#A18 INTEGERS 26 (2026)

JOINT DISTRIBUTIONS OF 10 CLASSICAL STATISTICS OVER PERMUTATIONS AVOIDING TWO PATTERNS OF LENGTH 3

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Received: 4/20/24, Revised: 6/9/25, Accepted: 1/2/26, Published: 1/19/26

Abstract

Finding the distributions of permutation statistics across classes of permutations that avoid certain patterns has attracted significant interest in the literature. In particular, Bukata et al. found the distribution of a single statistic among ascents (asc), descents (des), double ascents (dasc), double descents (ddes), peaks (pk), and valleys (vl) over permutations avoiding any two patterns of length 3. Moreover, Han and Kitaev found the joint distribution of six classical statistics (asc, des, lrmax, lrmin, rlmax, rlmin) over the same classes of permutations. In this paper, we generalize all of these results by deriving explicit formulas for the joint distributions of all of the above-mentioned statistics over permutations avoiding any two patterns of length 3. All the multivariate generating functions we obtain are rational.

1. Introduction

Let $[n] := \{1, 2, \dots, n\}$. A permutation of length n is a rearrangement of the set $[n]$. Denote by S_n the set of permutations of $[n]$. A permutation $\pi_1\pi_2\cdots\pi_n \in S_n$ avoids a *pattern* $p = p_1p_2\cdots p_k \in S_k$ if there is no subsequence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ such that $\pi_{i_j} < \pi_{i_m}$ if and only if $p_j < p_m$. For example, the permutation 45123 avoids the pattern 132. Let $S_n(\tau, \rho)$ denote the set of permutations in S_n that avoid patterns τ and ρ. For a comprehensive review of the field of permutation patterns, which has received considerable attention in the literature, we direct the reader to [7] and references therein.

Of interest to us are the following classical permutation statistics. For $1 \leq i \leq n-1$, we say that i is an *ascent* (resp., *descent*) in $\pi \in S_n$ if $\pi_i < \pi_{i+1}$ (resp., $\pi_i > \pi_{i+1}$) and $\mathrm{asc}(\pi)$ (resp., $\mathrm{des}(\pi)$) is the number of ascents (resp., descents) in π. Also, π_i is a *right-to-left maximum* (resp., *right-to-left minimum*) in π if π_i is greater (resp., smaller) than any element to its right. Note that π_n is always a right-to-left maximum and a right-to-left minimum. Denote by $\mathrm{rlmax}(\pi)$ and $\mathrm{rlmin}(\pi)$

DOI: 10.5281/zenodo.18305021

the number of right-to-left maxima and right-to-left minima in π, respectively. We define *left-to-right maxima*, counted by $\mathrm{lrmax}(\pi)$, and *left-to-right minima*, counted by $\mathrm{lrmin}(\pi)$, in a similar way. For $1 \le i \le n-2$, we say that i is a *double ascent* (resp., *double descent*) in $\pi \in S_n$ if $\pi_i < \pi_{i+1} < \pi_{i+2}$ (resp., $\pi_i > \pi_{i+1} > \pi_{i+2}$). We denote by $\mathrm{dasc}(\pi)$ (resp., $\mathrm{ddes}(\pi)$) the number of double ascents (resp., double descents) in π. For $1 \le i \le n-2$, i is a *peak* (resp., *valley*) in $\pi \in S_n$ if $\pi_i < \pi_{i+1} > \pi_{i+2}$ (resp., $\pi_i > \pi_{i+1} < \pi_{i+2}$), and $\mathrm{pk}(\pi)$ (resp., $\mathrm{vl}(\pi)$) is the number of peaks (resp., valleys) in π. For example, if $\pi = 453126$, then $\mathrm{des}(\pi) = 2$, $\mathrm{lrmin}(\pi) = \mathrm{lrmax}(\pi) = \mathrm{rlmin}(\pi) = \mathrm{asc}(\pi) = 3$, and $\mathrm{rlmax}(\pi) = \mathrm{dasc}(\pi) = \mathrm{ddes}(\pi) = \mathrm{pk}(\pi) = \mathrm{vl}(\pi) = 1$.

In the literature, there is a research trend focused on determining the distributions of permutation statistics within classes of permutations avoiding certain patterns (see, for example, [1, 2, 3, 4, 6] and the references therein). In particular, Bukata et al. [3] found the distribution of a *single* statistic in the set $\{\mathrm{asc}, \mathrm{des}, \mathrm{dasc}, \mathrm{ddes}, \mathrm{pk}, \mathrm{vl}\}$ over permutations avoiding any two patterns of length 3. Moreover, Han and Kitaev [6] found the joint distribution of six statistics (asc, des, lrmax, lrmin, rlmax, rlmin) over the same classes of permutations. In this paper, we generalize all of these results by finding explicit formulas for joint distribution of (asc, des, lrmax, lrmin, rlmax, rlmin, dasc, ddes, pk, vl) over permutations avoiding any two patterns of length 3. All the multivariate generating functions we have derived are rational.

In what follows, we let g.f. stand for "generating function". We will derive closed form expressions for the following g.f.'s:

$$F_{(\tau,\rho)}(x,p,q,u,v,s,t,y,z,\ell,m) := \sum_{n\ge 0} \sum_{\pi \in S_n(\tau,\rho)} x^n p^{\mathrm{asc}(\pi)} q^{\mathrm{des}(\pi)} u^{\mathrm{lrmax}(\pi)} v^{\mathrm{rlmax}(\pi)} s^{\mathrm{lrmin}(\pi)} t^{\mathrm{rlmin}(\pi)} y^{\mathrm{dasc}(\pi)} z^{\mathrm{ddes}(\pi)} \ell^{\mathrm{pk}(\pi)} m^{\mathrm{vl}(\pi)}$$

for all τ and ρ in S_3.

The following results appear in [8].

Theorem 1 ([8]). *Let $A_n(\tau,\rho)$ be the number of elements in $S_n(\tau,\rho)$. Then,*

(a) $A_n(123,132) = A_n(123,213) = A_n(321,231) = A_n(321,312) = 2^{n-1}$;

(b) $A_n(231,312) = A_n(132,213) = 2^{n-1}$;

(c) $A_n(213,312) = A_n(132,231) = 2^{n-1}$;

(d) $A_n(213,231) = A_n(132,312) = 2^{n-1}$;

(e) $A_n(132,321) = A_n(123,231) = A_n(123,312) = A_n(213,321) = 1 + \binom{n}{2}$;

(f) $A_n(123,321) = \begin{cases} 0 & \text{if } n \ge 5 \\ n & \text{if } n = 1 \text{ or } n = 2 \\ 4 & \text{if } n = 3 \text{ or } n = 4. \end{cases}$

In order to determine the distribution of the statistics over $S_n(\tau,\rho)$, for every

$\tau, \rho \in S_3$, based on the properties of the g.f.'s discussed in [3, 6] (which are established using *trivial bijections*), it is sufficient to examine the distributions of the statistics over the first pair in each of (a)–(e) in Theorem 1 (the case (f) is trivial and does not require consideration).

This paper is organized as follows. In Section 2, we derive all our distribution results, which are summarized in Table 1. To illustrate the applicability of our general formulas, we specialize them to pairs of corresponding statistics that appear in [3]; see Table 2 for references. Finally, in Section 3, we provide concluding remarks.

	(asc, des, lrmax, lrmin, rlmax, rlmin, dasc, ddes, pk, vl)
$S_n(123, 132)$	Theorem 2
$S_n(132, 321)$	Theorem 3
$S_n(231, 312)$	Theorem 4
$S_n(213, 231)$	Theorem 5
$S_n(213, 312)$	Theorem 6

Table 1: G.f.'s for joint distributions of the statistics over $S_n(\tau, \rho)$

	(asc, des)	(dasc, ddes)	(pk, vl)
$S_n(123, 132)$	(5)	(6)	(7)
$S_n(132, 321)$	(13)	(14)	(15)
$S_n(231, 312)$	(22)	(23)	(24)
$S_n(213, 231)$	(30)	(31)	(32)
$S_n(213, 312)$	(34)	(35)	(36)

Table 2: References to formulas for the g.f.'s of the distributions of pairs of statistics over $S_n(\tau, \rho)$

2. Distributions over $S_n(\tau, \rho)$

In this section, we find joint distribution of 10 classical statistics over five classes of pattern-avoiding permutations. Given permutations $\alpha \in S_a$ and $\beta \in S_b$, let $\alpha \oplus \beta \in S_{a+b}$ denote the *direct sum* of α and β, and let $\alpha \ominus \beta \in S_{a+b}$ denote the *skew-sum* of α and β, defined as follows [3]:

$$\alpha \oplus \beta = \begin{cases} \alpha(i), & 1 \le i \le a \\ a + \beta(i-a), & a+1 \le i \le a+b, \end{cases}$$

$$\alpha \ominus \beta = \begin{cases} \alpha(i) + b, & 1 \le i \le a \\ \beta(i-a), & a+1 \le i \le a+b. \end{cases}$$

For example, if $\alpha = 231 \in S_3$ and $\beta = 3241 \in S_4$, then $\alpha \oplus \beta = 2316574$ and $\alpha \ominus \beta = 6753241$.

2.1. Permutations in $S_n(123, 132)$

We first describe the structure of a $(123, 132)$-avoiding permutation. Let $\pi = \pi_1 \cdots \pi_n \in S_n(123, 132)$. If $\pi_k = n, 1 < k \leq n$, then $\pi_1 > \pi_2 > \cdots > \pi_{k-1}$ in order to avoid 123. On the other hand, in order to avoid 132, if $i < k$, then $\pi_i > n - k$. Hence, $\pi_i = n - i$ for $1 \leq i \leq k - 1$, while $\pi_{k+1}\pi_{k+2} \cdots \pi_n$ must be a (123, 132)-avoiding permutation in S_{n-k}. So $\pi = (\alpha \oplus 1) \ominus \beta$, where $\alpha \in S_{k-1}$ is a decreasing permutation and $\beta \in S_{n-k}$ is a (123,132)-avoiding permutation. We use the structure of π to prove the following theorem.

Theorem 2. *For $S_n(123, 132)$, we have*

$$F_{(123,132)}(x, p, q, u, v, s, t, y, z, \ell, m) = \frac{A}{1 - \ell mpqsvx^2 - qsxz - qsvxz + q^2 s^2 vx^2 z^2}, \tag{1}$$

where

$$\begin{aligned} A =& 1 + sx(tuv + pt^2u^2vx - q(\ell mpvx + z + vz)) + q^2 s^3 tuv^2 x^3 (\ell pux(m - z) \\ &+ z(-1 - mptux + z + ptuxz)) + qs^2vx^2(qz^2 + pt^2ux(m(u + v) - u(1 + v)z) \\ &+ tu(-z - v(-1 + \ell p(m - u)x + z))). \end{aligned}$$

Proof. Let $\pi = \pi_1 \cdots \pi_n \in S_n(123, 132)$. If $n = 0$, the empty permutation contributes 1 to $F_{(123,132)}(x, p, q, u, v, s, t, y, z, \ell, m)$. If $n = 1$, the only permutation contributes $xuvst$ to $F_{(123,132)}(x, p, q, u, v, s, t, y, z, \ell, m)$. Now, we consider three cases based on where the element n, $n \geq 2$, appears in π.

`Case (a)`: $\pi_1 = n$. In this case, we let the g.f. for these permutations be

$$G_{(123,132)}(x, p, q, u, v, s, t, y, z, \ell, m) := \sum_{n\geq 2} \sum_{\substack{\pi\in S_n(132,321)\\ \pi_1=n}} x^n p^{\mathrm{asc}(\pi)} q^{\mathrm{des}(\pi)} u^{\mathrm{lrmax}(\pi)} v^{\mathrm{rlmax}(\pi)} s^{\mathrm{lrmin}(\pi)} t^{\mathrm{rlmin}(\pi)} y^{\mathrm{dasc}(\pi)} z^{\mathrm{ddes}(\pi)} \ell^{\mathrm{pk}(\pi)} m^{\mathrm{vl}(\pi)}.$$

`Case (b)`: $\pi_n = n$. In this case, $\pi = (n-1)(n-2) \cdots 1n$, and we have the following three subcases.

`Subcase (1)`: $n = 2$. In this case, the term corresponding to $\pi = 12$ is $x^2 pu^2 vst^2$.
`Subcase (2)`: $n = 3$. In this case, $x^3 pqu^2 vs^2 t^2 m$ corresponds to $\pi = 213$.

`Subcase (3)`: $n \geq 4$. In this case, the corresponding g.f. is

$$\sum_{i\geq 4} x^i pq^{i-2} u^2 vs^{i-1} t^2 z^{i-3} m = \frac{x^4 pq^2 u^2 vs^3 t^2 zm}{1 - xqzs},$$

where positions $1, 2, \dots, (n-3)$ are double descents, and $\pi_{n-2}\pi_{n-1}\pi_n = 21n$ contributes to $\mathrm{vl}(\pi)$. So the g.f. for permutations in Case (b) is

$$x^2pu^2vst^2 + x^3pqu^2vs^2t^2m + \frac{x^4pq^2u^2vs^3t^2zm}{1-xqzs} = x^2pu^2vst^2 + \frac{x^3pqu^2vs^2t^2m}{1-xqzs}.$$

`Case (c)`: $\pi_k = n$ and $1 < k < n$. In this case, we have $\pi_1 > \pi_2 > \cdots > \pi_{k-1}$ and $\pi_i = n-i$ for $1 \le i \le k-1$. Then $\pi = (\alpha \oplus 1) \ominus \beta$, where the structure of $1 \ominus \beta$ is the same as in Case (a). Note that, in this case, $\pi_2 = n$ is not a left-to-right minimum. Next, we consider three subcases based on k.

`Subcase (1)`: $k = 2$. In this case, $\pi_1 = n-1 < \pi_2 = n$, and we obtain the g.f.

$$xpu\ell G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m),$$

where $\pi_1 = n-1 < \pi_2 = n > \pi_3$ contributes to $\mathrm{pk}(\pi)$.

`Subcase (2)`: $k = 3$. In this case, $\pi_1 = n-1 > \pi_2 = n-2 < \pi_3 = n$ and $\pi_2 = n-2 < \pi_3 = n > \pi_4$. We see that the corresponding g.f. in this case is

$$x^2pqusm\ell G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m).$$

`Subcase (3)`: $k \ge 4$. In this case, positions $1, 2, \dots, k-3$ are double descents. We see that the corresponding g.f. in this case is

$$\begin{aligned}&\sum_{i\ge 3} x^i pq^{i-1}us^{i-1}z^{i-2}m\ell G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m)\\ &\quad= \frac{x^3pq^2us^2zm\ell}{1-xqsz} G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m),\end{aligned}$$

where $\pi_{k-2}\pi_{k-1}\pi_k$ contributes to $\mathrm{vl}(\pi)$ and $\pi_{k-1}\pi_k\pi_{k+1}$ contributes to $\mathrm{pk}(\pi)$. So the g.f. for permutations in Case (c) is

$$\begin{aligned}&xpu\ell G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m) + x^2pqusm\ell G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m)\\ &\quad+ \frac{x^3pq^2us^2zm\ell}{1-xqsz} G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m)\\ &=xpu\ell G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m)\\ &\quad+ \frac{x^2pqusm\ell}{1-xqsz} G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m).\end{aligned}$$

Summarizing Cases (a)–(c) yields

$$\begin{aligned}F_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m) =& 1 + xuvst + G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m)\\ &+ xpu\ell G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m)\\ &+ \frac{x^2pqusm\ell}{1-xqsz} G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m)\\ &+ x^2pu^2vst^2 + \frac{x^3pqu^2vs^2t^2m}{1-xqzs}.\end{aligned} \tag{2}$$

Next we compute $G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m)$, which appears in Case (a). If $\pi_1 = n$, then $\pi = 1 \ominus \beta$, and the element n is the only left-to-right maximum, a left-to-right minimum, and a right-to-left maximum, and we do not need to consider left-to-right maxima for β. Note that, in this case, $n-1$ is not a left-to-right minimum. If $n = 2$, then $\pi = 21$, and the respective term in the g.f. is $x^2quv^2s^2t$. Next, we consider three subcases based on where the element $n-1$, $n \geq 3$, appears in π.

`Subcase (1)`: $\pi_2 = n-1$. In this case, the structure of β is the same as in Case (a), and we obtain the g.f.

$$xquvszG_{(123,132)}(x,p,q,1,v,s,t,y,z,\ell,m),$$

where $\pi_1 > \pi_2 > \pi_3$ contributes to $\mathrm{ddes}(\pi)$.

`Subcase (2)`: $\pi_{k'} = n-1$. In this case, $\beta = (\zeta \oplus 1) \ominus \gamma$, where $\zeta = (k'-1)\cdots 1$, and the structure of $1 \ominus \gamma$ is the same as in Case (a). We then consider the following two subsubcases.

`Subsubcase (i)`: $k' = 3$. In this case, $\pi_1 = n > \pi_2 = n-2 < \pi_3 = n-1$ contributes to $\mathrm{vl}(\pi)$ and $\pi_2 = n-2 < \pi_3 = n-1 > \pi_4$ contributes to $\mathrm{pk}(\pi)$. So the term corresponding to π is

$$x^2pquvsm\ell G_{(123,132)}(x,p,q,1,v,s,t,y,z,\ell,m).$$

`Subsubcase (ii)`: $k' \geq 4$. In this case, positions $1,2,\ldots,(k'-3)$ are double descents, so the term corresponding to π is

$$\sum_{i\geq 3} x^ipq^{i-1}uvs^{i-1}z^{i-2}m\ell G_{(123,132)}(x,p,q,1,v,s,t,y,z,\ell,m)$$
$$= \frac{x^3pq^2uvs^2zm\ell}{1-xqsz} G_{(123,132)}(x,p,q,1,v,s,t,y,z,\ell,m).$$

So the g.f. for permutations in Subcase (2) is given by

$$\left(x^2pquvsm\ell + \frac{x^3pq^2uvs^2zm\ell}{1-xqsz}\right) G_{(123,132)}(x,p,q,1,v,s,t,y,z,\ell,m)$$
$$= \frac{x^2pquvsm\ell}{1-xqsz} G_{(123,132)}(x,p,q,1,v,s,t,y,z,\ell,m).$$

`Subcase (3)`: $\pi_n = n-1$. In this case, the structure is the same as in Case (c). The corresponding g.f. is

$$\sum_{i\geq 3} x^ipq^{i-2}uv^2s^{i-1}t^2z^{i-3}m = \frac{x^3pquv^2s^2t^2m}{1-xqzs}.$$

Summarizing Subcases (1)–(3) yields

$$\begin{aligned}G_{(123,132)}(x,p,q,u,v,s,t,y,z,\ell,m) =& xquvszG_{(123,132)}(x,p,q,1,v,s,t,y,z,\ell,m)\\&+\frac{x^2pquvsm\ell}{1-xqsz}G_{(123,132)}(x,p,q,1,v,s,t,y,z,\ell,m)\\&+\frac{x^3pquv^2s^2t^2m}{1-xqzs}+x^2quv^2s^2t.\end{aligned}\tag{3}$$

Letting $u=1$ in Equation (3), we obtain

$$\begin{aligned}G_{(123,132)}(x,p,q,1,v,s,t,y,z,\ell,m) =& xqvszG_{(123,132)}(x,p,q,1,v,s,t,y,z,\ell,m)\\&+\frac{x^2pqvsm\ell}{1-xqsz}G_{(123,132)}(x,p,q,1,v,s,t,y,z,\ell,m)\\&+\frac{x^3pqv^2s^2t^2m}{1-xqzs}+x^2qv^2s^2t.\end{aligned}\tag{4}$$

By simultaneously solving (2)–(4) we obtain the desired result. □

Corollary 1. *Let $u=v=s=t=1$. By setting four of the six variables p, q, y, z, ℓ, and m equal to 1 in Equation (1), we obtain the joint distributions of* $(\mathrm{asc},\mathrm{des})$, $(\mathrm{dasc},\mathrm{ddes})$, *and* $(\mathrm{pk},\mathrm{vl})$ *over* $S_n(123,132)$*:*

$$\sum_{n\geq 0}\sum_{\pi\in S_n(123,132)}x^np^{\mathrm{asc}(\pi)}q^{\mathrm{des}(\pi)}=\frac{1+x-2qx+px^2-qx^2-pqx^2+q^2x^2}{1-2qx-pqx^2+q^2x^2},\tag{5}$$

$$\sum_{n\geq 0}\sum_{\pi\in S_n(123,132)}x^ny^{\mathrm{dasc}(\pi)}z^{\mathrm{ddes}(\pi)}=\frac{1+x^2+x^3-xz-x^2z-x^3z}{1-x-xz},\tag{6}$$

$$\sum_{n\geq 0}\sum_{\pi\in S_n(123,132)}x^n\ell^{\mathrm{pk}(\pi)}m^{\mathrm{vl}(\pi)}=\frac{1-x+(1-\ell m)x^2-(1-m)x^3\,(2-\ell+(\ell-1)x)}{1-2x+x^2-\ell mx^2}.\tag{7}$$

2.2. Permutations in $S_n(132,321)$

We first describe the structure of a $(132,321)$-avoiding permutation. Let $\pi=\pi_1\cdots\pi_n\in S_n(132,321)$. If $\pi_1=n$, then $\pi=n12\cdots(n-1)$. If $\pi_k=n$ and $1<k<n$, then $\pi_{k+1}<\pi_{k+2}<\cdots<\pi_n$ in order to avoid 321. On the other hand, in order to avoid 132, if $1\leq i\leq k-1$, then $\pi_i=n-k+i$. If $\pi_n=n$, then $\pi_1\pi_2\cdots\pi_{n-1}\in S_{n-1}(132,321)$. So $\pi=(\alpha\oplus 1)\ominus\beta$, where $\alpha\oplus 1\in S_k$ and $\beta\in S_{n-k}$ are two increasing (132, 321)-avoiding permutations. We use the structure of π to prove the following theorem.

Theorem 3. *For $S_n(132,321)$, we have*

$$F_{(132,321)}(x,p,q,u,v,s,t,y,z,\ell,m)=\frac{A}{(1-ptxy)(1-puxy)(1-ptuxy)},\tag{8}$$

where

$$\begin{aligned}A =&1 + stuvx + pst^2u^2vx^2 + qs^2tuv^2x^2 + mpqs^2t^2u^2vx^3 + mpqs^2t^2uv^2x^3 + \ell pqs^2tu^2v^2x^3\\&+ \ell mp^2qs^2t^2u^3vx^4 + \ell mp^2qs^2t^2u^2v^2x^4 - ptxy - puxy - pqs^2tu^2v^2x^3y - pst^2uvx^2y\\&- pstu^2vx^2y - pst^2u^2vx^2y - p^2st^3u^2vx^3y - p^2st^2u^3vx^3y - pqs^2t^2uv^2x^3y - ptuxy\\&- pqs^2t^2u^2v^2x^3y - mp^2qs^2t^2u^3vx^4y - \ell p^2qs^2t^2u^2v^2x^4y - mp^2qs^2t^2u^2v^2x^4y\\&- mp^2qs^2t^3u^2v^2x^4y - \ell p^2qs^2t^2u^3v^2x^4y - \ell mp^3qs^2t^3u^3v^2x^5y + p^2tux^2y^2 + p^2t^2ux^2y^2\\&+ p^2tu^2x^2y^2 + p^2st^2u^2vx^3y^2 + p^2st^3u^2vx^3y^2 + p^2st^2u^3vx^3y^2 + p^3st^3u^3vx^4y^2\\&+ p^2qs^2t^2u^2v^2x^4y^2 + p^2qs^2t^3u^2v^2x^4y^2 + p^2qs^2t^2u^3v^2x^4y^2 + \ell p^3qs^2t^3u^3v^2x^5y^2\\&+ mp^3qs^2t^3u^3v^2x^5y^2 - p^3t^2u^2x^3y^3 - p^3st^3u^3vx^4y^3 - p^3qs^2t^3u^3v^2x^5y^3.\end{aligned}$$

Proof. Let $\pi = \pi_1 \cdots \pi_n \in S_n(132, 321)$. If $n \le 1$, then the corresponding g.f. is $1 + xuvst$. For $n \ge 2$, we consider the following three cases.
`Case (a)`: $\pi_1 = n$. In this case, $\pi = n12\cdots(n-1)$. The element n is the only left-to-right maximum, a left-to-right minimum, and a right-to-left maximum. So $\mathrm{lrmax}(\pi) = 1$, and we consider the following three subcases.

`Subcase (1)`: $n = 2$. In this case, $\pi = 21$ and the g.f. is $x^2quv^2s^2t$.

`Subcase (2)`: $n = 3$. In this case, $\pi = 312$ and 1 is a valley, so the g.f. is $x^3pquv^2s^2t^2m$.

`Subcase (3)`: $n \ge 4$. In this case, 1 is a valley and $2, 3, \ldots, (n-2)$ are double ascents, so the g.f. in this case is

$$\sum_{i\ge4} x^ip^{i-2}quv^2s^2t^{i-1}my^{i-3} = \frac{x^4p^2quv^2s^2t^3my}{1 - xpty}.$$

Hence, the g.f. of permutations with $\pi_1 = n$ is given by

$$x^2quv^2s^2t + x^3pquv^2s^2t^2m + \frac{x^4p^2quv^2s^2t^3my}{1 - xpty} = x^2quv^2s^2t + \frac{x^3pquv^2s^2t^2m}{1 - xpty}.$$

`Case (b)`: $\pi_k = n$ and $1 < k < n$. In this case, $\pi = (\alpha \oplus 1) \ominus \beta$, where $\alpha \in S_{k-1}$ and $\beta \in S_{n-k}$ are two increasing (132, 321)-avoiding permutations. Note that $\pi_{k-1}\pi_k$ is an ascent, and when $k \ge 3$, we have that $1, 2, \ldots, (k-2)$ are double ascents. The g.f. for the permutation $\alpha \in S_{k-1}$ is

$$xpus\ell + \sum_{i\ge2} x^ip^iu^is\ell y^{i-1} = \frac{xpus\ell}{1 - xpuy}.$$

By Case (a), we obtain that the g.f. for $1 \ominus \beta \in S_{n-k+1}$ is

$$x^2quv^2t + \frac{x^3pquv^2t^2m}{1 - xpty},$$

noting that, in this case, n is not a left-to-right minimum. So the g.f. of the permutations in Case (b) is

$$\frac{xpus\ell}{1-xpuy}\left(x^2quv^2t+\frac{x^3pquv^2t^2m}{1-xpty}\right).$$

`Case (c)`: $\pi_n = n$. In this case, we let the g.f. for these permutations be

$$G_{(132,321)}(x,p,q,u,v,s,t,y,z,\ell,m) := \sum_{n\geq 2}\sum_{\substack{\pi\in\\ S_n(132,321)\\ \pi_n=n}} x^n p^{\mathrm{asc}(\pi)} q^{\mathrm{des}(\pi)} u^{\mathrm{lrmax}(\pi)} v^{\mathrm{rlmax}(\pi)} s^{\mathrm{lrmin}(\pi)} t^{\mathrm{rlmin}(\pi)} y^{\mathrm{dasc}(\pi)} z^{\mathrm{ddes}(\pi)} \ell^{\mathrm{pk}(\pi)} m^{\mathrm{vl}(\pi)}.$$

Considering Cases (a)–(c), we conclude that

$$\begin{aligned} F_{(132,321)}(x,p,q,u,v,s,t,y,z,\ell,m) =& 1+xuvst+x^2quv^2s^2t+\frac{x^3pquv^2s^2t^2m}{1-xpty}\\ &+\frac{xpus\ell}{1-xpuy}\left(x^2quv^2t+\frac{x^3pquv^2t^2m}{1-xpty}\right)\\ &+G_{(132,321)}(x,p,q,u,v,s,t,y,z,\ell,m). \end{aligned} \tag{9}$$

Next we evaluate $G_{(132,321)}(x,p,q,u,v,s,t,y,z,\ell,m)$, which appears in Case (c). If $\pi_n = n$, then $\pi = \alpha \oplus 1$ and $\mathrm{rlmax}(\pi) = 1$. Any non-empty permutation in $S_{n-1}(132, 321)$ is possible for α and we do not need to consider right-to-left maxima of α. We now divide the permutations into three classes depending on the position of $n-1$.

`Class A`: $\pi_1 = n-1$. In this case, we have the following three subclasses.

`Subclass (1)`: $n = 2$. In this case, $\pi = 12$, and the g.f. is $x^2pu^2vst^2$.

`Subclass (2)`: $n = 3$. In this case, $\pi = 213$ and 1 is a valley, so the g.f. is $x^3pqu^2vs^2t^2m$.

`Subclass (3)`: $n \geq 4$. In this case, 1 is a valley and $2, 3, \ldots, (n-2)$ are double ascents, so the g.f. in this case is

$$\sum_{i\geq 4} x^i p^{i-2} qu^2vs^2t^{i-1}my^{i-3} = \frac{x^4p^2qu^2vs^2t^3my}{1-xpty}.$$

Hence, the g.f. for the permutations in Class A is

$$x^2pu^2vst^2+x^3pqu^2vs^2t^2m+\frac{x^4p^2qu^2vs^2t^3my}{1-xpty} = x^2pu^2vst^2+\frac{x^3pqu^2vs^2t^2m}{1-xpty}.$$

`Class B`: $\pi_{k'} = n-1$ and $1 < k' < n-1$. In this case, we have $\pi = ((\zeta\oplus 1)\ominus\gamma)\oplus 1$, where ζ and γ are non-empty increasing permutations. Now, $\pi_{k'-1} < \pi_{k'} = n-1 > \pi_{k'+1}$ and $1, \ldots, (k'-2), (k'+1), \ldots, (n-2)$ are double ascents. So the g.f. for ζ is

$$\sum_{i\geq 1} x^i p^i u^i s\ell y^{i-1} = \frac{xpus\ell}{1-xpuy}.$$

The g.f. for γ is similar to that for Class A. Note that $n-1$ is not a left-to-right minimum, so the g.f. for the permutations in this case is

$$\frac{xpus\ell}{1-xpuy}\cdot\frac{x^3pqu^2vst^2m}{1-xpty}=\frac{x^4p^2qu^3vs^2t^2m\ell}{(1-xpuy)(1-xpty)}.$$

`Class C`: $\pi_{n-1}=n-1$. In this case, we let the g.f. for these permutations be
$GG_{(132,321)}(x,p,q,u,v,s,t,y,z,\ell,m):=$
$$\sum_{n\geq 3}\sum_{\substack{\pi\in\\ S_n(132,321)\\ \pi_n=n\\ \pi_{n-1}=n-1}} x^n p^{\mathrm{asc}(\pi)}q^{\mathrm{des}(\pi)}u^{\mathrm{lrmax}(\pi)}v^{\mathrm{rlmax}(\pi)}s^{\mathrm{lrmin}(\pi)}t^{\mathrm{rlmin}(\pi)}y^{\mathrm{dasc}(\pi)}z^{\mathrm{ddes}(\pi)}\ell^{\mathrm{pk}(\pi)}m^{\mathrm{vl}(\pi)}.$$

Considering Classes A–C, we conclude that

$$\begin{aligned}G_{(132,321)}(x,p,q,u,v,s,t,y,z,\ell,m)=&x^2pu^2vst^2+\frac{x^3pqu^2vs^2t^2m}{1-xpty}\\&+\frac{x^4p^2qu^3vs^2t^2m\ell}{(1-xpuy)(1-xpty)}+GG_{(132,321)}(x,p,q,u,v,s,t,y,z,\ell,m).\end{aligned}\tag{10}$$

Next, we evaluate $GG_{(132,321)}(x,p,q,u,v,s,t,y,z,\ell,m)$, which appears in Class C. Now $\pi=\alpha'\oplus 12$, and any non-empty permutation in $S_{n-2}(132,321)$ is possible for α'. We do not need to consider right-to-left maxima of α', since $\mathrm{rlmax}(\pi)=1$. We divide the permutations into three subclasses depending on the position of $n-2$.

`Subclass (1)`: $\pi_1=n-2$. In this case, we have the following two subsubcases.

`Subsubcase (i)`: $n=3$. In this case, $\pi=123$ and 1 is a double ascent, so the g.f. is $x^3p^2u^3vst^3y$.

`Subsubcase (ii)`: $n\geq 4$. In this case, $\pi=(n-2)12\cdots(n-3)(n-1)n$, so 1 is a valley and $2,3,\ldots,(n-2)$ are double ascents. Thus, the g.f. in this case is

$$\sum_{i\geq 4}x^ip^{i-2}qu^3vs^2t^{i-1}my^{i-3}=\frac{x^4p^2qu^3vs^2t^3my}{1-xpty}.$$

So the g.f. for the permutations in Subclass (1) is

$$x^3p^2u^3vst^3y+\frac{x^4p^2qu^3vs^2t^3my}{1-xpty}.$$

`Subclass (2)`: $\pi_{k''}=n-2$ and $1<k'<n-2$. In this case, we have $\pi=((\zeta'\oplus 1)\ominus\gamma')\oplus 12$, where $\zeta'\in S_{k''-1}$ and $\gamma'\in S_{n-k''-2}$ are non-empty increasing permutations. Now, $\pi_{k''-1}<\pi_{k''}=n-2>\pi_{k''+1}$ and $1,\ldots,(k''-2),(k''+1),\ldots,(n-2)$ are double ascents. So the g.f. for ζ' is

$$\sum_{i\geq 1}x^ip^iu^is\ell y^{i-2}=\frac{xpus\ell}{1-xpuy}.$$

The g.f. for γ' is $\frac{x^4p^2qu^3vst^3my}{1-xpty}$. Note that $n-2$ is not a left-to-right minimum, so the g.f. for the permutations in this case is

$$\frac{xpus\ell}{1-xpuy}\frac{x^4p^2qu^3vst^3my}{1-xpty}=\frac{x^5p^3qu^4vs^2t^3m\ell y}{(1-xpuy)(1-xpty)}.$$

Subclass (3): $\pi_{n-2}=n-2$. In this case, the g.f. for these permutations is

$$xpuvtyGG_{(132,321)}(x,p,q,u,1,s,t,y,z,\ell,m),$$

where n gives $xpuvty(\pi_{n-2}=n-2<\pi_{n-1}=n-1<\pi_n=n)$ and we do not need to consider right-to-left maxima of $\pi_1\cdots\pi_{n-1}$.

Considering Subclasses (1)–(3), we conclude that

$$\begin{aligned}GG_{(132,321)}(x,p,q,u,v,s,t,y,z,\ell,m)=&xpuvtyGG_{(132,321)}(x,p,q,u,1,s,t,y,z,\ell,m)\\&+x^3p^2u^3vst^3y+\frac{x^4p^2qu^3vs^2t^3my}{1-xpty}\\&+\frac{x^5p^3qu^4vs^2t^3m\ell y}{(1-xpuy)(1-xpty)}.\end{aligned}\tag{11}$$

Letting $v=1$ in Equation (11), we get

$$\begin{aligned}GG_{(132,321)}(x,p,q,u,1,s,t,y,z,\ell,m)=&xputyGG_{(132,321)}(x,p,q,u,1,s,t,y,z,\ell,m)\\&+x^3p^2u^3st^3y+\frac{x^4p^2qu^3s^2t^3my}{1-xpty}\\&+\frac{x^5p^3qu^4s^2t^3m\ell y}{(1-xpuy)(1-xpty)}.\end{aligned}\tag{12}$$

By simultaneously solving (9)–(12), we obtain the desired result. □

Corollary 2. *Let $u=v=s=t=1$. Then, by setting four out of the six variables p, q, y, z, ℓ, and m equal to one individually in Equation* (8)*, we obtain the joint distributions of* (asc, des)*,* (dasc, ddes)*, and* (pk, vl) *over $S_n(132,321)$:*

$$\sum_{n\geq 0}\sum_{\pi\in S_n(132,321)}x^np^{\mathrm{asc}(\pi)}q^{\mathrm{des}(\pi)}=\frac{1+x-3px-2px^2+3p^2x^2+qx^2+p^2x^3-p^3x^3}{(1-px)^3},\tag{13}$$

$$\sum_{n\geq 0}\sum_{\pi\in S_n(132,321)}x^ny^{\mathrm{dasc}(\pi)}z^{\mathrm{ddes}(\pi)}=\frac{B}{(1-xy)^3},\tag{14}$$

$$\sum_{n\geq 0}\sum_{\pi\in S_n(132,321)}x^n\ell^{\mathrm{pk}(\pi)}m^{\mathrm{vl}(\pi)}=\frac{C}{(1-x)^3},\tag{15}$$

where

$B=1+x(1-3y)+x^2(2-3y+3y^2)+x^3(3-5y+3y^2-y^3)+x^4(2-y)(1-y)^2-x^5(1-y)^2y$,

and $C=1-2x+2x^2-(3-\ell-2m)x^3+(3-2\ell)(1-m)x^4-(1-\ell-m+\ell m)x^5$.

2.3. Permutations in $S_n(231, 312)$

We first describe the structure of a $(231, 312)$-avoiding permutation. Let $\pi = \pi_1 \cdots \pi_n \in S_n(231, 312)$. If $\pi_1 = n$, then $\pi = n(n-1)\cdots 21$. If $\pi_k = n$ and $1 < k < n$, then $\pi_{k+1} > \pi_{k+2} > \cdots > \pi_n$ in order to avoid 312. On the other hand, in order to avoid 231, $\pi_i = n + k - i$ if $k + 1 \le i \le n$, and $\pi_1\pi_2\cdots\pi_{k-1}$ must be a permutation in $S_{k-1}(231, 312)$. If $\pi_n = n$, then $\pi_1\pi_2\cdots\pi_{n-1}$ must be a permutation in $S_{n-1}(231, 312)$. Namely, for $\pi \in S_n(231, 312)$, its structure is $\pi = \alpha \oplus (1 \ominus \beta)$, where $\alpha \in S_{k-1}(231, 312)$ and $1 \ominus \beta \in S_{n-k+1}$ is a decreasing (231, 312)-avoiding permutation. We use the structure of π to prove the following theorem.

Theorem 4. *For $S_n(231, 312)$, we have*

$$F_{(231,312)}(x, p, q, u, v, s, t, y, z, \ell, m) = \frac{A}{B} \tag{16}$$

where

$$\begin{aligned} A =&(1 - qvxz)(\ell mpqtux^2 - (1 - ptuxy)(1 - qxz)) + q^2s^3tuv^2x^3z(mptux((1 - qxz)(1 - qvxz)\\ &+ \ell qx(-1 + v + z - qvxz^2)) + (1 - qxz)((1 - z)(1 - qvxz) - ptux(z(1 + \ell qvx - qvxz)\\ &+ y(1 - z)(1 - qvxz)))) + sx(q(1 + v)z(1 - qxz)(1 - qvxz) + v(-1 + qx(1 - py)z\\ &+ pt^2u^2vx((1 - y)(1 - qxz)(-1 + qvxz) + \ell qx(m - mqvxz + v(-1 + qxz)))\\ &- tu(-1 + qvxz)(pqxz(-\ell mqx + y(-1 + qxz)) + pq^2x^2z(-\ell m + yz))))\\ &+ qs^2vx^2(-qz^2(1 - qxz)(1 - qvxz) + tu(-1 + qvxz)(v(1 - z)(1 - qxz)\\ &+ z(-1 + qx(1 - py)z + pq^2x^2z(-\ell m + yz))) + pt^2u^2x(-((-1 + qxz)(z - yz\\ &+ qv^2x(\ell + y(-1 + z) - z)z + v(y + z + \ell qxz - yz - qxz^2 + qxyz^2)))\\ &+ m(-1 - (-1 + \ell)q(1 + v)xz + q^2vx^2z(-z + \ell(1 - v(1 - z) + z))))), \end{aligned}$$

and

$$B = (1 - qsxz)(1 - qvxz)(1 - qsvxz)(\ell mqtux^2 - (1 - ptuxy)(1 - qxz)).$$

Proof. Let $\pi = \pi_1 \cdots \pi_n \in S_n(231, 312)$. If $n \le 1$, then the corresponding g.f. is $1 + xuvst$. For $n \ge 2$, the permutations are divided into three cases depending on the position of n.

Case (a): $\pi_1 = n$. In this case, $\pi = n(n-1)\cdots 21$. If $n = 2$, then the corresponding g.f. is $x^2quv^2s^2t$. If $n \ge 3$, then positions $1, 2, \ldots, (n-2)$ are double descents, so the corresponding g.f. is

$$\sum_{i\ge 3} x^i q^{i-1} uv^i s^i tz^{i-2} = \frac{x^3q^2uv^3s^3tz}{1 - xqvsz}.$$

Therefore, the g.f. in Case (a) is

$$x^2quv^2s^2t + \frac{x^3q^2uv^3s^3tz}{1 - xqvsz} = \frac{x^2quv^2s^2t}{1 - xqvsz}.$$

`Case (b)`: $\pi_k = n$ and $1 < k < n$. In this case, we let the g.f. be

$$G_{(231,312)}(x,p,q,u,v,s,t,y,z,\ell,m) := \sum_{n\geq 3} \sum_{\substack{\pi\in\\ S_n(231,312)\\ \pi_k=n}} x^n p^{\mathrm{asc}(\pi)} q^{\mathrm{des}(\pi)} u^{\mathrm{lrmax}(\pi)} v^{\mathrm{rlmax}(\pi)} s^{\mathrm{lrmin}(\pi)} t^{\mathrm{rlmin}(\pi)} y^{\mathrm{dasc}(\pi)} z^{\mathrm{ddes}(\pi)} \ell^{\mathrm{pk}(\pi)} m^{\mathrm{vl}(\pi)}.$$

`Case (c)`: $\pi_n = n$. In this case, we let the g.f. for these permutations be

$$GG_{(231,312)}(x,p,q,u,v,s,t,y,z,\ell,m) := \sum_{n\geq 2} \sum_{\substack{\pi\in\\ S_n(231,312)\\ \pi_n=n}} x^n p^{\mathrm{asc}(\pi)} q^{\mathrm{des}(\pi)} u^{\mathrm{lrmax}(\pi)} v^{\mathrm{rlmax}(\pi)} s^{\mathrm{lrmin}(\pi)} t^{\mathrm{rlmin}(\pi)} y^{\mathrm{dasc}(\pi)} z^{\mathrm{ddes}(\pi)} \ell^{\mathrm{pk}(\pi)} m^{\mathrm{vl}(\pi)}.$$

Combining Cases (a)–(c), we have

$$\begin{aligned} F_{(231,312)}(x,p,q,u,v,s,t,y,z,\ell,m) =& GG_{(231,312)}(x,p,q,u,v,s,t,y,z,\ell,m) \\ &+ G_{(231,312)}(x,p,q,u,v,s,t,y,z,\ell,m) \\ &+ 1 + xuvst + \frac{x^2quv^2s^2t}{1-xqvsz}. \end{aligned} \tag{17}$$

Next we evaluate $G_{(231,312)}(x,p,q,u,v,s,t,y,z,\ell,m)$, which appears in Case (b). If $\pi_k = n$ and $1 < k < n$, then $\pi = \alpha \oplus (1 \ominus \beta)$, where $\alpha \in S_{k-1}(231,312)$ and $1 \ominus \beta \in S_{n-k+1}$ is a decreasing non-empty (231, 312)-avoiding permutation. Note that we do not need to consider left-to-right minima for $1\ominus\beta$ and right-to-left maxima for α. For $n \geq 3$, we divide the permutations into three subcases depending on the position of $k-1$ which is the largest element of α.

`Subcase (1)`: $\pi_1 = k-1$. In this case, we have the following two subsubcases.

`Subsubcase (i)`: $k = 2$. In this case, we have $\pi = 1n\cdots 2$. For $1\ominus\beta = (n-1)\cdots 1$, similarly to Case (a), we see that the corresponding g.f. is

$$xpustl\frac{x^2quv^2t}{1-xqvz}.$$

`Subsubcase (ii)`: $k \neq 2$. In this case, we have $\pi = (k-1)\cdots 1n\cdots k$. Similarly to Case (a), we see that the corresponding g.f. is

$$\ell mp\frac{x^2qus^2t}{1-xqsz}\frac{x^2quv^2t}{1-xqvz} = \frac{x^4pq^2u^2v^2s^2t^2\ell m}{(1-xqsz)(1-xqvz)},$$

where $\pi_{k-2}\pi_{k-1}\pi_k = 21n$ contributes to $\mathrm{vl}(\pi)$, $\pi_{k-1}\pi_k\pi_{k+1} = 1n(n-1)$ contributes to $\mathrm{pk}(\pi)$, and $\pi_{k-1}\pi_k = 1n$ contributes to $\mathrm{asc}(\pi)$.

So the g.f. in Subcase (1) is

$$xpust\ell\frac{x^2quv^2t}{1-xqvz} + \frac{x^4pq^2u^2v^2s^2t^2\ell m}{(1-xqsz)(1-xqvz)}.$$

`Subcase (2)`: $\pi_{k'} = k-1$ and $1 < k' < k-1$. In this case, $\alpha = \gamma \oplus (1 \ominus \zeta)$, where $\gamma \in S_{k'-1}(231, 312)$ and $1 \ominus \zeta \in S_{k-k'}$ is a decreasing non-empty (231, 312)-avoiding permutation. For $\alpha = \gamma \oplus (1 \ominus \zeta)$, because the structure is the same as in Case (b), we obtain the g.f. $G_{(231,312)}(x, p, q, u, 1, s, t, y, z, \ell, m)$. For $1 \ominus \beta$, similarly to Case (a), we see that the corresponding g.f. in Subcase (2) is

$$\ell mpG_{(231,312)}(x, p, q, u, 1, s, t, y, z, \ell, m)\frac{x^2quv^2t}{1-xqvz},$$

where $\pi_{k-2}\pi_{k-1}\pi_k$ contributes to $\mathrm{vl}(\pi)$, $\pi_{k-1}\pi_k\pi_{k+1}$ contributes to $\mathrm{pk}(\pi)$, and $\pi_{k-1}\pi_k$ contributes to $\mathrm{asc}(\pi)$.

`Subcase (3)`: $\pi_{k-1} = k-1$. In this case, the structure of α is the same as in Case (c), so the g.f. in Subcase (3) is

$$py\ell GG_{(231,312)}(x, p, q, u, 1, s, t, y, z, \ell, m)\frac{x^2quv^2t}{1-xqvz},$$

where $\pi_{k-2}\pi_{k-1}\pi_k$ contributes to $\mathrm{dasc}(\pi)$, $\pi_{k-1}\pi_k\pi_{k+1}$ contributes to $\mathrm{pk}(\pi)$, and $\pi_{k-1}\pi_k$ contributes to $\mathrm{asc}(\pi)$.

Combining Subcases (1)–(3), we have

$$\begin{aligned}G_{(231,312)}(x, p, q, u, v, s, t, y, z, \ell, m) =& py\ell GG_{(231,312)}(x, p, q, u, 1, s, t, y, z, \ell, m)\frac{x^2quv^2t}{1-xqvz}\\ &+ \ell mpG_{(231,312)}(x, p, q, u, 1, s, t, y, z, \ell, m)\frac{x^2quv^2t}{1-xqvz}\\ &+ xpust\ell\frac{x^2quv^2t}{1-xqvz} + \frac{x^4pq^2u^2v^2s^2t^2\ell m}{(1-xqsz)(1-xqvz)}. \qquad (18)\end{aligned}$$

Letting $v = 1$ in Equation (18), we get

$$\begin{aligned}G_{(231,312)}(x, p, q, u, 1, s, t, y, z, \ell, m) =& py\ell GG_{(231,312)}(x, p, q, u, 1, s, t, y, z, \ell, m)\frac{x^2qut}{1-xqz}\\ &+ \ell mpG_{(231,312)}(x, p, q, u, 1, s, t, y, z, \ell, m)\frac{x^2qut}{1-xqz}\\ &+ xpust\ell\frac{x^2qut}{1-xqz} + \frac{x^4pq^2u^2s^2t^2\ell m}{(1-xqsz)(1-xqz)}. \qquad (19)\end{aligned}$$

Next we evaluate $GG_{(231,312)}(x, p, q, u, v, s, t, y, z, \ell, m)$, which appears in Case (c). If $\pi_n = n$, then $\pi = \alpha \oplus 1$, where $\alpha \in S_{n-1}(231, 312)$ is any non-empty (231, 312)-avoiding permutation. Note that we do not need to consider right-to-left maxima for α. For $n \geq 2$, we divide the permutations into three classes depending on the position of $n-1$, which is the largest element of α.

`Class A`: $\pi_1 = n-1$. In this case, we have the following subclasses.

`Subclass 1`: $n = 2$. In this case, we have $\pi = 12$ and the g.f. is $x^2pu^2vst^2$.

`Subclass 2`: $n \neq 2$. In this case, we have $\alpha = (n-1)\cdots 1$. Similarly to Case (a), we see that the corresponding g.f. is

$$xpuvtm\frac{x^2qus^2t}{1-xqsz},$$

where $\pi_{n-2}\pi_{n-1}\pi_n = 21n$ contributes to $\mathrm{vl}(\pi)$. So the g.f. for Class A is

$$x^2pu^2vst^2 + xpuvtm\frac{x^2qus^2t}{1-xqsz}.$$

`Class B`: $\pi_{k'} = n-1$ and $1 < k' < n-1$. In this case, $\alpha = \gamma \oplus (1 \ominus \zeta)$, where $\gamma \in S_{k'-1}(231,312)$ and $1\ominus\zeta \in S_{n-k'}$ is a decreasing non-empty (231, 312)-avoiding permutation. For $\alpha = \gamma \oplus (1 \ominus \zeta)$, because the structure is the same as in Case (b), the g.f. is $G_{(231,312)}(x,p,q,u,1,s,t,y,z,\ell,m)$. We see that the corresponding g.f. in Class B is

$$xpuvtmG_{(231,312)}(x,p,q,u,1,s,t,y,z,\ell,m),$$

where $\pi_{n-2}\pi_{n-1}\pi_n$ contributes to $\mathrm{vl}(\pi)$.

`Class C`: $\pi_{n-1} = n-1$. In this case, the structure of α is the same as in Case (c), so the g.f. for Class C is

$$xpuvtyGG_{(231,312)}(x,p,q,u,1,s,t,y,z,\ell,m),$$

where $\pi_{n-2}\pi_{n-1}\pi_n$ contributes to $\mathrm{dasc}(\pi)$. Classes A–C together give

$$\begin{aligned} GG_{(231,312)}&(x,p,q,u,v,s,t,y,z,\ell,m) \\ &= xpuvtyGG_{(231,312)}(x,p,q,u,1,s,t,y,z,\ell,m) \\ &\quad + xpuvtmG_{(231,312)}(x,p,q,u,1,s,t,y,z,\ell,m) \\ &\quad + x^2pu^2vst^2 + xpuvtm\frac{x^2qus^2t}{1-xqsz}. \end{aligned} \tag{20}$$

Letting $v = 1$ in Equation (20), we get

$$\begin{aligned} GG_{(231,312)}(x,p,q,u,1,s,t,y,z,\ell,m) =& xputyGG_{(231,312)}(x,p,q,u,1,s,t,y,z,\ell,m) \\ &+ xputmG_{(231,312)}(x,p,q,u,1,s,t,y,z,\ell,m) \\ &+ x^2pu^2st^2 + xputm\frac{x^2qus^2t}{1-xqsz}. \end{aligned} \tag{21}$$

Solving Equations (17)–(21) simultaneously, we obtain (16). □

Corollary 3. *Let $u = v = s = t = 1$. Then, by setting four out of the six variables p, q, y, z, ℓ, and m equal to one individually in Equation* (16)*, we obtain the joint* *distributions of* (asc, des), (dasc, ddes), *and* (pk, vl) *over* $S_n(231, 312)$*:*

$$\sum_{n\geq 0} \sum_{\pi\in S_n(231,312)} x^n p^{\mathrm{asc}(\pi)} q^{\mathrm{des}(\pi)} = \frac{C}{(1-qx)(1-qx(1+x)-px(1-qx))}, \tag{22}$$

$$\sum_{n\geq 0} \sum_{\pi\in S_n(231,312)} x^n y^{\mathrm{dasc}(\pi)} z^{\mathrm{ddes}(\pi)} = \frac{1+x^2(1+x)(1-y)(1-z)+x(1-y-z)}{1-x(y+z)-x^2(1-yz)}, \tag{23}$$

$$\sum_{n\geq 0} \sum_{\pi\in S_n(231,312)} x^n \ell^{\mathrm{pk}(\pi)} m^{\mathrm{vl}(\pi)} = \frac{D}{(1-x)(1-2x+(1-\ell m)x^2-(1-\ell)mx^3)}, \tag{24}$$

where $C = 1+(1-p-2q)x - q(2-2p-q)x^2 - (1-p)(1-q)qx^3 - (1-p)pqx^4$, *and* $D = 1-2x+(2-\ell m)x^2 - (2-\ell-\ell m)x^3 + (1-\ell-m+\ell m)x^4$.

2.4. Permutations in $S_n(213, 231)$

We first describe the structure of a $(213, 231)$-avoiding permutation. Let $\pi = \pi_1 \cdots \pi_n \in S_n(213, 231)$. If $\pi_1 = n$, then $\pi = n(n-1)\cdots 21$. If $\pi_k = n$ and $1 < k < n$, then $\pi_1 < \pi_2 < \cdots < \pi_{k-1}$ in order to avoid 213. On the other hand, in order to avoid 231, if $k+1 \leq i \leq n$, then $\pi_i > \pi_{k-1}$. If $\pi_n = n$, then $\pi = 12\cdots n$. So, for $\pi \in S_n(213, 231)$, its structure is $\pi = \alpha \oplus (1 \ominus \beta)$, where $\alpha \in S_{k-1}$ is an increasing (213, 231)-avoiding permutation and $1 \ominus \beta \in S_{n-k+1}(213, 231)$. We use the structure of π to prove the following theorem.

Theorem 5. *For* $S_n(213, 231)$*, we have*

$$F_{(213,231)}(x, p, q, u, v, s, t, y, z, \ell, m) = \frac{A}{B}, \tag{25}$$

where

$$\begin{aligned} A =& svx(qz(qvxz-1) + p^2t^3u^2x^2y(\ell(m-1)qvx - (y-1)(qvxz-1)) \\ &+ t(u(1-qvxz)(1+pqxyz) + pqxz(\ell mqvx + y - qvxyz)) \\ &- pt^2ux(-\ell quvx - u(y-1)(qvxz-1) + (1+pqxyz)(\ell mqvx + y - qvxyz))) \\ &- qs^2tuv^2x^2(mptx(ptuxy-1)(1+(\ell-1)qvxz) - (ptxy-1)((1-z)(qvxz-1) \\ &+ ptux(z(1+\ell qvx - qvxz) + y(z-1)(qvxz-1)))) \\ &+ (ptuxy-1)(\ell mpqtvx^2 - (1-ptxy)(1-qvxz)) \end{aligned}$$

and

$$B = (1-ptuxy)(1-qsvxz)((1-ptxy)(1-qvxz) - \ell mpqtvx^2).$$

Proof. Let $\pi = \pi_1 \cdots \pi_n \in S_n(213, 231)$. If $n \leq 1$, then we have the formula $F_{(213,231)}(x, p, q, u, v, s, t, y, z, \ell, m) = 1 + xuvst$. For $n \geq 2$, the permutations are divided into three cases depending on the position of n.

`Case (a)`: If $\pi_1 = n$, then $\pi = 1 \ominus \beta$, where $\beta \in S_{n-1}(213, 231)$ is non-empty. We let the g.f. for the permutations in Case (a) be

$$G_{(213,231)}(x,p,q,u,v,s,t,y,z,\ell,m) :=$$
$$\sum_{n\geq 2} \sum_{\substack{\pi\in\\ S_n(213,231)\\ \pi_1=n}} x^n p^{\mathrm{asc}(\pi)} q^{\mathrm{des}(\pi)} u^{\mathrm{lrmax}(\pi)} v^{\mathrm{rlmax}(\pi)} s^{\mathrm{lrmin}(\pi)} t^{\mathrm{rlmin}(\pi)} y^{\mathrm{dasc}(\pi)} z^{\mathrm{ddes}(\pi)} \ell^{\mathrm{pk}(\pi)} m^{\mathrm{vl}(\pi)}.$$

`Case (b)`: $\pi_k = n$ and $1 < k < n$. In this case, $\pi = \alpha \oplus (1 \ominus \beta)$, where $\alpha = 12\cdots(k-1)$ and $1 \ominus \beta \in S_{n-k+1}$ is the same as in Case (a). Note that we do not need to consider left-to-right minima for $1 \ominus \beta$. Next, we consider the following two subcases.
`Subcase (1)`: $k = 2$. In this case, $\alpha = 1$, and the g.f. for π is

$$xpust\ell G_{(213,231)}(x,p,q,u,v,1,t,y,z,\ell,m),$$

where $\pi_1\pi_2\pi_3$ contributes to $\mathrm{pk}(\pi)$.
`Subcase (2)`: $k > 2$. In this case, $\alpha = 12\cdots(k-1)$, and the g.f. is

$$\begin{aligned} G_{(213,231)}(x,p,q,u,v,1,t,y,z,\ell,m) &\sum_{i\geq 2} x^i p^i u^i s t^i y^{i-1}\ell \\ &= G_{(213,231)}(x,p,q,u,v,1,t,y,z,\ell,m)\frac{x^2p^2u^2st^2y\ell}{1-xputy}, \end{aligned}$$

where $\pi_{k-2}\pi_{k-1}\pi_k = (k-2)(k-1)n$ contributes to $\mathrm{dasc}(\pi)$ and $\pi_{k-1}\pi_k\pi_{k+1}$ contributes to $\mathrm{pk}(\pi)$. So the g.f. in Case (b) is

$$\begin{aligned} &xpust\ell G_{(213,231)}(x,p,q,u,v,1,t,y,z,\ell,m) \\ &\quad + G_{(213,231)}(x,p,q,u,v,1,t,y,z,\ell,m)\frac{x^2p^2u^2st^2y\ell}{1-xputy} \\ &= G_{(213,231)}(x,p,q,u,v,1,t,y,z,\ell,m)\frac{xpust\ell}{1-xputy}. \end{aligned}$$

`Case (c)`: $\pi_n = n$. In this case, $\pi = 12\cdots n$.

`Subcase (1)`: $n = 2$. In this case, $\pi = 12$ and the g.f. is $x^2pu^2vst^2$.

`Subcase (2)`: If $n > 2$, then positions $1, 2, \ldots, (n-2)$ are double ascents and the g.f. for π is

$$\sum_{i\geq 3} x^i p^{i-1} u^i v s t^i y^{i-2} = \frac{x^3p^2u^3vst^3y}{1-xputy}.$$

So the g.f. in Case (c) is

$$x^2pu^2vst^2 + \frac{x^3p^2u^3vst^3y}{1-xputy} = \frac{x^2pu^2vst^2}{1-xputy}.$$

Combining Cases (a)–(c), we have $F_{(213,231)}(x,p,q,u,v,s,t,y,z,\ell,m)$

$$\begin{aligned}&= G_{(213,231)}(x,p,q,u,v,s,t,y,z,\ell,m)\\ &+ G_{(213,231)}(x,p,q,u,v,1,t,y,z,\ell,m)\frac{xpust\ell}{1-xputy}\\ &\quad + 1 + xuvst + \frac{x^2pu^2vst^2}{1-xputy}. \end{aligned} \tag{26}$$

Next we evaluate $G_{(213,231)}(x,p,q,u,v,s,t,y,z,\ell,m)$, which appears in Case (a). If $\pi_1 = n$, then $\pi = 1 \ominus \beta$, where $\beta \in S_{n-1}(213,231)$ is non-empty. Note that we do not need to consider left-to-right maxima for β. For $n \geq 2$, we divide the permutations into three classes depending on the position of $n-1$, which is the largest element of β.

`Class A`: $\pi_2 = n-1$. In this case, we consider the following two subclasses.

`Subclass (1)`: $n = 2$. In this case, we have $\pi = 21$, and the g.f. is $x^2quv^2s^2t$.

`Subclass (2)`: $n \geq 2$. In this case, we have $\beta = 1 \ominus \zeta$, where $\zeta \in S_{n-2}(213,231)$ is non-empty. Similarly to Case (a), we see that the corresponding g.f. is

$$xquvszG_{(213,231)}(x,p,q,1,v,s,t,y,z,\ell,m),$$

where $\pi_1\pi_2\pi_3$ contributes to $\mathrm{ddes}(\pi)$. So the g.f. for Class A is

$$x^2quv^2s^2t + xquvszG_{(213,231)}(x,p,q,1,v,s,t,y,z,\ell,m).$$

`Class B`: $\pi_{k'} = n-1$ and $2 < k' < n$. In this case, $\beta = \gamma' \oplus (1 \ominus \zeta')$, where $1 \ominus \zeta' \in S_{n-k'+1}$ is the same as in Case (a) and $\gamma' = 1 \cdots (k'-2)$ is a non-empty permutation. Note that we do not need to consider left-to-right maxima and left-to-right minima for $1 \ominus \zeta'$. We consider the following two subclasses of Class B.

`Subclass (1)`: $k' = 3$. In this case, we have $\gamma' = 1$, and the g.f. is

$$x^2pquvs^2t\ell mG_{(213,231)}(x,p,q,1,v,1,t,y,z,\ell,m),$$

where $\pi_1\pi_2\pi_3 = n1(n-1)$ contributes to $\mathrm{vl}(\pi)$ and $\pi_2\pi_3\pi_4$ contributes to $\mathrm{pk}(\pi)$.

`Subclass (2)`: $k' > 3$. In this case, we have position 1 is a valley, positions $2,3,\ldots,(k'-2)$ are double ascents, and position $k'-1$ is a peak. The g.f. for π is

$$\begin{aligned}&G_{(213,231)}(x,p,q,1,v,1,t,y,z,\ell,m)\sum_{i\geq 3} x^ip^{i-1}quvs^2t^{i-1}y^{i-2}\ell\\ &=G_{(213,231)}(x,p,q,1,v,1,t,y,z,\ell,m)\frac{x^3p^2quvs^2t^2y\ell m}{1-xpty}.\end{aligned}$$

So the g.f. for Case B is

$$G_{(213,231)}(x,p,q,1,v,1,t,y,z,\ell,m)\left(x^2pquvs^2t\ell m+\frac{x^3p^2quvs^2t^2y\ell m}{1-xpty}\right).$$

`Class C`: $\pi_n = n-1$. In this case, $\pi = n1\cdots(n-1)$, which is the same as Case (c), so the corresponding g.f. is

$$xquvsm\frac{x^2pvst^2}{1-xpty}.$$

Classes A–C together give

$$\begin{aligned}
&G_{(213,231)}(x,p,q,u,v,s,t,y,z,\ell,m)\\
&\quad= x^2quv^2s^2t + xquvszG_{(213,231)}(x,p,q,1,v,s,t,y,z,\ell,m) + xquvsm\frac{x^2pvst^2}{1-xpty}\\
&\qquad+ G_{(213,231)}(x,p,q,1,v,1,t,y,z,\ell,m)\left(x^2pquvs^2t\ell m+\frac{x^3p^2quvs^2t^2y\ell m}{1-xpty}\right). \qquad (27)
\end{aligned}$$

Letting $u=1$ in Equation (27), we get

$$\begin{aligned}
&G_{(213,231)}(x,p,q,1,v,s,t,y,z,\ell,m)\\
&\quad= x^2qv^2s^2t + xqvszG_{(213,231)}(x,p,q,1,v,s,t,y,z,\ell,m) + xqvsm\frac{x^2pvst^2}{1-xpty}\\
&\qquad+ G_{(213,231)}(x,p,q,1,v,1,t,y,z,\ell,m)\left(x^2pqvs^2t\ell m+\frac{x^3p^2qvs^2t^2y\ell m}{1-xpty}\right). \qquad (28)
\end{aligned}$$

Letting $s=1$ in Equation (28), we get

$$\begin{aligned}
&G_{(213,231)}(x,p,q,1,v,1,t,y,z,\ell,m)\\
&\quad= x^2qv^2t + xqvzG_{(213,231)}(x,p,q,1,v,1,t,y,z,\ell,m) + xqvm\frac{x^2pvt^2}{1-xpty}\\
&\qquad+ G_{(213,231)}(x,p,q,1,v,1,t,y,z,\ell,m)\left(x^2pqvt\ell m+\frac{x^3p^2qvt^2y\ell m}{1-xpty}\right). \qquad (29)
\end{aligned}$$

Solving Equations (26) and (29) simultaneously, we obtain (25). □

From Theorem 5 we have the following results.

Corollary 4. *Let $u=v=s=t=1$. Then, by setting four out of the six variables p, q, y, z, ℓ, and m equal to one individually in Equation (25), we obtain the joint* *distributions of* (asc, des), (dasc, ddes), *and* (pk, vl) *over* $S_n(213, 312)$*:*

$$\sum_{n\geq 0}\sum_{\pi\in S_n(213,231)} x^n p^{\mathrm{asc}(\pi)} q^{\mathrm{des}(\pi)} = \frac{1+x-px-qx}{1-px-qx}, \tag{30}$$

$$\sum_{n\geq 0}\sum_{\pi\in S_n(213,231)} x^n y^{\mathrm{dasc}(\pi)} z^{\mathrm{ddes}(\pi)} = \frac{C}{1-x^2-xy-xz+x^2yz}, \tag{31}$$

$$\sum_{n\geq 0}\sum_{\pi\in S_n(213,231)} x^n \ell^{\mathrm{pk}(\pi)} m^{\mathrm{vl}(\pi)} = \frac{1-x+x^2-\ell mx^2-x^3+\ell x^3+mx^3-\ell mx^3}{1-2x+x^2-\ell mx^2}, \tag{32}$$

where $C = 1+x+x^2+x^3-xy-x^2y-x^3y-xz-x^2z-x^3z+x^2yz+x^3yz.$

2.5. Permutations in $S_n(213, 312)$

We first describe the structure of a $(213, 312)$-avoiding permutation. Let $\pi = \pi_1\cdots\pi_n \in S_n(213, 312)$. If $\pi_i = n$, then $\pi_1 < \pi_2 < \cdots < \pi_{i-1}$ in order to avoid 213. On the other hand, in order to avoid 312, $\pi_{i+1} > \pi_{i+2} > \cdots > \pi_n$. We use the structure of π to prove the following theorem.

Theorem 6. *For* $S_n(213, 312)$*, we have*

$$\begin{aligned}F_{(213,312)}(x,p,q,u,v,s,t,y,z,\ell,m) =& 1+\frac{qs^2tuv^2x^2}{1-qsvxz}+pstu^2vx^2\frac{t}{1-ptuxy}\\ &+\frac{\ell qst^2uv^2x^2}{(1-ptuxy)(1-puxy-qvxz)}\\ &+\frac{\ell qsvx}{(1-puxy-qvxz)(1-qsvxz)}.\end{aligned} \tag{33}$$

Proof. Let $\pi = \pi_1\cdots\pi_n \in S_n(213, 312)$. If $n = 0$, then we have the term of 1 in $F_{(213,312)}(x, p, q, u, v, s, t)$. If $\pi = 1$, then the g.f. is $xuvst$. For $n \geq 2$, suppose that $\pi_1 = i$, $\pi_k = n$, and $\pi_n = j$. We consider the following cases.

If $i = n$, namely $k = 1$, then $\pi = n\cdots 1$. The corresponding g.f. is

$$\sum_{n=2}^{\infty} x^n q^{n-1} uv^n s^n tz^{n-2}.$$

If $j = n$, namely $k = n$, then $\pi = 1\cdots n$. The corresponding g.f. is

$$\sum_{n=2}^{\infty} x^n p^{n-1} u^n vst^n y^{n-2}.$$

Next, let $2 \leq i, j, k \leq n-1$. If $\pi_1 = 1$, then in order to avoid 312, there are $\binom{n-j-1}{n-k-1}$ permutations whose g.f. is $x^n p^{k-1} q^{n-k} u^k v^{n-k+1} st^j y^{k-2} z^{n-k-1}$, so the g.f.

in this case is

$$\sum_{n=2}^{\infty}\sum_{j=2}^{n-1}\sum_{k=j}^{n-1}\binom{n-j-1}{n-k-1}x^n p^{k-1}q^{n-k}u^k v^{n-k+1}st^j y^{k-2}z^{n-k-1}.$$

If $\pi_1 \neq 1$, then to avoid 213, there are $\binom{n-i-1}{k-2}$ permutations whose g.f. is

$$x^n p^{k-1}q^{n-k}u^k v^{n-k+1}s^i t y^{k-2}z^{n-k-1}\ell.$$

Hence, the g.f. in this case is

$$\sum_{n=2}^{\infty}\sum_{i=2}^{n-1}\sum_{k=2}^{n+1-i}\binom{n-i-1}{k-2}x^n p^{k-1}q^{n-k}u^k v^{n-k+1}s^i t y^{k-2}z^{n-k-1}\ell.$$

In conclusion,

$$\begin{aligned}
&F_{(213,312)}(x,p,q,u,v,s,t,y,z,\ell,m)\\
&= 1 + xtuvs + \sum_{n=2}^{\infty} x^n q^{n-1}uv^n s^n tz^{n-2} + \sum_{n=2}^{\infty} x^n p^{n-1}u^n vst^n y^{n-2}\\
&\quad + \sum_{n=2}^{\infty}\sum_{j=2}^{n-1}\sum_{k=j}^{n-1}\binom{n-j-1}{n-k-1}x^n p^{k-1}q^{n-k}u^k v^{n-k+1}st^j y^{k-2}z^{n-k-1}\\
&\quad + \sum_{n=2}^{\infty}\sum_{i=2}^{n-1}\sum_{k=2}^{n+1-i}\binom{n-i-1}{k-2}x^n p^{k-1}q^{n-k}u^k v^{n-k+1}s^i t y^{k-2}z^{n-k-1}\ell.
\end{aligned}$$

Simplifying $F_{(213,312)}(x,p,q,u,v,s,t,y,z,\ell,m)$, we obtain (33). □

Corollary 5. *Let $u = v = s = t = 1$. Then, by setting four out of the six variables p, q, y, z, ℓ, and m equal to one individually in Equation* (33)*, we obtain the joint distributions of* (asc, des)*,* (dasc, ddes)*, and* (pk, vl) *over $S_n(213, 312)$:*

$$\sum_{n\geq 0}\sum_{\pi\in S_n(213,312)} x^n p^{\mathrm{asc}(\pi)}q^{\mathrm{des}(\pi)} = \frac{(1-px+px^2)(1-px-qx+qx^2)}{(1-px)(1-px+qx)}, \tag{34}$$

$$\sum_{n\geq 0}\sum_{\pi\in S_n(213,312)} x^n y^{\mathrm{dasc}(\pi)}z^{\mathrm{ddes}(\pi)} = \frac{A}{(1-xz)(1-2xy+x^2y^2-xz+x^2yz)}, \tag{35}$$

$$\sum_{n\geq 0}\sum_{\pi\in S_n(213,312)} x^n \ell^{\mathrm{pk}(\pi)}m^{\mathrm{vl}(\pi)} = \frac{1-3x+4x^2-4x^3+\ell x^3+\ell x^4}{1-3x+2x^2}, \tag{36}$$

where

$$\begin{aligned}
A =& 1 - x^5 z - 2x(y+z) + x^2(2+y^2+3yz+z^2) + x^4(1+y^2+z^2+y(-1+2z))\\
&- x^3(-1+3z+y^2z+y(3+z^2)).
\end{aligned}$$

3. Concluding Remarks

In this paper, we determine the joint distributions of the statistics (asc, des, lrmax, lrmin, rlmax, rlmin, dasc, ddes, pk, vl) on permutations avoiding any two patterns of length 3. This generalizes several earlier results in [3, 4, 6], which considered, respectively, one, two, and six statistics on the same sets of permutations. All generating functions derived in this paper are rational. It is noteworthy that we are able to simultaneously control such a large number of statistics while still obtaining explicit distribution results. This is achieved by considering a more refined structure of the permutations in question, which results in a larger number of cases and subcases to be analyzed. We also note that, unlike the situation in [6], where combinatorial proofs of five equidistribution results are provided, we have not observed any equidistributions in our more general setting.

Finally, studying (joint) distributions of statistics in other permutation classes, such as those considered in the literature [7], is an interesting direction for further research. For instance, our approach should be applicable to extending the joint distribution results for *separable permutations* in [5], where up to four statistics are simultaneously controlled.

Acknowledgement. The author is grateful to Sergey Kitaev for numerous helpful discussions related to this paper.

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