#A16 INTEGERS 26 (2026)

A NOTE ON MULTISET RECONSTRUCTION FROM PAIRWISE PRODUCTS AND TOTAL SUM

Stella Jiahui Li
Department of Mathematics, Harvard University, Cambridge, Massachusetts
stellali@college.harvard.edu

Received: 8/5/25, Accepted: 12/31/25, Published: 1/19/26

Abstract

Ballantine, Beck, and Merca defined a map pre_2, which sends an integer partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ to the multiset $\{\!\{\lambda_i\lambda_j : 1 \le i < j \le \ell\}\!\}$ consisting of the pairwise products of parts of λ. These three authors conjectured that for each n, the map pre_2 is injective on the set of integer partitions of n. In this note, we prove their conjecture.

1. Background

An *integer partition* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of a positive integer n is a weakly decreasing sequence of positive integers whose sum is n. We refer to the λ_i's as the *parts* of a partition, $|\lambda| = n$ as the *size*, and $\ell(\lambda) = \ell$ as the *length*.

The jth *elementary symmetric polynomial* e_j is defined by

$$e_j(x_1, \ldots, x_n) = \begin{cases} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} x_{i_1} \cdots x_{i_j} & \text{if } j \le n; \\ 0 & \text{if } j > n. \end{cases}$$

We are interested in the partition whose parts are the summands of $e_j(\lambda_1, \ldots, \lambda_\ell)$.

Definition 1 ([1])**.** Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, we define $\text{pre}_k(\lambda)$ to be the partition whose multiset of parts is $\{\!\{\lambda_{i_1} \cdots \lambda_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le \ell\}\!\}$, where $\{\!\{\}\!\}$ denotes multiset. If $\ell(\lambda) < j$, then $\text{pre}_k(\lambda)$ is the empty partition $\emptyset$. We call $\text{pre}_k(\lambda)$ an *elementary symmetric partition*.

For example, consider the partition $\lambda = (4, 2, 1, 1)$ and $k = 2$. We can evaluate $e_2(4, 2, 1, 1)$ as

$$e_2(4, 2, 1, 1) = 4 \cdot 2 + 4 \cdot 1 + 4 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 1 \cdot 1,$$

and by looking at the summands, we get that $\text{pre}_2(\lambda) = (8, 4, 4, 2, 2, 1)$.

DOI: 10.5281/zenodo.18304958

The main conjecture, which first appeared in [1] and was noted again in [2], is that the map pre_2 is injective on the set of partitions of n for each n. Our main result is a proof of this conjecture.

Theorem 1. *If λ and μ are partitions of a positive integer n, then $\mathrm{pre}_2(\lambda) = \mathrm{pre}_2(\mu)$ if and only if $\lambda = \mu$.*

In fact, our proof establishes a stronger statement where we replace partitions of n by finite nonincreasing sequences of positive real numbers summing to n.

2. Proof

We start with an auxiliary lemma.

Lemma 1. *Let λ and μ be integer partitions of a positive integer n. If $\lambda_1 = \mu_1$ and $\mathrm{pre}_2(\lambda) = \mathrm{pre}_2(\mu)$, then $\lambda = \mu$.*

Proof. Observe that $\ell(\lambda) = \ell(\mu)$ because $\ell(\mathrm{pre}_2(\lambda)) = \binom{\ell(\lambda)}{2} = \binom{\ell(\mu)}{2} = \ell(\mathrm{pre}_2(\mu))$. We want to show $\lambda_i = \mu_i$ for all $1 \le i \le \ell(\lambda) = \ell(\mu)$, and we proceed by strong induction on i. For the base case, we know $\lambda_1 = \mu_1$ by assumption.

For the inductive step, observe that

$$\begin{aligned}\lambda_1\lambda_k &= \max\left(\mathrm{pre}_2(\lambda) \setminus \{\!\{\lambda_i\lambda_j : 1 \le i < j \le k-1\}\!\}\right)\\ &= \max\left(\mathrm{pre}_2(\mu) \setminus \{\!\{\mu_i\mu_j : 1 \le i < j \le k-1\}\!\}\right) = \mu_1\mu_k.\end{aligned}$$

Since $\lambda_1 = \mu_1$, we conclude that $\lambda_k = \mu_k$. □

With Lemma 1 established, we have the necessary tool to prove our main result.

Proof of Theorem 1. It suffices to show the identity $\sum_{1\le i\le \ell} \lambda_i^{2^m} = \sum_{1\le i\le \ell} \mu_i^{2^m}$ for all positive integers m because then

$$\lambda_1 = \lim_{m\to\infty}\left(\sum_{1\le i\le \ell} \lambda_i^{2^m}\right)^{1/2^m} = \lim_{m\to\infty}\left(\sum_{1\le i\le \ell} \mu_i^{2^m}\right)^{1/2^m} = \mu_1,$$

and we can apply Lemma 1.

We prove the identity by strong induction on m. For the base case, $\sum_i \lambda_i = \sum_i \mu_i = n$ by assumption. For the inductive step, observe that

$$\begin{aligned}\sum_{1\le i\le \ell} \lambda_i^{2^m} &= \left(\sum_{1\le i\le \ell} \lambda_i^{2^{m-1}}\right)^2 - 2\sum_{\lambda_i\lambda_j\in \mathrm{pre}_2(\lambda)} (\lambda_i\lambda_j)^{2^{m-1}}\\ &= \left(\sum_{1\le i\le \ell} \mu_i^{2^{m-1}}\right)^2 - 2\sum_{\mu_i\mu_j\in \mathrm{pre}_2(\mu)} (\mu_i\mu_j)^{2^{m-1}} = \sum_{1\le i\le \ell} \mu_i^{2^m},\end{aligned}$$

as desired. □

We remark that a classical result of Selfridge and Straus [3] states that the multiset $X = \{\!\{x_1, \ldots, x_\ell\}\!\} \subset \mathbb{C}$ is uniquely determined by the multiset of pairwise sums of elements of X if and only if ℓ is not a power of 2. An immediate consequence of this theorem is that pre_2 is injective on partitions with lengths that are not powers of 2, regardless of the sizes of the partitions. The strength of our result is that we can also handle partitions with lengths that are powers of 2 if we fix their sizes.

Acknowledgements. This research was conducted at the University of Minnesota Duluth REU with support from Jane Street Capital, NSF Grant 2409861, and donations from Ray Sidney and Eric Wepsic. I would like to thank Mitchell Lee, Noah Kravitz, and Katherine Tung for their invaluable guidance throughout the research process and helpful suggestions during the editing process. I am very grateful to Joe Gallian and Colin Defant for organizing the Duluth REU and for the invitation to participate.

References

[1] C. Ballantine, G. Beck, and M. Merca, Partitions and elementary symmetric polynomials: an experimental approach, *Ramanujan J.* **66** (34) (2025), 1-22.

[2] C. Ballantine, G. Beck, M. Merca, and B. Sagan, Elementary symmetric partitions, preprint, `arXiv: 2409.11268`.

[3] J. L. Selfridge and E. G. Straus, On the determination of numbers by their sums of a fixed order, *Pacific J. Math.* **8** (4) (1958), 847-856.