above proof, we consider the case $n = 6, m = 2$. We have that $\mathrm{si}((1, 2), (3, 4, 5, 6)) = 3, 4, 5, 1, 2, 6$ and that $s_{[321]}(3, 4, 5, 1, 2, 6) = 4, 5, 1, 6, 2, 3$, which is equivalent to swapping the adjacent elements $2, 6$ since $2 < m+1 < 6$ in the permutation $3, 4, 5, 1, 2, 6$ and then placing the element $m+1 = 3$ at the end. Then,

$$\mathrm{SC}_{[321]}(3, 4, 5, 1, 2, 6) = s(4, 5, 1, 6, 2, 3) = 4, 1, 5, 2, 3, 6 = \mathrm{si}((1, 2, 3), (4, 5, 6)).$$

We now prove Theorem 2 using Lemma 3.

Proof of Theorem 2. $\mathrm{SC}_{[321]}(\mathrm{id_n}) = (2, 3, \cdots, n-1, 1, n) = \mathrm{si}((1), (2, 3, \cdots, n))$. Then from Lemma 3, it follows that for all $0 \le m < \lceil \frac{2n-1}{3} \rceil$,

$$\mathrm{SC}^{\mathrm{m+1}}_{[321]}(\mathrm{id_n}) = \mathrm{SC}^{\mathrm{m}}_{[321]}(\mathrm{si}((1), (2, 3, \cdots, n))) = \mathrm{si}((1, 2, \cdots, m+1), (m+2, \cdots, n)).$$

Hence,

$$\mathrm{SC}^{\lceil \frac{2n-1}{3} \rceil}_{[321]}(\mathrm{id_n}) = \mathrm{si}((1, 2, \cdots, \lceil \frac{2n-1}{3} \rceil), (\lceil \frac{2n-1}{3} \rceil + 1, \lceil \frac{2n-1}{3} \rceil + 2, \cdots, n)).$$

Since $3\lceil \frac{2n-1}{3} \rceil > 2n-2$ implies $\lceil \frac{2n-1}{3} \rceil > 2(n - \lceil \frac{2n-1}{3} \rceil - 1)$, the first element of $\mathrm{SC}^{\lceil \frac{2n-1}{3} \rceil}_{[321]}(\mathrm{id_n})$ is 1. Thus, when applying $s_{[321]}$ to $\mathrm{SC}^{\lceil \frac{2n-1}{3} \rceil}_{[321]}(\mathrm{id_n})$, the element 1 is at the bottom of the stack. When any element $i > \lceil \frac{2n-1}{3} \rceil$ is appended to the stack, all elements currently on the stack are less than i, hence, all elements except 1 are removed from the stack to avoid forming a 321 pattern. The element i is removed from the stack, either right before the element $i+1$ is added onto the stack or when all elements are removed from the stack and appended to the output permutation in the order of removal. Meanwhile for any element $i < \lceil \frac{2n-1}{3} \rceil$, the element i is removed from the stack immediately after it is added, because the next element is always greater than i. Thus, applying $s_{[321]}$ to $\mathrm{SC}_{[321]}^{\lceil \frac{2n-1}{3} \rceil}(\mathrm{id_n})$ is equivalent to moving each element i where $\lceil \frac{2n-1}{3} \rceil < i < n$ to the original position of the element $i+1$, and then moving the elements $n, 1$ to the end of the output permutation since they are outputted last. Thus,

$$s_{[321]}(\mathrm{SC}_{[321]}^{\lceil \frac{2n-1}{3} \rceil}(\mathrm{id_n})) = \mathrm{si}((2, 3, \cdots, \lceil \frac{2n-1}{3} \rceil), (\lceil \frac{2n-1}{3} \rceil + 1, \cdots, n-1)), (n, 1)$$

which denotes the concatenation of the two sequences on the right side. Then it follows that

$$s(s_{[321]}(\mathrm{SC}_{[321]}^{\lceil \frac{2n-1}{3} \rceil}(\mathrm{id_n}))) = \mathrm{si}((2, \cdots, \lceil \frac{2n-1}{3} \rceil), (\lceil \frac{2n-1}{3} \rceil + 1, \cdots, n-2)), \mathrm{si}((1), (n-1, n))$$

which equals $\mathrm{SC}_{[321]}^{\lceil \frac{2n-1}{3} \rceil + 1}(\mathrm{id_n})$.

Similarly, by reusing the above reasoning we have that

$$\mathrm{SC}_{[321]}^{\lceil \frac{2n-1}{3} \rceil + 2}(\mathrm{id_n}) = \mathrm{si}(\tau), \mathrm{si}((1, 2), (n-3, n-2, \cdots, n)),$$

where $\tau = (3, \cdots, \lceil \frac{2n-1}{3} \rceil), (\lceil \frac{2n-1}{3} \rceil + 1, \cdots, n-4)$. Then from induction on m, for all m where $2m \leq n - 1 - \lceil \frac{2n-1}{3} \rceil$, it follows that

$$\mathrm{SC}_{[321]}^{\lceil \frac{2n-1}{3} \rceil + \mathrm{m}}(\mathrm{id_n}) = \mathrm{si}(\phi), \mathrm{si}((1, \cdots, m), (n-2m+1, \cdots, n)),$$

where $\phi = (m+1, \cdots, \lceil \frac{2n-1}{3} \rceil), (\lceil \frac{2n-1}{3} \rceil + 1, \cdots, n-2m)$.

Then by setting $m = \lceil \frac{n-1-\lceil \frac{2n-1}{3} \rceil}{2} \rceil$, through simple verification we obtain the equation

$$\mathrm{SC}_{[321]}^{\lceil \frac{2n-1}{3} \rceil + \mathrm{m}}(\mathrm{id_n}) = \mathrm{si}((1, \cdots, m), (m+1, \cdots, n)) = \mathrm{SC}_{[321]}^{\mathrm{m}}(\mathrm{id_n}).$$

Hence, for $m \geq \lceil \frac{n-1-\lceil \frac{2n-1}{3} \rceil}{2} \rceil$ we have that $\mathrm{SC}_{[321]}^{\lceil \frac{2n-1}{3} \rceil + \mathrm{m}}(\mathrm{id_n}) = \mathrm{SC}_{[321]}^{\mathrm{m}}(\mathrm{id_n})$. □

3.3. Proof of Theorem 3

In this section, we prove Theorem 3.

Proof of Theorem 3. Assume that there exists a permutation π such that $s_{[\underline{123}]}(\pi) = \mathrm{id_n}$. Note that π_1 must be the last element outputted, and hence $\pi_1 = n$.

If $\pi_2 = 1$, then when π_3 is added onto the stack, no previous element has been removed from the stack. Thus, π_3 is removed from the stack before π_2. Since $\pi_3 > \pi_2$, it contradicts $s_{[\underline{123}]}(\pi) = \mathrm{id_n}$.

If $1 < \pi_2 < n$, we consider two sub-cases based on the relative order of π_2 and π_3. In the case of $\pi_2 < \pi_3$, the element π_3 is added onto the stack without removing any elements from the top of the stack. Hence, π_3 is removed from the stack before π_2. Since $\pi_3 > \pi_2$, it contradicts $s_{[\underline{123}]}(\pi) = \mathrm{id_n}$. Now, in the case where $\pi_2 > \pi_3$, the sequence $\pi_3\pi_2\pi_1$ forms the 123 pattern, hence, π_2 is removed from the stack before π_3 is added. Since $\pi_2 > \pi_3$, it contradicts $s_{[\underline{123}]}(\pi) = \mathrm{id_n}$. □

3.4. Proof of Theorem 4

A sequence π is *good* if for any i with $1 \leq i \leq |\pi|$, we have that $\pi_i = \min(\pi_i, \pi_{i+1}, \ldots, \pi_{|\pi|})$ or $\pi_i = \max(\pi_i, \pi_{i+1}, \ldots, \pi_{|\pi|})$. For example, 51432 is a good sequence. We first prove the necessary condition for a permutation to be sorted by $s_{[\underline{321}]}$.

Lemma 4. *For all $n \geq 2$ and any $\pi \in \mathrm{Sort_n}(s_{[\underline{321}]})$, we must have that $\pi_1 = n$ and $\pi_{[2:n]}$ must be good.*

Proof. We have that $\pi_1 = n$ for any π in $\mathrm{Sort_n}(s_{[\underline{321}]})$ because π_1 is the last element of $s_{[\underline{321}]}(\pi)$. We prove that $\pi_{[2:n]}$ is good by induction. The base case $n = 2$ is clear. Assume that for all n where $1 \leq n \leq m$, we have that $\pi_{[2:n]}$ is good for any $\pi \in \mathrm{Sort_n}(s_{[321]})$. Now, we prove the same statement for $n = m + 1$.

If $\pi_2 = 1$ then π_2 is removed from the stack immediately before π_3 is added onto the stack, since $\pi_3, \pi_2, \pi_1 = \pi_3, 1, m+1$ would form a 213 pattern. Thus, the sequence $\pi_1\pi_{[3:m+1]}$ is sorted by $s_{[\underline{321}]}$ and as a result, from our inductive hypothesis it follows that $\pi_{[3:m+1]}$ is a good sequence. Since $\pi_2 = 1$, thus, $\pi_{[2:m+1]}$ is a good sequence.

If $\pi_2 = m$, then since π_1 and π_2 do not form consecutive patterns [321] with any other element, π_2 will not be removed from the stack until the final operation in which the stack is cleared. Hence, $\pi_{[2:m+1]}$ is sorted by $s_{[\underline{321}]}$, and by our inductive hypothesis $\pi_{[3:m+1]}$ is a good sequence. Thus, $\pi_{[2:m+1]}$ is also good.

If $1 < \pi_2 < m$, then π_2 must be removed from the stack before the final operation in which all elements in the stack are removed and appended to the output permutation in that order. Otherwise, π_2 becomes the second-to-last element of the output permutation, implying that π is not sortable by $s_{[\underline{321}]}$. Additionally, π_2 must be removed from the stack before any element $\pi_i > \pi_2$ is added onto the stack. Otherwise, π_i would be removed from the stack before π_2, which would contradict the sortability of π. Since the elements $1, 2, \cdots, \pi_2 - 1$ must be removed from the stack before π_2, thus, the elements $1, 2, \cdots, \pi_2 - 1$ must be added onto the stack before any element greater than π_2. However, the addition of any element less than π_2 to the stack cannot induce any consecutive pattern in [321] with π_1 and π_2. Hence, π_2 is not removed from the stack until right before the first element $\pi_i > \pi_2$ is added onto the stack. Thus, before π_i is added onto the stack there is still at least one element less than π_2 on the top of the stack, and it is impossible for all of them to be removed from the stack by patterns induced through π_i since π_i does not induce any consecutive pattern in [321] with an element less than π_2 and the element π_2. Thus, π_2 is not removed from the stack before the addition of $\pi_i > \pi_2$ to the stack, which leads to a contradiction. □

The converse of Lemma 4 follows from reversing the inductive process. Hence, the condition that $\pi_1 = n$ and $\pi_{[2:n]}$ is good is both sufficient and necessary for π

to be sorted by $s_{\underline{[321]}}$.

Corollary 2. *For all $n \geq 2$ and any $\pi \in S_n$, we have that $\pi \in \mathrm{Sort_n}(s_{\underline{[321]}})$ if and only if $\pi_1 = n$ and $\pi_{[2:n]}$ is good.*

We now finish by using Corollary 2 to prove Theorem 4.

Proof of Theorem 4. From Corollary 2, it follows that

$$\mathrm{Sort_n}(s_{\underline{[321]}}) = \{\pi \in S_n | \pi_1 = n \text{ and } \pi_{[2:n]} \text{ is good}\}$$

which has a size equal to $|\{\pi \in S_{n-1} | \pi \text{ is good}\}|$. For any good permutation π of size $n-1$, by definition $\pi_1 = \min(\{1, 2, \cdots, n-1\})$ or $\pi_1 = \max(\{1, 2, \cdots, n-1\})$. Then $\pi_2 = \min(\{1, 2, \cdots, n-1\} - \{\pi_1\})$ or $\pi_2 = \max(\{1, 2, \cdots, n-1\} - \{\pi_1\})$. Similarly,

$$\pi_i = \min(\{1, 2, \cdots, n-1\} - \{\pi_1, \cdots, \pi_{i-1}\})$$

or

$$\pi_i = \max(\{1, 2, \cdots, n-1\} - \{\pi_1, \cdots, \pi_{i-1}\}).$$

Hence, after fixing $\pi_1, \pi_2, \cdots, \pi_{i-1}$, each π_i has exactly two possible values, except for π_{n-1} which is fixed by the values of $\pi_1, \pi_2, \cdots, \pi_{n-2}$ since

$$|\{1, 2, \cdots, n-1\} - \{\pi_1, \cdots, \pi_{n-2}\}| = 1.$$

Hence, $|\{\pi \in S_{n-1} | \pi \text{ is good}\}| = 2^{n-2}$ which implies that $|\mathrm{Sort_n}(s_{\underline{[321]}})| = 2^{n-2}$ for all $n \geq 2$. □

4. Future Directions

We conclude with the following conjectures. Our first conjecture concerns the number of permutations that are sortable by $\mathrm{SC}_{[123]}$ and has been verified by code for $n \leq 9$.

Conjecture 1. For all $n \geq 1$, the sequence $|\mathrm{Sort}_n(\mathrm{SC}_{[123]})|$ is enumerated by OEIS A006318 [8], the sequence of Large Schröder numbers.

In Theorem 3, we proved that no permutations of length at least 4 are sortable by $s_{\underline{[123]}}$. The conjecture below describes the output permutation that shares the longest prefix with $\mathrm{id_n}$.

Conjecture 2. The permutation $1, 2, ..., \lfloor \frac{n}{2} \rfloor, n, n-1, ..., \lfloor \frac{n}{2} \rfloor + 1$ is the unique length n permutation in $\{s_{\underline{[123]}}(\pi) | \pi \in S_n\}$ that shares the longest prefix with $\mathrm{id_n}$.

The last two conjectures are about the number of input permutations that are mapped to the conjectured unique output permutation that shares the longest prefix with $\mathrm{id_n}$.

Conjecture 3. For all $n \geq 4$ when n is even, there exists a unique input permutation $\pi \in S_n$ such that $s_{[\underline{123}]}(\pi) = 1, 2, ..., \lfloor \frac{n}{2} \rfloor, n, n-1, ..., \lfloor \frac{n}{2} \rfloor + 1$.

Conjecture 4. For all $n \geq 4$ when n is odd, there exist $\lceil \frac{n}{2} \rceil$ input permutations $\pi \in S_n$ such that $s_{[\underline{123}]}(\pi) = 1, 2, ..., \lfloor \frac{n}{2} \rfloor, n, n-1, ..., \lfloor \frac{n}{2} \rfloor + 1$.

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