A SHORT PROOF OF KNUTH'S OLD SUM, GENERALIZATIONS AND RELATED SUMS

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Abstract

We give a short proof of the well-known Knuth's old sum and provide some generalizations. Our approach utilizes the binomial theorem and integration formulas derived using the Beta function. Several new polynomial identities and combinatorial identities are derived.

1. Introduction

There appears to be a renewed interest [1, 2, 7, 11, 14] in the famous Knuth's old sum (also known as the Reed Dawson identity),

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k}\binom{2k}{k}=\begin{cases}2^{-n}\binom{n}{n/2}, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd.}\end{cases} \tag{1}$$

Many different proofs of this identity and various generalizations exist in the literature (see [10] for a survey).

In this paper we give a very short proof of Equation (1) and offer the following generalization:

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k-m}\binom{2(k+m)}{k+m}\\ &\qquad=\begin{cases}\sum\limits_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}2^{-n-2k}\binom{2k+n}{(2k+n)/2}, & \text{if } n \text{ is even;}\\ -\sum\limits_{k=1}^{\lceil m/2\rceil}\binom{m}{2k-1}2^{-n-2k+1}\binom{2k+n-1}{(2k+n-1)/2}, & \text{if } n \text{ is odd;}\end{cases}\end{aligned} \tag{2}$$

where m and n are non-negative integers and, as usual, $\lfloor z\rfloor$ is the greatest integer less than or equal to z while $\lceil z\rceil$ is the smallest integer greater than or equal to z.

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The following special cases of Equation (2) were also reported by [12, p.72, Problem 4(b)]:

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{n-2k}\binom{2k}{k}=\binom{2n}{n}, \tag{3}$$

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}2^{n-2k}\binom{2k}{k}=\frac{1}{2}\binom{2n+2}{n+1}-\binom{2n}{n}=\frac{n}{n+1}\binom{2n}{n}. \tag{4}$$

Equation (3) corresponds to setting $n=0$ in Equation (2) and re-labeling m as n; while Equation (4) follows from setting $n=1$ in Equation (2).

In Section 5, we will derive the following complements of Knuth's old sum:

$$\sum_{k=0}^{n}(-1)^k\binom{2k}{k}\binom{2\,(n-k)}{n-k}=\begin{cases}2^n\binom{n}{n/2}, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd;}\end{cases}$$

and

$$\sum_{k=0}^{n}\binom{2\,(n-k)}{n-k}\binom{2k}{k}=2^{2n}. \tag{5}$$

Equation (5) is the famous combinatorial identity concerning the convolution of central binomial coefficients. Many different proofs of this identity exist in the literature, (see Mikić [8] and the many references therein).

Equation (2) is itself a particular case of a more general identity, stated in Theorem 2, which has many interesting consequences, including another generalization of Knuth's old sum, namely,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k}\binom{2k+v}{(2k+v)\,/2}\binom{k+v}{v/2}^{-1}$$
$$=\begin{cases}2^{-n}\binom{n}{n/2}\binom{(n+v)/2}{v/2}^{-1}, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd;}\end{cases}$$

where v is a real number; as well as simple, apparently new combinatorial identities such as

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}2^{n-2k}C_k=\frac{1}{2}C_{n+2}-C_{n+1};$$

where, here and throughout this paper,

$$C_j=\frac{1}{j+1}\binom{2j}{j},$$

defined for every non-negative integer j, is a Catalan number.

Based on the binomial theorem, we will derive, in Section 7, some presumably new polynomial identities, including the following:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} 2^{-k} \binom{2k}{k} (1-x)^{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{-2k} \binom{2k}{k} x^{n-2k}. \tag{6}$$

Equation (6) subsumes Knuth's old sum, Equation (1), (at $x = 0$), as well as Equation (3) (at $x = 1$).

Finally, in Section 8, the polynomial identities will facilitate the derivation of apparently new combinatorial identities such as

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{1}{2k+1} = \frac{2^{n-1}}{2^n - 1} \sum_{k=1}^{\lceil n/2 \rceil} \binom{n}{2k-1} \frac{1}{k}, \quad n \neq 0,$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{-2k} C_k = \frac{2^{-n+1}}{n+2} (2n+1) C_n,$$

and

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{2(2k+1)}{k+2} C_k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k.$$

2. Required Identities

In order to give the short proof of Knuth's old sum, we need a couple of definite integrals which we establish in Lemma 1.

The binomial coefficients are defined, for non-negative integers m and n, by

$$\binom{m}{n} = \begin{cases} \dfrac{m!}{n!(m-n)!}, & m \geq n \\ 0, & m < n, \end{cases}$$

the number of distinct sets of n objects that can be chosen from m distinct objects.

Generalized binomial coefficients are defined for complex numbers u and v, excluding the set of negative integers, by

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\,\Gamma(u-v+1)}, \tag{7}$$

where $\Gamma(z)$ is the Gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = \int_0^\infty (\log(1/t))^{z-1} dt$$

and extended to the rest of the complex plain, excluding the non-positive integers, by analytic continuation.

Lemma 1. *Let u and v be complex numbers such that $\Re u > -1$ and $\Re v > -1$. Let m be a non-negative integer. Then*

$$\int_0^\pi \cos^u(x/2)\,dx = 2^{-u}\,\pi\binom{u}{u/2} = \int_0^\pi \sin^u(x/2)\,dx, \tag{8}$$

$$\int_0^\pi \cos^m x\,dx = \begin{cases} 2^{-m}\,\pi\binom{m}{m/2}, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd;} \end{cases} \tag{9}$$

and, more generally,

$$I(u,v) := \int_0^\pi \cos^u\left(\frac{x}{2}\right)\sin^v\left(\frac{x}{2}\right)\,dx = 2^{-u-v}\,\pi\binom{u}{u/2}\binom{v}{v/2}\binom{(u+v)/2}{u/2}^{-1}, \tag{10}$$

and

$$J(m,v) := \int_0^\pi \cos^m x \sin^v x\,dx = \begin{cases} 2^{-m-v}\,\pi\binom{m}{m/2}\binom{v}{v/2}\binom{(m+v)/2}{m/2}^{-1}, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases} \tag{11}$$

Obviously $I(v,u) = I(u,v)$, a symmetry property that is not possessed by $J(m,v)$.

Proof. Identities (10) and (11) are immediate consequences of the well-known Beta function integral [6, Entry 3.621.5]:

$$K(u,v) := \int_0^{\pi/2} \cos^u x \sin^v x\,dx = 2^{-u-v-1}\,\pi\binom{u}{u/2}\binom{v}{v/2}\binom{(u+v)/2}{u/2}^{-1}, \tag{12}$$

valid for $\Re u > -1$, $\Re v > -1$, with the symmetry property $K(u,v) = K(v,u)$.

Equation (10) is obtained via a simple change of the integration variable from x to y in Equation (12), with $x = y/2$.

To prove Equation (11), write

$$J(m,v) = \int_0^\pi \cos^m x \sin^v x\,dx = \int_0^{\pi/2} \cos^m x \sin^v x\,dx + \int_{\pi/2}^\pi \cos^m x \sin^v x\,dx.$$

Change the integration variable in the second integral on the right-hand side from x to y via $x = y + \pi/2$; this gives

$$\begin{aligned} J(m,v) &= \int_0^{\pi/2} \cos^m x \sin^v x\,dx + (-1)^m \int_0^{\pi/2} \sin^m y \cos^v y\,dy \\ &= K(m,v) + (-1)^m K(v,m) \\ &= (1 + (-1)^m)\,K(m,v); \end{aligned}$$

and hence Equation (11). □

Remark 1. Since, for a real number u,

$$1+(-1)^u = 2\cos^2\left(\frac{\pi u}{2}\right) + i\sin\left(\pi u\right),$$

the $J(m,v)$ stated in Equation (11) is a special case of the following more general result:

$$\begin{aligned} J\left(u,v\right) &= \int_0^{\pi} \cos^u x \sin^v x\, dx \\ &= \frac{\pi}{2^{u+v+1}} \binom{u}{u/2} \binom{v}{v/2} \binom{\left(u+v\right)/2}{u/2}^{-1} \left(2\cos^2\left(\frac{\pi u}{2}\right) + i\sin\left(\pi u\right)\right), \end{aligned}$$

which is valid for $u > -1$ and $\Re v > -1$.

3. A Short Proof of Knuth's Old Sum

Theorem 1. *If n is a non-negative integer, then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{-k} \binom{2k}{k} = \begin{cases} 2^{-n}\binom{n}{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Substitute $-\cos x - 1$ for y in the binomial theorem

$$\sum_{k=0}^{n} \binom{n}{k} y^k = (1+y)^n,$$

to obtain

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^k \cos^{2k}(x/2) = (-1)^n \cos^n x. \tag{13}$$

Thus

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^k \int_0^{\pi} \cos^{2k}(x/2)\, dx = (-1)^n \int_0^{\pi} \cos^n x\, dx,$$

and hence Equation (1) on account of Equations (8) and (9). □

4. A Generalization of Knuth's Old Sum

In this section we extend Equation (1) by introducing an arbitrary non-negative integer m and a real number v.

Theorem 2. *If m and n are non-negative integers and v is a real number, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k-m}\binom{2k+2m+v}{(2k+2m+v)/2}\binom{k+m+v}{v/2}^{-1}$$
$$=\begin{cases}\sum\limits_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}2^{-n-2k}\binom{2k+n}{(2k+n)/2}\binom{(2k+n+v)/2}{(2k+n)/2}^{-1}, & \text{if } n \text{ is even};\\ -\sum\limits_{k=1}^{\lceil m/2\rceil}\binom{m}{2k-1}2^{-n-2k+1}\binom{2k+n-1}{(2k+n-1)/2}\binom{(2k+n-1+v)/2}{(2k+n-1)/2}^{-1}, & \text{if } n \text{ is odd}.\end{cases} \tag{14}$$

In particular,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k-m}\binom{2(k+m)}{k+m}$$
$$=\begin{cases}\sum\limits_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}2^{-n-2k}\binom{2k+n}{(2k+n)/2}, & \text{if } n \text{ is even};\\ -\sum\limits_{k=1}^{\lceil m/2\rceil}\binom{m}{2k-1}2^{-n-2k+1}\binom{2k+n-1}{(2k+n-1)/2}, & \text{if } n \text{ is odd}.\end{cases}$$

Proof. Since

$$(1+\cos x)^m = 2^m\cos^{2m}\left(\frac{x}{2}\right) = \sum_{k=0}^{m}\binom{m}{k}\cos^k x$$

and

$$\sin^v x = 2^v\sin^v\left(\frac{x}{2}\right)\cos^v\left(\frac{x}{2}\right),$$

multiplication of the left-hand side of Equation (13) by

$$2^{m+v}\cos^{2m+v}\left(\frac{x}{2}\right)\sin^v\left(\frac{x}{2}\right)$$

and the right-hand side by

$$\sin^v x\sum_{k=0}^{m}\binom{m}{k}\cos^k x$$

gives

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{k+m+v}\cos^{2k+2m+v}(x/2)\sin^v(x/2)$$
$$= (-1)^n\sum_{k=0}^{m}\binom{m}{k}\cos^{k+n}x\sin^v x.$$

Thus, we have

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{k+m+v}\cos^{2k+2m+v}(x/2)\sin^v(x/2)\\&\quad=(-1)^n\sum_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}\cos^{2k+n}x\sin^v x\\&\qquad+(-1)^n\sum_{k=1}^{\lceil m/2\rceil}\binom{m}{2k-1}\cos^{2k-1+n}x\sin^v x.\end{aligned}$$

Equation (14) now follows by termwise integration from 0 to π, according to the parity of n, using Lemma 1. □

Corollary 1. *If n is a non-negative integer and v is a real number, then*

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}\\&\quad=\begin{cases}2^{-n}\binom{n}{n/2}\binom{(n+v)/2}{v/2}^{-1}, & \text{if } n \text{ is even;}\\0, & \text{if } n \text{ is odd.}\end{cases}\end{aligned}\tag{15}$$

Corollary 2. *If n is a non-negative integer and v is a real number, then*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1}=2^{-n}\binom{2n+v}{(2n+v)/2}\binom{n+v}{v/2}^{-1},\tag{16}$$

and

$$\begin{aligned}&\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}2^{n-2k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1}\\&\quad=\frac{1}{2}\binom{2n+v+2}{(2n+v+2)/2}\binom{n+v+1}{v/2}^{-1}-\binom{2n+v}{(2n+v)/2}\binom{n+v}{v/2}^{-1}.\end{aligned}\tag{17}$$

Proof. Equation (16) is obtained by setting $n=0$ in Equation (14) and re-labeling m as n while Equation (17) is the evaluation of Equation (14) at $n=1$ with a re-labeling of m as n. □

Proposition 1. *If n is a non-negative integer, then*

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\frac{1}{2k+1}=\frac{2^{n+1}}{n+2}-\frac{2^n}{n+1},\tag{18}$$

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}2^{n-2k}C_k=\frac{1}{2}C_{n+2}-C_{n+1}.\tag{19}$$

Proof. Evaluation of Equation (17) at $v = 1$ gives Equation (18) while evaluation at $v = 2$ yields Equation (19). In deriving Equation (18), we used the following relationships between binomial coefficients:

$$\binom{r}{1/2} = \frac{2^{2r+1}}{\pi} \binom{2r}{r}^{-1}, \tag{20}$$

$$\binom{r}{r/2} = \frac{2^{2r}}{\pi} \binom{r}{(r-1)/2}^{-1}, \tag{21}$$

$$\binom{r+1/2}{r} = (2r+1)\, 2^{-2r} \binom{2r}{r}, \tag{22}$$

and

$$r\binom{s}{r} = s\binom{s-1}{r-1}; \tag{23}$$

all of which can be derived by using the Gamma function identities:

$$\Gamma\left(u + \frac{1}{2}\right) = \sqrt{\pi}\, 2^{-2u} \binom{2u}{u} \Gamma(u+1),$$

and

$$\Gamma\left(-u + \frac{1}{2}\right) = (-1)^u\, 2^{2u} \binom{2u}{u}^{-1} \frac{\sqrt{\pi}}{\Gamma(u+1)},$$

together with the definition of the generalized binomial coefficients as given in Equation (7). □

Proposition 2. *If m and n are non-negative integers, then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^{k+m}}{k+m+1} = \begin{cases} \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} \frac{1}{2k+n+1}, & \text{if } n \text{ is even;} \\ -\sum_{k=1}^{\lceil m/2 \rceil} \binom{m}{2k-1} \frac{1}{2k+n}, & \text{if } n \text{ is odd.} \end{cases}$$

In particular,

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} 2^k}{k+1} = \begin{cases} \frac{1}{n+1}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd;} \end{cases}$$

and

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} 2^{k+1}}{k+2} = \begin{cases} \frac{1}{n+1}, & \text{if } n \text{ is even;} \\ -\frac{1}{n+2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Evaluate Equation (14) at $v = 1$. □

5. Complements of Knuth's Old Sum

Theorem 3. *If n is a non-negative integer, then*

$$\sum_{k=0}^{n}(-1)^k\binom{2k}{k}\binom{2\,(n-k)}{n-k}=\begin{cases}2^n\binom{n}{n/2}, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd.}\end{cases}\tag{24}$$

Proof. Set $a=\cos^2(x/2)$ and $b=-\sin^2(x/2)$ in the binomial theorem:

$$\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}=(a+b)^n\,,\tag{25}$$

to obtain

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\cos^{2k}\left(\frac{x}{2}\right)\sin^{2n-2k}\left(\frac{x}{2}\right)=\cos^n x,\tag{26}$$

from which Equation (24) follows by term-wise integration using Lemma 1. □

Theorem 4. *If n is a non-negative integer, then*

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}=2^{2n}.$$

Proof. Set $a=\cos^2(x/2)$ and $b=\sin^2(x/2)$ in Equation (25) to obtain

$$\sum_{k=0}^{n}\binom{n}{k}\cos^{2k}\left(\frac{x}{2}\right)\sin^{2n-2k}\left(\frac{x}{2}\right)=1,\tag{27}$$

from which the stated identity follows by term-wise integration using Lemma 1. □

Next, we present a generalization of Equation (24).

Theorem 5. *If n is a non-negative integer and v is a real number, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k+v}{(2k+v)\,/2}\binom{2n-2k+v}{(2n-2k+v)\,/2}\binom{n+v}{(2k+v)\,/2}^{-1}$$
$$=\begin{cases}2^n\binom{n}{n/2}\binom{v}{v/2}\binom{(n+v)/2}{v/2}^{-1}, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Multiply through Equation (26) by $\sin^v x$ and integrate from 0 to π, using Lemma 1. □

We conclude this section with a generalization of Equation (5).

Theorem 6. *If n is a non-negative integer and v is a real number, then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k+v}{(2k+v)/2}\binom{2n-2k+v}{(2n-2k+v)/2}\binom{n+v}{(2k+v)/2}^{-1}=2^{2n}\binom{v}{v/2}.$$

Proof. Multiply through Equation (27) by $\sin^v x$ and integrate from 0 to π, using Lemma 1. □

6. Combinatorial Identities Associated with Polynomial Identities of a Certain Type

In this section we derive some combinatorial identities associated with any polynomial identity having the following form:

$$\sum_{k=s}^{n} f(k)\,(1+t)^{p(k)}=\sum_{k=m}^{r} g(k)\,t^{q(k)}; \tag{28}$$

where m, n, r, and s are non-negative integers, $p(k)$ and $q(k)$ are sequences of non-negative integers, $f(k)$ and $g(k)$ are sequences, and t is a complex variable.

Theorem 7. *Consider the polynomial identity given in Equation* (28)*. Let u and v be arbitrary complex numbers such that $\Re u>-1$ and $\Re v>-1$. Then*

$$\begin{aligned}&\sum_{k=s}^{n} f(k)\binom{p(k)+u+v+1}{u+1}^{-1}\\&\qquad=\frac{u+1}{v+1}\sum_{k=m}^{r}(-1)^{q(k)}g(k)\binom{q(k)+u+v+1}{v+1}^{-1}.\end{aligned} \tag{29}$$

In particular,

$$\sum_{k=s}^{n}\frac{f(k)}{p(k)+1}=\sum_{k=m}^{r}\frac{(-1)^{q(k)}g(k)}{q(k)+1}.$$

Proof. Write $-t$ for t in Equation (28) and multiply through by $t^u(1-t)^v$ to obtain

$$\sum_{k=s}^{n} f(k)\,(1-t)^{p(k)+v}\,t^u=\sum_{k=m}^{r}(-1)^{q(k)}g(k)\,(1-t)^v\,t^{q(k)+u};$$

from which Equation (29) follows after integrating from 0 to 1, using the Beta function (variant of Equation (12)):

$$\int_0^1 (1-t)^x\,t^y\,dt=\frac{1}{x+1}\binom{x+y+1}{x+1}^{-1}, \tag{30}$$

for $\Re x>-1$ and $\Re y>-1$. □

Theorem 8. *Consider the polynomial identity given in Equation* (28)*. Let u and v be arbitrary complex numbers such that $\Re v > -1$, $\Re(2(u-p(j))+v) > -1$, $\Re(2(u-q(j))+v) > -1$, and $2q(j)+\Re v > -1$ for every non-negative integer j. Then*

$$\sum_{k=s}^{n} f(k)2^{2p(k)}\binom{2\left(u-p(k)\right)+v}{\left(2\left(u-p(k)\right)+v\right)/2}\binom{u-p(k)+v}{v/2}^{-1} = \binom{v}{v/2}^{-1}\sum_{k=m}^{r} g(k)A(u,v,q(k)), \tag{31}$$

and

$$\sum_{k=m}^{r} (-1)^{q(k)}g(k)2^{2q(k)}\binom{2\left(u-q(k)\right)+v}{\left(2\left(u-q(k)\right)+v\right)/2}\binom{u-q(k)+v}{v/2}^{-1} = \binom{v}{v/2}^{-1}\sum_{k=s}^{n} (-1)^{p(k)}f(k)B(u,v,p(k)), \tag{32}$$

where

$$A(u,v,q(k)) = \binom{2\left(u-q(k)\right)+v}{\left(2\left(u-q(k)\right)+v\right)/2}\binom{2q(k)+v}{\left(2q(k)+v\right)/2}\binom{u+v}{\left(2q(k)+v\right)/2}^{-1}$$

and

$$B(u,v,p(k)) = \binom{2\left(u-p(k)\right)+v}{\left(2\left(u-p(k)\right)+v\right)/2}\binom{2p(k)+v}{\left(2p(k)+v\right)/2}\binom{u+v}{\left(2p(k)+v\right)/2}^{-1}.$$

In particular,

$$\sum_{k=s}^{n} f(k)2^{2p(k)}\binom{2\left(u-p(k)\right)}{u-p(k)} = \sum_{k=m}^{r} g(k)\binom{2\left(u-q(k)\right)}{u-q(k)}\binom{2q(k)}{q(k)}\binom{u}{q(k)}^{-1}$$

and

$$\sum_{k=m}^{r}(-1)^{q(k)}g(k)2^{2q(k)}\binom{2\left(u-q(k)\right)}{u-q(k)}=\sum_{k=s}^{n}(-1)^{p(k)}f(k)\binom{2\left(u-p(k)\right)}{u-p(k)}\binom{2p(k)}{p(k)}\binom{u}{p(k)}^{-1}.$$

Proof. Substituting $t=y/x$ in Equation (28) and multiplying through by x^w gives

$$\sum_{k=s}^{n} f(k)x^{u-p(k)}\left(x+y\right)^{p(k)} = \sum_{k=m}^{r} g(k)x^{u-q(k)}y^{q(k)}. \tag{33}$$

Writing $\cos^2 x$ for x and $\sin^2 x$ for y in Equation (33), multiplying through by $\sin^v x$ and integrating from 0 to $\pi/2$ using Lemma 1 gives Equation (31). Equation (32) follows from the fact that the transformation $y \to y - x$ followed by $x \to -x$ causes Equation (28) to become

$$\sum_{k=m}^{r} (-1)^{u-q(k)} g(k) x^{u-q(k)} (x+y)^{q(k)} = \sum_{k=s}^{n} (-1)^{u-p(k)} f(k) x^{u-p(k)} y^{p(k)}.$$

□

Theorem 9. *Consider the polynomial identity given in Equation* (28)*. Let u and v be arbitrary complex numbers such that $\Re v > -1$, $\Re u - p(j) > -1$, $p(j) + \Re(v) > -1$, and $\Re u - q(j) > -1$ for every non-negative integer j. Then*

$$\sum_{k=s}^{n} (-1)^{p(k)} f(k) \binom{u+v}{u-p(k)}^{-1} = \frac{u+v+1}{v+1} \sum_{k=m}^{r} (-1)^{q(k)} g(k) \binom{u-q(k)+1}{v+1}^{-1}. \tag{34}$$

In particular,

$$\sum_{k=s}^{n} (-1)^{p(k)} f(k) \binom{u}{p(k)}^{-1} = (u+1) \sum_{k=m}^{r} (-1)^{q(k)} \frac{g(k)}{u-q(k)+1}.$$

Proof. Set $y = -1$ in Equation (33) and multiply through by $(1-x)^v$ to obtain

$$\sum_{k=s}^{n} (-1)^{p(k)} f(k) x^{u-p(k)} (1-x)^{p(k)+v} = \sum_{k=m}^{r} (-1)^{q(k)} g(k) x^{u-q(k)} (1-x)^{v},$$

which upon integration from 0 to 1, using Equation (30), gives Equation (34). □

Theorem 10. *Consider the polynomial identity given in Equation* (28)*. Let u and v be arbitrary complex numbers such that $\Re u > -1$ and $\Re v > -1$. Then*

$$\begin{aligned} &\sum_{k=s}^{n} \frac{f(k)}{2^{2p(k)}} \binom{v}{v/2} \binom{2p(k)+u}{(2p(k)+u)/2} \binom{(2p(k)+u+v)/2}{v/2}^{-1} \\ &\qquad = \sum_{k=m}^{r} \frac{(-1)^{q(k)} g(k)}{2^{2q(k)}} \binom{u}{u/2} \binom{2q(k)+v}{(2q(k)+v)/2} \binom{(2q(k)+u+v)/2}{u/2}^{-1}. \end{aligned} \tag{35}$$

In particular,

$$\sum_{k=s}^{n} \frac{f(k)}{2^{2p(k)}} \binom{2p(k)}{p(k)} = \sum_{k=m}^{r} \frac{(-1)^{q(k)} g(k)}{2^{2q(k)}} \binom{2q(k)}{q(k)}.$$

Proof. Write $-\sin^2 t$ for t in Equation (28) and multiply through by $\cos^u t \sin^v t$ to obtain

$$\sum_{k=s}^{n} f(k) \cos^{2p(k)+u} t \sin^v t = \sum_{k=m}^{r} (-1)^{q(k)} g(k) \cos^u t \sin^{2q(k)+v} t,$$

from which Equation (35) follows upon integration from 0 to $\pi/2$ using Lemma 1.

□

Theorem 11. *Consider the polynomial identity given in Equation (28). Let v be an arbitrary complex number such that $\Re v > -1$.*

1. *Suppose that, for every integer j, each of $q(2j)$ and $q(2j-1)$ is a sequence of non-negative integers having a particular parity but such that the parity of $q(2j)$ is different from the parity of $q(2j-1)$ for every integer j.*

   If $q(2j)$ is an even integer for every integer j, then

$$\sum_{k=s}^{n} \frac{f(k)}{2^{p(k)}} \binom{2p(k)+v}{(2p(k)+v)/2} \binom{p(k)+v}{v/2}^{-1} = \sum_{k=\lfloor (m+1)/2 \rfloor}^{\lfloor r/2 \rfloor} \frac{g(2k)}{2^{q(2k)}} \binom{q(2k)}{q(2k)/2} \binom{(q(2k)+v)/2}{v/2}^{-1}, \tag{36}$$

   while if $q(2j)$ is an odd integer for every integer j, then

$$\sum_{k=s}^{n} \frac{f(k)}{2^{p(k)}} \binom{2p(k)+v}{(2p(k)+v)/2} \binom{p(k)+v}{v/2}^{-1} = \sum_{k=\lfloor (m+2)/2 \rfloor}^{\lceil r/2 \rceil} \frac{g(2k-1)}{2^{q(2k-1)}} \binom{q(2k-1)}{q(2k-1)/2} \binom{(q(2k-1)+v)/2}{v/2}^{-1}. \tag{37}$$

2. *Suppose that, for every integer j, each of $p(2j)$ and $p(2j-1)$ is a sequence of non-negative integers having a particular parity but such that the parity of $p(2j)$ is different from the parity of $p(2j-1)$ for every integer j.*

   If $p(2j)$ is an even integer for every integer j, then

$$\sum_{k=m}^{r} \frac{g(k)(-1)^{q(k)}}{2^{q(k)}} \binom{2q(k)+v}{(2q(k)+v)/2} \binom{q(k)+v}{v/2}^{-1} = \sum_{k=\lfloor (s+1)/2 \rfloor}^{\lfloor n/2 \rfloor} \frac{(-1)^{p(2k)} f(2k)}{2^{p(2k)}} \binom{p(2k)}{p(2k)/2} \binom{(p(2k)+v)/2}{v/2}^{-1} \tag{38}$$

   while if $p(2j)$ is an odd integer for every integer j, then

$$\sum_{k=m}^{r} \frac{g(k)(-1)^{q(k)}}{2^{q(k)}} \binom{2q(k)+v}{(2q(k)+v)/2} \binom{q(k)+v}{v/2}^{-1} = \sum_{k=\lfloor (s+2)/2 \rfloor}^{\lceil n/2 \rceil} \frac{(-1)^{p(2k-1)} f(2k-1)}{2^{p(2k-1)}} \binom{p(2k-1)}{p(2k-1)/2} \binom{(p(2k-1)+v)/2}{v/2}^{-1}. \tag{39}$$

Proof. Set $t = \cos x$ in Equation (28) and multiply through by $\sin^v x$ to obtain

$$\sum_{k=s}^{n} 2^{p(k)+v} f(k) \cos^{2p(k)+v}\left(\frac{x}{2}\right) \sin^v\left(\frac{x}{2}\right)$$
$$= \sum_{k=\lfloor (m+1)/2 \rfloor}^{\lfloor r/2 \rfloor} g(2k) \cos^{q(2k)} x \sin^v x + \sum_{k=\lfloor (m+2)/2 \rfloor}^{\lceil r/2 \rceil} g(2k-1) \cos^{q(2k-1)} x \sin^v x,$$

from which Equation (36) and Equation (37) follow after term-wise integration from 0 to π, using Lemma 1. Equations (38) and (39) are obtained from Equations (36) and (37) since Equation (28) can be written in the following equivalent form:

$$\sum_{k=m}^{r} (-1)^{q(k)} g(k)\,(1+t)^{q(k)} = \sum_{k=s}^{n} (-1)^{p(k)} f(k) t^{p(k)}.$$

□

Remark 2. We offer the following remarks.

(a) Suppose that, for every integer j, each of $q(2j)$ and $q(2j-1)$ is a sequence of non-negative integers having a particular parity but such that the parity of $q(2j)$ is different from the parity of $q(2j-1)$ for every integer j. If $q(2j)$ is an even integer for every integer j, then

$$\sum_{k=s}^{n} \frac{f(k)}{2^{p(k)}} \binom{2p(k)}{p(k)} = \sum_{k=\lfloor (m+1)/2 \rfloor}^{\lfloor r/2 \rfloor} \frac{g(2k)}{2^{q(2k)}} \binom{q(2k)}{q(2k)/2},$$

while if $q(2j)$ is an odd integer for every integer j, then

$$\sum_{k=s}^{n} \frac{f(k)}{2^{p(k)}} \binom{2p(k)}{p(k)} = \sum_{k=\lfloor (m+2)/2 \rfloor}^{\lceil r/2 \rceil} \frac{g(2k-1)}{2^{q(2k-1)}} \binom{q(2k-1)}{q(2k-1)/2}.$$

(b) Suppose that, for every integer j, each of $p(2j)$ and $p(2j-1)$ is a sequence of non-negative integers having a particular parity but such that the parity of $p(2j)$ is different from the parity of $p(2j-1)$ for every integer j. If $p(2j)$ is an even integer for every integer j, then

$$\sum_{k=m}^{r} \frac{g(k)(-1)^{q(k)}}{2^{q(k)}} \binom{2q(k)}{q(k)} = \sum_{k=\lfloor (s+1)/2 \rfloor}^{\lfloor n/2 \rfloor} \frac{f(2k)}{2^{p(2k)}} \binom{p(2k)}{p(2k)/2},$$

while if $p(2j)$ is an odd integer for every integer j, then

$$\sum_{k=m}^{r} \frac{g(k)(-1)^{q(k)}}{2^{q(k)}} \binom{2q(k)}{q(k)} = \sum_{k=\lfloor (s+2)/2 \rfloor}^{\lceil n/2 \rceil} \frac{(-1)^{p(2k-1)} f(2k-1)}{2^{p(2k-1)}} \binom{p(2k-1)}{p(2k-1)/2}.$$

Corollary 3. *Let an arbitrary polynomial identity have the following form:*

$$\sum_{k=s}^{n} f(k)\,(1+t)^{k} = \sum_{k=m}^{r} g(k)\,t^{k}, \tag{40}$$

where m, n, r, and s are non-negative integers, $f(k)$ and $g(k)$ are sequences, and t is a complex variable. Let v be an arbitrary real number. Then

$$\sum_{k=s}^{n} \frac{f(k)}{2^k} \binom{2k+v}{(2k+v)/2} \binom{k+v}{v/2}^{-1} = \sum_{k=\lfloor (m+1)/2 \rfloor}^{\lfloor r/2 \rfloor} \frac{g(2k)}{2^{2k}} \binom{2k}{k} \binom{(2k+v)/2}{v/2}^{-1}, \tag{41}$$

and

$$\sum_{k=m}^{r} \frac{g(k)(-1)^k}{2^k} \binom{2k+v}{(2k+v)/2} \binom{k+v}{v/2}^{-1} = \sum_{k=\lfloor (s+1)/2 \rfloor}^{\lfloor n/2 \rfloor} \frac{f(2k)}{2^{2k}} \binom{2k}{k} \binom{(2k+v)/2}{v/2}^{-1}.$$

In particular,

$$\sum_{k=s}^{n} \frac{f(k)}{2^k} \binom{2k}{k} = \sum_{k=\lfloor (m+1)/2 \rfloor}^{\lfloor r/2 \rfloor} \frac{g(2k)}{2^{2k}} \binom{2k}{k},$$

and

$$\sum_{k=m}^{r} \frac{g(k)(-1)^k}{2^k} \binom{2k}{k} = \sum_{k=\lfloor (s+1)/2 \rfloor}^{\lfloor n/2 \rfloor} \frac{f(2k)}{2^{2k}} \binom{2k}{k}.$$

7. Polynomial Identities

In this section, by following the procedures outlined in Section 6, we derive new polynomial identities associated with the binomial theorem.

Theorem 12. *Let u and v be arbitrary complex numbers such that $\Re u > -1$ and $\Re v > -1$. Let x be a complex variable. If n is a non-negative integer, then*

$$\begin{aligned} &\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{k+u+v+1}{u+1}^{-1} (1-x)^{n-k} \\ &\quad = \frac{u+1}{v+1} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{k+u+v+1}{v+1}^{-1} x^{n-k}. \end{aligned} \tag{42}$$

In particular,

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{\binom{n}{k}}{k+1} (1-x)^{n-k} = \sum_{k=0}^{n} (-1)^{k} \frac{\binom{n}{k}}{k+1} x^{n-k}.$$

Proof. Consider the following variation on the binomial theorem:

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1+t)^{k}(1-x)^{n-k}=\sum_{k=0}^{n}\binom{n}{k}t^{k}x^{n-k}. \tag{43}$$

Use Equation (29) with

$$f(k)=(-1)^{n-k}\binom{n}{k}(1-x)^{n-k},\quad g(k)=\binom{n}{k}x^{n-k},\quad s=0=m,\quad r=n,$$

to obtain Equation (42). □

Theorem 13. *If n is a non-negative integer, v is a real number, and x is a complex variable, then*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{n-k}&\binom{n}{k}2^{-k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}(1-x)^{n-k}\\&=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1}x^{n-2k}.\end{aligned} \tag{44}$$

Proof. Making use of Equation (40) and with Equation (43) in mind, use Equation (41) with

$$f(k)=(-1)^{n-k}\binom{n}{k}(1-x)^{n-k},\quad g(k)=\binom{n}{k}x^{n-k},\quad s=0=m,\quad r=n,$$

to obtain Equation (44). □

Corollary 4. *If n is a non-negative integer and x is a complex variable, then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{2^{k}}{k+1}(1-x)^{n-k}=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{n}{2k}}{2k+1}x^{n-2k}, \tag{45}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{\binom{n}{k}}{k+2}2^{-k}\binom{2(k+1)}{k+1}(1-x)^{n-k}=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{n}{2k}}{k+1}2^{-2k}\binom{2k}{k}x^{n-2k}. \tag{46}$$

Proof. Equations (6), (45), and (46) correspond to the evaluation of Equation (44) at $v=0$, $v=1$, and $v=2$, respectively.

In deriving Equation (45), we use Equations (20)–(23) to obtain

$$\binom{2k+1}{k+1/2}=\frac{2^{4k+4}}{\pi(k+1)}\binom{2(k+1)}{k+1}^{-2}\binom{2k+1}{k},$$

$$\binom{k+1}{1/2}=\frac{2^{2k+3}}{\pi}\binom{2(k+1)}{k+1}^{-1},$$

and
$$\binom{2k+1}{k} = \frac{1}{2}\binom{2\left(k+1\right)}{k+1}.$$
□

Theorem 14. *Let u and v be complex numbers such that $\Re u > -1$ and $\Re v > -1$. If n is a non-negative integer, then*
$$\begin{aligned}\binom{v}{v/2}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}2^{-2k}\binom{2k+u}{\left(2k+u\right)/2}\binom{\left(2k+u+v\right)/2}{v/2}^{-1}\left(1-x\right)^{n-k}\\ =\binom{u}{u/2}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}2^{-2k}\binom{2k+v}{\left(2k+v\right)/2}\binom{\left(2k+u+v\right)/2}{u/2}^{-1}x^{n-k}.\end{aligned} \tag{47}$$

Proof. With Equation (43) in mind, use
$$f(k) = (-1)^{n-k}\binom{n}{k}(1-x)^{n-k}, \quad g(k) = \binom{n}{k}x^{n-k}, \quad s = 0 = m, \quad r = n,$$
and $p(k) = k = q(k)$ in Equation (35). □

Corollary 5. *If n is a non-negative integer, then*
$$\sum_{k=0}^{n}(-1)^{n-k}\binom{2k}{k}2^{-2k}\binom{n}{k}\left(1-x\right)^{n-k} = \sum_{k=0}^{n}(-1)^{k}\binom{2k}{k}2^{-2k}\binom{n}{k}x^{n-k}, \tag{48}$$
$$\sum_{k=0}^{n}(-1)^{n-k}2^{-2k}\binom{n}{k}C_{k+1}\left(1-x\right)^{n-k} = \sum_{k=0}^{n}(-1)^{k}2^{-2k}\binom{n}{k}C_{k+1}x^{n-k}. \tag{49}$$

Proof. Evaluate Equation (47) at $u = 0 = v$ and at $u = 2 = v$. □

Theorem 15. *If n is a non-negative integer, v is a real number and x is a complex variable, then*
$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}2^{-k}\binom{2k+v}{\left(2k+v\right)/2}\binom{k+v}{v/2}^{-1}\left(1-x\right)^{k}x^{n-k}\\ =\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}\binom{\left(2k+v\right)/2}{k}^{-1}(1-x)^{2k}.\end{aligned} \tag{50}$$

Proof. Consider another variation on the binomial theorem:
$$\sum_{k=0}^{n}\binom{n}{k}\left(1-x\right)^{k}\left(1+y\right)^{k}x^{n-k} = \sum_{k=0}^{n}\binom{n}{k}y^{k}\left(1-x\right)^{k}. \tag{51}$$

This identity has the form of Equation (40). Use Equation (41) with

$$f(k) = \binom{n}{k}(1-x)^k x^{n-k}, \quad g(k) = \binom{n}{k}(1-x)^k, \quad s = 0 = m, \quad r = n,$$

to obtain Equation (50). □

Corollary 6. *If n is a non-negative integer and x is a complex variable, then*

$$\sum_{k=0}^{n} \binom{n}{k} 2^{-k} \binom{2k}{k} (1-x)^k x^{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{-2k} \binom{2k}{k} (1-x)^{2k}, \tag{52}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{2^k}{k+1} (1-x)^k x^{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\binom{n}{2k}}{2k+1} (1-x)^{2k}, \tag{53}$$

$$\sum_{k=0}^{n} \binom{n}{k} 2^{-k} C_{k+1} (1-x)^k x^{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{-2k} C_k (1-x)^{2k}. \tag{54}$$

Proof. Equations (52), (53), and (54) correspond to the evaluation of Equation (50) at $v = 0$, $v = 1$, and $v = 2$, respectively. □

Remark 3. The reader is invited to employ the procedures established in Theorems 7–11 to discover more polynomial identities associated with Equation (51).

8. More Combinatorial Identities

8.1. Identities from the Binomial Theorem

Theorem 16. *Let u and v be complex numbers such that $\Re u > -1$ and $\Re v > -1$. If n is a non-negative integer, then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+u+v+1}{u+1}^{-1} = \frac{u+1}{v+1} \binom{n+u+v+1}{v+1}^{-1}.$$

Proof. Set $x = 0$ in Equation (42). □

Theorem 17. *If n is an integer and v is a real number, then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-2k} \binom{2k+v}{(2k+v)/2} \binom{k+v}{v/2}^{-1}$$
$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{-2k} \binom{2k}{k} \binom{(2k+v)/2}{k}^{-1},$$

and

$$\sum_{k=0}^{n}\binom{n}{k}2^{-k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}$$
$$=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{n-4k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1}.$$

Proof. Evaluate Equation (44) at $x=-1$ and $x=2$, respectively. □

Remark 4. Setting $x=0$ in Equation (44) reproduces Equation (15) while setting $x=1$ reproduces Equation (16).

Proposition 3. *If n is a non-negative integer, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-2k}\binom{2k}{k}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k},$$
$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{1}{2k+1}=\frac{2^{n-1}}{2^n-1}\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\frac{1}{k},\quad n\neq 0,$$
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2k+1}{k+2}2^{n-2k+1}C_k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}C_k.$$

Proof. Set $x=-1$ in Equations (6), (45), and (46). □

Proposition 4. *If n is a non-negative integer, then*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}=2^{-n}\binom{2n}{n},$$
$$\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{n}{2k}}{2k+1}=\frac{2^n}{n+1},$$
$$\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{n}{2k}}{k+1}2^{-2k}\binom{2k}{k}=\frac{2^{-n+1}}{n+2}(2n+1)C_n.$$

Proof. Set $x=1$ in Equations (6), (45), and (46). □

Proposition 5. *If n is a non-negative integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}2^{-k}\binom{2k}{k}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{n-4k}\binom{2k}{k},$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{2^{n-2k+1}-2^{2k+1}}{2k+1}=\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\frac{2^{2k-1}}{k},$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{2k+1}{k+2}\,2^{-k+1}C_k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{n-4k}C_k.$$

Proof. Set $x=2$ in Equations (6), (45), and (46). □

Theorem 18. *If n is a non-negative integer and v is a real number, then*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}\\ =\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1}.\end{aligned}\tag{55}$$

Proof. Set $x=-1$ in Equation (50). □

Proposition 6. *If n is a non-negative integer, then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{2k}{k}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k},$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{2^{2k}}{k+1}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{2^{2k}}{2k+1},$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{2(2k+1)}{k+2}C_k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k.$$

Proof. Set $x=-1$ in each of Equations (52)–(54) or what is the same thing, $v=0$, $v=1$, and $v=2$ in Equation (55). □

Theorem 19. *If n is a non-negative integer and v is a real number, then*

$$\begin{aligned}\sum_{k=0}^{n}\binom{2n}{2k}\binom{2(n-k)}{n-k}\binom{2k+v}{(2k+v)/2}\binom{(2n+v)/2}{n-k}^{-1}\\ =\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{2n-2k}\binom{2k}{k}\binom{2k+v}{(2k+v)/2}\binom{(4k+v)/2}{k}^{-1}\end{aligned}\tag{56}$$

and

$$\sum_{k=1}^{n}\binom{2n}{2k-1}\binom{2(n-k+1)}{n-k+1}\binom{2k-1+v}{(2k-1+v)/2}\binom{(2n+v+1)/2}{n-k+1}^{-1}$$
$$=\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}2^{2n+1-2k}\binom{2k}{k}\binom{2k-1+v}{(2k-1+v)/2}\binom{(4k+v-1)/2}{k}^{-1}. \quad (57)$$

In particular,

$$\sum_{k=0}^{n}\binom{2n}{2k}\binom{2(n-k)}{n-k}\binom{2k}{k}\binom{n}{k}^{-1}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{2n-2k}\binom{2k}{k}$$

and

$$\sum_{k=1}^{n}\binom{2n}{2k-1}\binom{2(n-k+1)}{n-k+1}\binom{2k-1}{(2k-1)/2}\binom{(2n+1)/2}{n-k+1}^{-1}$$
$$=\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}2^{2n+1-2k}\binom{2k}{k}\binom{2k-1}{(2k-1)/2}\binom{(4k-1)/2}{k}^{-1}.$$

Proof. Since

$$\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)^{2m}=(1+\sin x)^{m},$$

the binomial theorem gives

$$\sum_{k=0}^{2n}\binom{2n}{k}\cos^{2n-k}\left(\frac{x}{2}\right)\sin^{k}\left(\frac{x}{2}\right)=\sum_{k=0}^{n}\binom{n}{k}\sin^{k}x.$$

Thus,

$$\sum_{k=0}^{2n}\binom{2n}{k}\cos^{2n-k}x\sin^{k}x=\sum_{k=0}^{n}\binom{n}{k}2^{k}\cos^{k}x\sin^{k}x$$

and, therefore,

$$\sum_{k=0}^{n}\binom{2n}{2k}\sin^{2k+v}x\cos^{2n-2k}x+\sum_{k=1}^{n}\binom{2n}{2k-1}\sin^{2k-1+v}x\cos^{2n-2k+1}x$$
$$=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{2k}\cos^{2k}x\sin^{2k+v}x+\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}2^{2k-1}\cos^{2k-1}x\sin^{2k-1+v}x.$$

Integrating from 0 to π using Lemma 1 gives Equation (56) while multiplying through by $\cos x$ and integrating from 0 to π gives Equation (57). □

8.2. Identities from Waring's Formulas

Waring's formula and its dual [4, Equations (22) and (1)] are

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n}{n-k}\binom{n-k}{k}(xy)^k(x+y)^{n-2k}=x^n+y^n \tag{58}$$

and

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}(xy)^k(x+y)^{n-2k}=\frac{x^{n+1}-y^{n+1}}{x-y}\,. \tag{59}$$

Equation (58) holds for every positive integer n while Equation (59) holds for every non-negative integer n.

Theorem 20. *If n is a non-negative integer and v is a real number, then*

$$\begin{aligned}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n}{n-k}\binom{n-k}{k}2^{-4k}\binom{2k+v}{(2k+v)/2}\\=\binom{2n+v}{(2n+v)/2}\binom{v}{v/2}2^{1-2n}\binom{n+v}{v/2}^{-1}.\end{aligned} \tag{60}$$

In particular,

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n}{n-k}\binom{n-k}{k}2^{-4k}\binom{2k}{k}=2^{-2n+1}\binom{2n}{n}.$$

Proof. Write $\cos^2(x/2)$ for x and $\sin^2(y/2)$ for y in Equation (58) and multiply through by $\sin^v x$ to obtain

$$\begin{aligned}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n}{n-k}\binom{n-k}{k}2^{-2k}\sin^{2k+v}x&=2^v\cos^{2n+v}\left(\frac{x}{2}\right)\sin^v\left(\frac{x}{2}\right)\\&\quad+2^v\sin^{2n+v}\left(\frac{x}{2}\right)\cos^v\left(\frac{x}{2}\right),\end{aligned}$$

from which, upon term-wise integration from 0 to π, Equation (60) follows. □

By writing $\cos^2(x/2)$ for x and $-\sin^2(y/2)$ for y, the reader is invited to discover a combinatorial identity associated with Equation (59).

8.3. Identities from an Identity of Simons

Simons [13] proved an identity that is equivalent to the following:

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}(1+t)^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}t^k. \tag{61}$$

On choosing

$$f(k) = (-1)^{n-k}\binom{n}{k}\binom{n+k}{k}, \quad g(k) = \binom{n}{k}\binom{n+k}{k},$$

$s = m = 0$, and $r = n$ in Equation (40), Equation (41) gives the result stated in the next proposition.

Proposition 7. *If n is a non-negative integer and v is a real number, then*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}2^{-k}\binom{n+k}{k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}\\ = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{n+2k}{2k}\binom{2k}{k}\binom{(2k+v)/2}{v/2}^{-1}.\end{aligned} \tag{62}$$

In particular,

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}2^{-k}\binom{n+k}{k}\binom{2k}{k} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{n+2k}{2k}\binom{2k}{k}.$$

The same set of sequences and parameters, $f(k)$ etc. that led to Equation (62), when used in Equation (35) gives the following result.

Proposition 8. *If n is a non-negative integer and u and v are real numbers, then*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}2^{-2k}\binom{n+k}{k}\binom{v}{v/2}\binom{2k+u}{(2k+u)/2}\binom{(2k+u+v)/2}{v/2}^{-1}\\ = \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}2^{-2k}\binom{n+k}{k}\binom{u}{u/2}\binom{2k+v}{(2k+v)/2}\binom{(2k+u+v)/2}{u/2}^{-1}.\end{aligned}$$

Remark 5. Chapman [3], Gould [5], Munarini [9], and many other authors gave different proofs and generalizations of the identity of Simons, Equation (61). Results similar to those stated in Propositions 7 and 8 can be derived from their results.

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