POWER SERIES ASSOCIATED WITH RECIPROCALS OF GENERALIZED CENTRAL BINOMIAL COEFFICIENTS

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Abstract

We explore power series representing reciprocals of central binomial coefficients multiplied by linear terms of differences or their squares, incorporating polynomial functions, the arcsine function, and square roots. The results of this study generalize those established by Sprugnoli.

1. Introduction

Central binomial coefficients are defined for $n \in \mathbb{N}$ as

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!}.$$

They possess many combinatorial properties and contribute to solving numerous counting problems. For instance, Vandermonde's identity, $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$, illustrates that the n-th central binomial coefficient equals the sum of the squares of the coefficients from the n-th row of Pascal's triangle. These coefficients are also related to Catalan numbers, defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1},$$

which represent, for example, the number of ways of dividing an $(n+2)$-gon into triangles formed by connecting the vertices with non-crossing line segments. There are many other interpretations for Catalan numbers; see for example [13].

In [4], it is shown that the generating function associated with the reciprocals of the Catalan numbers is given by

$$\sum_{k=0}^{\infty} \frac{x^k}{C_k} = \frac{2(x+8)}{(4-x)^2} + \frac{24\sqrt{x}}{(4-x)^{5/2}} \arcsin\left(\frac{\sqrt{x}}{2}\right).$$

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This series also appears in [2, 12, 14, 20].

There exist many results [2, 3, 7, 10, 11, 19] regarding finite or infinite sums that involve central binomial coefficients (or their reciprocals), Catalan numbers, Fibonacci numbers, Lucas numbers, harmonic numbers, and other special sequences that arise. Boyadzhiev [5, 6], for example, studied series of the forms

$$\sum_{k=0}^{\infty} \frac{4^k H_k x^k}{\binom{2k}{k}} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k} x^k}{k+m+1},$$

where H_k stands for the k-th harmonic number, expressing them in terms of elementary functions.

In 2006, Sprugnoli [17] proved, among other results, that

$$\sum_{k=2}^{\infty} \frac{4^k x^k}{(k-1)\binom{2k}{k}} = 2(2x-1)\sqrt{\frac{x}{1-x}} \arctan\left(\sqrt{\frac{x}{1-x}}\right) + 2x,$$

and

$$\sum_{k=2}^{\infty} \frac{4^k x^k}{(k-1)^2\binom{2k}{k}} = 4(1-x)\sqrt{\frac{x}{1-x}} \arctan\left(\sqrt{\frac{x}{1-x}}\right) + 4x\left(\arctan\sqrt{\frac{x}{1-x}}\right)^2 - 4x.$$

Both equalities hold for $|x| < 1$. These results provided the primary initial motivation for the developments presented in this work. In fact, in the aforementioned article, Sprugnoli does not specify the domains of validity for several of the results he presents concerning the power series he studies. A more rigorous determination of these domains can be obtained by differentiating and integrating the result of Theorem 1 of [1], an article in which additional issues in Sprugnoli's work were also emphasized and discussed.

In 2018, Chu and Esposito [8, 9] obtained exact evaluations, in terms of elementary functions, for series of the form

$$\sum_{k=0}^{\infty} \frac{\Lambda_m(k)}{\binom{2mk+2\gamma}{mk+\gamma}} (2x)^{2mk},$$

where $\Lambda_m(k)$ denotes a polynomial of degree m in k expressed in terms of falling factorials.

In this paper, we find in Theorems 1 and 2 general results for sums of the form

$$\sum_{k=n+1}^{\infty} \frac{4^k x^k}{(k-n)\binom{2mk}{mk}} \quad \text{and} \quad \sum_{k=n+1}^{\infty} \frac{4^k x^k}{(k-n)^2\binom{2mk}{mk}},$$

with $m, n \in \mathbb{N}$ fixed. These series, which generalize Sprugnoli's results, express themselves using polynomial functions, inverse trigonometric functions, and radicals, enabling more familiar manipulation of the expressions. The main idea is to apply a multisection to the series

$$\sum_{k=0}^{\infty} \frac{4^k x^k}{\binom{2k}{k}} = \frac{\sqrt{x}\arcsin\left(\sqrt{x}\right)}{(1-x)^{3/2}} + \frac{1}{1-x}$$

established by Lehmer [15]. Subsequently, by integrating such expressions, exact results are obtained in terms of elementary functions for certain integrals that appear in the process. Moreover, appropriate manipulations are performed to simplify summations, thus yielding the main results of the present work. As a special case, we can compute infinite sums of the form

$$\sum_{k=n+1}^{\infty} \frac{1}{(k-n)\binom{2mk}{mk}} \quad \text{and} \quad \sum_{k=n+1}^{\infty} \frac{1}{(k-n)^2\binom{2mk}{mk}},$$

or their respective alternating series.

The present work complements and extends the author's previous study [21], in which general results were obtained in terms of elementary functions, for series of the form

$$\sum_{k=0}^{\infty} \frac{x^k}{(2k+n)\binom{2mk}{mk}} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{x^k}{k^2\binom{2mk}{mk}}.$$

2. Definitions

Definition 1. Let $n \geq 1$ be an integer. We define the polynomials

$$S_n(x) := \sum_{k=0}^{n} s_k x^k$$

of degree n such that their coefficients satisfy the following recursive relation

$$s_0 := \frac{1}{1/2-n};$$
$$s_k := \left(\frac{k-1-n}{k-n+1/2}\right) \cdot s_{k-1}, \quad \text{for } k = 1, 2, \ldots, n.$$

Remark 1. We have

$$s_n = \frac{2^{2n+1}}{\binom{2n}{n}},$$

which implies

$$s_{n-1} = -\frac{2^{2n}}{\binom{2n}{n}}.$$

Indeed, by using Definition 1, we obtain

$$s_n = \frac{1}{1/2-n}\prod_{k=1}^{n}\left(\frac{k-1-n}{k-n+1/2}\right) = \frac{2^{2n+1}n!^2}{(2n)!} = \frac{2^{2n+1}}{\binom{2n}{n}}.$$

Table 1 shows the first few polynomials $S_n(x)$.

n	$S_n(x)$
1	$-2+4x$
2	$-\frac{2}{3}-\frac{8}{3}x+\frac{16}{3}x^2$
3	$-\frac{2}{5}-\frac{4}{5}x-\frac{16}{5}x^2+\frac{32}{5}x^3$
4	$-\frac{2}{7}-\frac{16}{35}x-\frac{32}{35}x^2-\frac{128}{35}x^3+\frac{256}{35}x^4$
5	$-\frac{2}{9}-\frac{20}{63}x-\frac{32}{63}x^2-\frac{64}{63}x^3-\frac{256}{63}x^4+\frac{512}{63}x^5$
6	$-\frac{2}{11}-\frac{8}{33}x-\frac{80}{231}x^2-\frac{128}{231}x^3-\frac{256}{231}x^4-\frac{1024}{231}x^5+\frac{2048}{231}x^6$

Table 1: The first six polynomials $S_n(x)$.

Definition 2. Let $n \geq 2$ be an integer. We define the constant $c_n := \frac{s_n}{2} - 1$. We also define the polynomials

$$T_{n-2}(x) := \sum_{k=0}^{n-2} t_k x^k$$

of degree $n-2$, such that their coefficients satisfy the recursive relation

$$t_{n-2} := -\frac{s_{n-1}}{2} - c_n; \quad t_{k-1} := \frac{(k+1-n)t_k + s_k/2}{k-n}, \quad \text{for } k = n-2, n-3, \ldots, 1.$$

For n $= 1$, we agree $c_1 := 1$ and $T_{-1}(x) := 0$.

Table 2 shows the first few values for c_n and the first few polynomials $T_{n-2}(x)$.

Remark 2. By Remark 1, for $n \in \mathbb{N}$ we have

$$c_n = \frac{4^n}{\binom{2n}{n}} - 1.$$

Definition 3. Let $n \geq 1$ be an integer. We define the polynomials

$$M_{n-1}(x) := \sum_{k=0}^{n-1} m_k x^k$$

of degree $n-1$, such that their coefficients satisfy the recursive relation

$$m_0 := \frac{2}{2n-1};$$
$$m_k := \frac{2(k-n)}{2k-2n+1} \cdot m_{k-1}, \quad \text{for } k = 1, 2, \ldots, n-1.$$

Table 3 shows the first few polynomials $M_{n-1}(x)$.

n	c_n	$T_{n-2}(x)$
1	1	0
2	5/3	$-1/3$
3	11/5	$-1/10-3/5x$
4	93/35	$-\frac{1}{21}-\frac{13}{70}x-\frac{29}{35}x^2$
5	193/63	$-\frac{1}{36}-\frac{17}{189}x-\frac{11}{42}x^2-\frac{65}{63}x^3$
6	793/231	$-\frac{1}{55}-\frac{7}{132}x-\frac{89}{693}x^2-\frac{51}{154}x^3-\frac{281}{231}x^4$

Table 2: The first six constants c_n and polynomials $T_{n-2}(x)$.

n	$M_{n-1}(x)$
1	2
2	$\frac{2}{3}+\frac{4}{3}x$
3	$\frac{2}{5}+\frac{8}{15}x+\frac{16}{15}x^2$
4	$\frac{2}{7}+\frac{12}{35}x+\frac{16}{35}x^2+\frac{32}{35}x^3$
5	$\frac{2}{9}+\frac{16}{63}x+\frac{32}{105}x^2+\frac{128}{315}x^3+\frac{256}{315}x^4$
6	$\frac{2}{11}+\frac{20}{99}x+\frac{160}{693}x^2+\frac{64}{231}x^3+\frac{256}{693}x^4+\frac{512}{693}x^5$

Table 3: The first six polynomials $M_{n-1}(x)$.

Definition 4. Let $n \geq 2$ be an integer. We define the constant $d_n := \frac{m_{n-1}}{2}$. We also define the polynomials

$$N_{n-2}(x) := \sum_{k=0}^{n-2} n_k x^k$$

of degree $n-2$ by

$$N_{n-2}(x) := \frac{x^{n-1}}{2} \cdot \int \left(\frac{M_{n-1}(x) - m_{n-1}x^{n-1}}{x^n} \right) dx = \frac{1}{2} \sum_{k=0}^{n-2} \frac{m_k x^k}{-n+k+1}.$$

For $n = 1$, we agree $d_1 := 1$ and $N_{-1}(x) := 0$.

Table 4 shows the first few constants d_n and the first few polynomials $N_{n-2}(x)$

Remark 3. Let $n \in \mathbb{N}$. We can prove that

$$d_n = \frac{4^n}{2n\binom{2n}{n}}.$$

n	d_n	$N_{n-2}(x)$
1	1	0
2	2/3	$-1/3$
3	8/15	$-(8x+3)/30$
4	16/35	$-(24x^2+9x+5)/105$
5	128/315	$-(768x^3+288x^2+160x+105)/3780$
6	256/693	$-(3840x^4+1440x^3+800x^2+525x+378)/20790$

Table 4: The first six polynomials $N_{n-2}(x)$.

3. Auxiliary Results

Lemma 1. *For $n \geq 1$, we have*

$$\int x^{-n-1/2}\frac{\arcsin(\sqrt{x})}{(1-x)^{3/2}}\,dx = \log(x-1)+c_n\log(x)+\frac{T_{n-2}(x)}{x^{n-1}}+\frac{\arcsin(\sqrt{x})}{x^{n-1/2}\sqrt{1-x}}S_n(x), \tag{1}$$

where the constant c_n and the polynomials $T_{n-2}(x)$ and $S_n(x)$ are as in Definitions 1 and 2.

Proof. For $n = 1$, the proof is straightforward. For $n \geq 2$, we must prove that the derivative of the right-hand side of Equation (1) is equal to the integrand of the left-hand side.

On the one hand, we have

$$\frac{d}{dx}\left(\frac{T_{n-2}(x)}{x^{n-1}}\right) = \sum_{k=0}^{n-2}(-n+k+1)t_k x^{-n+k}.$$

On the other hand, we have

$$\begin{aligned}
&\frac{d}{dx}\Big(\frac{\arcsin(\sqrt{x})S_n(x)}{x^{n-1/2}\sqrt{1-x}}\Big)\\
&\quad= \frac{d}{dx}\left(\frac{\arcsin(\sqrt{x})}{x^{n-1/2}\sqrt{1-x}}\right)S_n(x)+\frac{\arcsin(\sqrt{x})}{x^{n-1/2}\sqrt{1-x}}\frac{d}{dx}(S_n(x))\\
&\quad= \frac{S_n(x)}{2x^n(1-x)}+\frac{\arcsin(\sqrt{x})}{x^{n+1/2}(1-x)^{3/2}}\cdot\Big((nx-n+1/2)S_n(x)+x(1-x)S_n'(x)\Big).
\end{aligned}$$

Let us prove that

$$x(1-x)S_n'(x)+(nx-n+1/2)S_n(x) = 1. \tag{2}$$

Indeed, Equation (2) holds if and only if

$$x(1-x)(s_1+2s_2x+3s_3x^2+\cdots+ns_nx^{n-1})+(nx-n+1/2)(s_0+s_1x+\cdots+s_nx^n) = 1,$$

if and only if

$$s_0(-n+1/2)=1, \quad ks_k-(k-1)s_{k-1}+(-n+1/2)s_k+ns_{k-1}=0,$$

for all $1 \le k \le n$, and $-ns_n+ns_n=0$, if and only if

$$s_0=\frac{1}{1/2-n} \quad \text{and} \quad s_k=\frac{k-1-n}{k-n+1/2}\cdot s_{k-1}.$$

These two last equalities hold by the definition of the coefficients s_k.

By letting $x=1$ in Equation (2), we have

$$\sum_{k=0}^{n} s_k = 2. \tag{3}$$

The proof is complete if we can prove that

$$\frac{1}{x-1}+\frac{c_n}{x}+\sum_{k=0}^{n-2}(-n+k+1)t_k x^{-n+k}+\frac{S_n(x)}{2x^n(1-x)}=0.$$

Indeed, by comparing coefficients, this last equality holds if and only if

$$x^n+c_nx^{n-1}(x-1)+(x-1)\sum_{k=0}^{n-2}(-n+k+1)t_kx^k-\frac{S_n(x)}{2}=0,$$

if and only if

$$1+c_n-\frac{s_n}{2}=0, \quad -c_n-t_{n-2}-\frac{s_{n-1}}{2}=0, \quad (-n+k)t_{k-1}-(-n+k+1)t_k-\frac{s_k}{2}=0,$$

for all $1 \le k \le n-2$, and

$$(n-1)t_0-\frac{s_0}{2}=0$$

if and only if

$$c_n=\frac{s_n}{2}-1, \quad t_{n-2}=-\frac{s_{n-1}}{2}-c_n, \quad t_{k-1}=\frac{(-n+k+1)t_k+s_k/2}{k-n}, \tag{4}$$

for all $1 \le k \le n-2$, and

$$t_0=\frac{s_0/2}{n-1}. \tag{5}$$

The equations in (4) hold by definition of the coefficients c_n and t_k. Let us prove (5) holds. Indeed, by using the recursive relation defining t_k, we have

$$\frac{s_0/2}{n-1}=t_0=\frac{(2-n)t_1+s_1/2}{1-n},$$

if and only if
$$\frac{(s_0+s_1)/2}{n-2} = t_1 = \frac{(3-n)t_2 + s_2/2}{2-n},$$
if and only if
$$\frac{(s_0+s_1+s_2)/2}{n-3} = t_2 = \frac{(4-n)t_3 + s_3/2}{3-n},$$
if and only if ...
$$t_{n-2} = \frac{(s_0+s_1+\cdots+s_{n-2})/2}{n-(n-2+1)},$$
if and only if
$$\frac{(s_0+s_1+\cdots+s_{n-2})}{2} = -\frac{s_{n-1}}{2} - c_n,$$
if and only if
$$c_n = -\frac{\sum_{k=0}^{n-1} s_k}{2}.$$
By Equation (3), this last equality holds if and only if
$$c_n = -\frac{(2-s_n)}{2} = \frac{s_n}{2} - 1,$$
which in turn holds by definition of c_n. □

Lemma 2. *For $n \geq 1$, we have*
$$\int x^{-n-1/2} \frac{\arcsin(\sqrt{x})}{\sqrt{1-x}}\, dx = d_n \log(x) - \frac{\arcsin(\sqrt{x})\sqrt{x(1-x)}}{x^n} M_{n-1}(x) + \frac{N_{n-2}(x)}{x^{n-1}}, \tag{6}$$
where the constant d_n and the polynomials $M_{n-1}(x)$ and $N_{n-2}(x)$ are as in Definitions 3 and 4.

Remark 4. For $n = 0$,
$$\int x^{-1/2} \frac{\arcsin(\sqrt{x})}{\sqrt{1-x}}\, dx = (\arcsin(\sqrt{x}))^2.$$

Proof. For $n = 1$, the proof is straightforward. For $n \geq 2$, we must prove that the derivative of the right-hand side of Equation (6) equals the integrand of the left-hand side, i.e.,
$$\frac{d_n}{x} - \left(\frac{1}{2} - \frac{\arcsin(\sqrt{x})(2x-1)}{2\sqrt{x(1-x)}}\right) M_{n-1}(x) x^{-n} - \arcsin(\sqrt{x})\sqrt{x(1-x)}\, x^{-n} M'_{n-1}(x)$$
$$+ n \arcsin(\sqrt{x})\sqrt{x(1-x)}\, x^{-n-1} M_{n-1}(x) + \frac{N'_{n-2}(x)}{x^{n-1}} + (-n+1)\frac{N_{n-2}(x)}{x^n}$$
$$= x^{-n-1/2} \frac{\arcsin(\sqrt{x})}{\sqrt{1-x}}.$$

To do this, let us prove that

$$\frac{(2x-1)}{2\sqrt{x(1-x)}}M_{n-1}(x)x^{-n}+n\sqrt{x(1-x)}x^{-n-1}M_{n-1}(x)-\sqrt{x(1-x)}x^{-n}M'_{n-1}(x)$$
$$=\frac{x^{-n-1/2}}{\sqrt{1-x}} \tag{7}$$

and

$$\frac{d_n}{x}-\frac{1}{2}M_{n-1}(x)x^{-n}+\frac{N'_{n-2}(x)}{x^{n-1}}+(-n+1)\frac{N_{n-2}(x)}{x^n}=0. \tag{8}$$

Let us first consider Equation (8). We must prove that

$$\big((2-2n)x+(2n-1)\big)M_{n-1}(x)+(2x^2-2x)M'_{n-1}(x)=2. \tag{9}$$

By comparing coefficients, we see that Equation (9) holds if and only if

$$2m_{n-1}-2nm_{n-1}+2m_{n-1}(n-1)=0,$$
$$(2-2n)m_{k-1}+(2n-1)m_k+2(k-1)m_{k-1}-2km_k=0, \quad \text{for all } 1\le k\le n,$$
$$(2n-1)m_0=2,$$

if and only if

$$m_k=\Big(\frac{-2n+2k}{2k-2n+1}\Big)m_{k-1}, \quad \text{for all } 1\le k\le n,$$
$$m_0=\frac{2}{2n-1},$$

which holds by definition of the coefficients of $M_{n-1}(x)$. Replacing $x=1$ in Equation (9) we have

$$\sum_{k=0}^{n-1}m_k=2. \tag{10}$$

Let us now consider Equation (7), which is equivalent to

$$N'_{n-2}(x)=\frac{n-1}{x}N_{n-2}(x)+\left(\frac{M_{n-1}(x)}{2x}-d_nx^{n-2}\right).$$

This represents a first-order differential equation, whose solution is given by

$$N_{n-2}(x)=x^{n-1}\int\left(\frac{M_{n-1}(x)}{2x^n}-\frac{d_n}{x}\right)dx.$$

By Definition 4 we know that $d_n=\frac{m_{n-1}}{2}$. Then

$$N_{n-2}(x)=x^{n-1}\int\Big(\frac{M_{n-1}(x)-m_{n-1}x^{n-1}}{2x^n}\Big)\,dx,$$

which holds by Definition 4. □

Lemma 3. *For every fixed integer $m \geq 1$ and for all $|x| < 4^{m-1}$, we have*

$$\sum_{k=0}^{\infty} \frac{4^k x^k}{\binom{2mk}{mk}} = \frac{1}{m} \sum_{j=0}^{m-1} \left(\frac{4\sqrt{y}\arcsin(\sqrt{y}/2)}{(4-y)^{3/2}} + \frac{4}{4-y} \right), \tag{11}$$

where $y = w^j (4x)^{1/m}$ depends on j, and $w = e^{2\pi i/m}$ is the m-th root of unity.

Proof. We know from Lehmer [15] that, for all $|x| < 1$,

$$\sum_{k=0}^{\infty} \frac{4^k x^k}{\binom{2k}{k}} = \frac{\sqrt{x}\arcsin(\sqrt{x})}{(1-x)^{3/2}} + \frac{1}{1-x}.$$

This implies that, for $|x| < 4$,

$$\sum_{k=0}^{\infty} \frac{x^k}{\binom{2k}{k}} = \frac{4\sqrt{x}\arcsin(\sqrt{x}/2)}{(4-x)^{3/2}} + \frac{4}{4-x}.$$

Then we have for $|x| < 4^{m-1}$,

$$\begin{aligned} \frac{1}{m} \sum_{j=0}^{m-1} \left(\frac{4\sqrt{y}\arcsin(\sqrt{y}/2)}{(4-y)^{3/2}} + \frac{4}{4-y} \right) &= \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} \frac{w^{kj}(4x)^{k/m}}{\binom{2k}{k}} \\ &= \frac{1}{m} \sum_{k=0}^{\infty} \frac{(4x)^{k/m}}{\binom{2k}{k}} \sum_{j=0}^{m} w^{kj}. \end{aligned}$$

Finally, since

$$\sum_{j=0}^{m-1} w^{kj} = \begin{cases} 0 & \text{if } k \not| m, \\ m & \text{if } k | m, \end{cases}$$

we have that

$$\frac{1}{m} \sum_{k=0}^{\infty} \frac{(4x)^{k/m}}{\binom{2k}{k}} \sum_{j=0}^{m-1} w^{kj} = \sum_{l=0}^{\infty} \frac{(4x)^l}{\binom{2ml}{ml}},$$

which proves the result. □

4. Main Results

Theorem 1. *For any fixed integers $n \geq 1, m \geq 1$, and for $|x| < 4^{m-1}$, we have*

$$\sum_{k=n+1}^{\infty} \frac{4^k x^k}{(k-n)\binom{2mk}{mk}}$$
$$= 4^{1/m-1}x^{1/m}\sum_{j=0}^{m-1} T_{mn-2}(z)w^j + \sum_{j=0}^{m-1}\sqrt{\frac{z}{1-z}}\arcsin(\sqrt{z})S_{mn}(z)$$
$$+ \sum_{k=1}^{n-1}\frac{4^k x^k}{(k-n)}\left(\frac{1}{4^{km}} - \frac{1}{\binom{2mk}{mk}}\right) - \left(\frac{4^{n(1-m)}m}{2}\sum_{k=1}^{mn}\frac{4^k}{k\binom{2k}{k}}s_{mn-k}\right)x^n,$$

where $z = 4^{1/m-1}w^j x^{1/m}$ depends on j, $w = e^{2\pi i/m}$ is the m-th root of unity, the polynomials T_{mn-2} and S_{mn} of degrees $mn-2$ and mn, respectively, are as in Definitions 1 and 2, and the coefficients s_{mn-k} correspond to those of the polynomial S_{mn}.

Proof. We start by considering Equation (11), which we multiply by x^{n-1} and then integrate with respect to x, which leads to

$$\sum_{k\geq n+1}\frac{4^k x^k}{(k-n)\binom{2mk}{mk}}$$
$$= \frac{x^n}{m}\sum_{j=0}^{m-1}\underbrace{\int x^{-n-1}\left(\frac{4\sqrt{y}\arcsin(\sqrt{y}/2)}{(4-y)^{3/2}}\right)dx}_{I_1} + \frac{x^n}{m}\sum_{j=0}^{m-1}\underbrace{\int \frac{4}{x^{n+1}(4-y)}\,dx}_{I_2} \quad (12)$$
$$+ Cx^n - \sum_{k=0}^{n-1}\frac{4^k x^k}{(k-n)\binom{2mk}{mk}} - \frac{4^n x^n \log(x)}{\binom{2mn}{mn}},$$

where C is a constant to be determined.

By letting $y = w^j(4x)^{1/m} = 4z$, which implies that $x = 4^{m-1}z^m$, we have on the one hand

$$I_1 = 4^{-n(m-1)}m\int z^{-mn-1/2}\frac{\arcsin(\sqrt{z})}{(1-z)^{3/2}}\,dz.$$

Then Lemma 1 leads to

$$I_1 = 4^{-n(m-1)}m\left(\log(z-1) + c_{mn}\log(z) + \frac{T_{mn-2}(z)}{z^{mn-1}} + \frac{\arcsin(\sqrt{z})}{z^{mn-1/2}\sqrt{1-z}}S_{mn}(z)\right).$$

On the other hand, by the same change of variable as before,

$$I_2 = m\int\frac{4^{m-1}z^{m-1}}{4^{(m-1)(n+1)}(1-z)z^{m(n+1)}}\,dz$$
$$= \frac{m}{4^{(m-1)n}}\int\frac{1}{z^{mn+1}(1-z)}\,dz.$$

By decomposing into partial fractions, we have

$$\begin{aligned} I_2 &= \frac{m}{4^{(m-1)n}} \int \left(\frac{1}{z} - \frac{1}{z-1} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots + \frac{1}{z^{mn+1}} \right) dz \\ &= \frac{m}{4^{(m-1)n}} \cdot \left(\log(z) - \log(z-1) - \frac{1}{z} - \frac{1}{2z^2} - \frac{1}{3z^3} - \cdots - \frac{1}{mnz^{mn}} \right). \end{aligned}$$

Continuing the calculations on Equation (12),

$$\begin{aligned} &\sum_{k=n+1}^{\infty} \frac{4^k x^k}{(k-n)\binom{2mk}{mk}} \\ &= x^n 4^{-n(m-1)} \cdot \sum_{j=0}^{m-1} \left(\log(z-1) + c_{mn} \log(z) + \frac{T_{mn-2}(z)}{z^{mn-1}} + \frac{\arcsin\left(\sqrt{z}\right)}{z^{mn-1/2}\sqrt{1-z}} S_{mn}(z) \right) \\ &\quad + \frac{x^n}{4^{(m-1)n}} \cdot \sum_{j=0}^{m-1} \left(\log(z) - \log(z-1) - \frac{1}{z} - \frac{1}{2z^2} - \frac{1}{3z^3} - \cdots - \frac{1}{mnz^{mn}} \right) \\ &\quad + Cx^n - \sum_{k=0}^{n-1} \frac{4^k x^k}{(k-n)\binom{2mk}{mk}} - \frac{4^n x^n \log(x)}{\binom{2mn}{mn}}. \end{aligned}$$

Then, by Remark 2,

$$\begin{aligned} &\sum_{k=n+1}^{\infty} \frac{4^k x^k}{(k-n)\binom{2mk}{mk}} \\ &\quad = x^n 4^{-(m-1)n} \sum_{j=0}^{m-1} \left(\frac{4^{mn}}{\binom{2mn}{mn}} \log(z) \right) - \frac{4^n x^n \log(x)}{\binom{2mn}{mn}} - x^n 4^{-n(m-1)} \sum_{j=0}^{m-1} \sum_{k=1}^{mn} \frac{1}{kz^k} \\ &\quad\quad + x^n 4^{-n(m-1)} \sum_{j=0}^{m-1} \left(\frac{T_{mn-2}(z)}{z^{mn-1}} + \frac{\arcsin\left(\sqrt{z}\right)}{z^{mn-1/2}\sqrt{1-z}} S_{mn}(z) \right) + Cx^n \\ &\quad\quad - \sum_{k=0}^{n-1} \frac{4^k x^k}{(k-n)\binom{2mk}{mk}}. \end{aligned}$$

By letting $z = 4^{1/m-1} w^j x^{1/m}$, and grouping and simplifying the first and second terms on the right-hand side of the above expression, we obtain

$$\frac{x^n 4^n}{\binom{2mn}{mn}} \sum_{j=0}^{m-1} \left((\frac{1}{m} - 1) \log(4) + \frac{2\pi i j}{m} \right) = \frac{4^n x^n}{\binom{2mn}{mn}} \Big((1-m)\log(4) + \pi \mathrm{i}(m-1) \Big).$$

On the other hand,

$$\begin{aligned}
x^n 4^{-n(m-1)} \sum_{j=0}^{m-1} \sum_{k=1}^{mn} \frac{1}{kz^k} &= x^n 4^{-n(m-1)} \sum_{s=1}^{mn} \frac{1}{s} \sum_{j=0}^{m-1} \frac{1}{(4^{1/m-1}x^{1/m})^s w^{sj}} \\
&= x^n 4^{-n(m-1)} \sum_{s=1}^{mn} \left(\frac{1}{4^{(1/m-1)s} x^{s/m} s} \cdot \sum_{j=0}^{m-1} w^{-sj} \right) \\
&= x^n 4^{-n(m-1)} \sum_{k=1}^{n} \frac{1}{k 4^{(1-m)k} x^k} \\
&= \sum_{k=1}^{n} \frac{x^{n-k}}{k 4^{(k-n)(1-m)}} \\
&= \sum_{k=0}^{n-1} \frac{x^k}{(n-k) 4^{k(m-1)}}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \frac{4^k x^k}{(k-n)\binom{2mk}{mk}} \\
&\quad = \frac{4^n x^n}{\binom{2mn}{mn}} \Big((1-m)\log(4) + \pi i (m-1) \Big) - \sum_{k=0}^{n-1} \frac{x^k}{(n-k)4^{k(m-1)}} \\
&\qquad + 4^{1/m-1} x^{1/m} \sum_{j=0}^{m-1} \left(T_{mn-2}(z) w^j + w^j \frac{\arcsin(\sqrt{z})}{z} \sqrt{\frac{z}{1-z}} S_{mn}(z) \right) + C x^n \\
&\qquad - \sum_{k=0}^{n-1} \frac{4^k x^k}{(k-n)\binom{2mk}{mk}}.
\end{aligned} \tag{13}$$

The third term on the right-hand side equals

$$\left(4^{1/m-1} x^{1/m} \sum_{j=0}^{n-1} T_{mn-2}(z) w^j \right) + \sum_{j=0}^{m-1} \arcsin(\sqrt{z}) \sqrt{\frac{z}{1-z}} S_{mn}(z).$$

By comparing the coefficients of x^n on both sides of Equation (13), we have

$$0 = C + \frac{4^n}{\binom{2mn}{mn}} \Big((1-m)\log(4) + \pi i (m-1) \Big) + \mathrm{Coef}_{x^n} \left(\sum_{j=0}^{m-1} \arcsin(\sqrt{z}) \sqrt{\frac{z}{1-z}} S_{mn}(z) \right).$$

We know that, from Melzak [16, p. 108] and Lehmer [15],

$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{4^k z^k}{k \binom{2k}{k}} = \sqrt{\frac{z}{1-z}} \arcsin(\sqrt{z}). \tag{14}$$

Therefore,

$$\begin{aligned}
C &= -\frac{4^n}{\binom{2mn}{mn}}\big((1-m)\log(4)+\pi i(m-1)\big) - \frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=0}^{mn-1}\frac{4^{mn-k}s_k(4^{1/m-1}w^j)^{mn}}{(mn-k)\binom{2(mn-k)}{mn-k}}\\
&= -\frac{4^n}{\binom{2mn}{mn}}\big((1-m)\log(4)+\pi i(m-1)\big) - \frac{m}{2}\sum_{k=0}^{mn-1}\frac{4^{n-k}s_k}{(mn-k)\binom{2(mn-k)}{mn-k}}\\
&= -\frac{4^n}{\binom{2mn}{mn}}\big((1-m)\log(4)+\pi i(m-1)\big) - \frac{m4^{n(1-m)}}{2}\sum_{k=1}^{mn}\frac{4^k s_{mn-k}}{k\binom{2k}{k}}.
\end{aligned}$$

By substituting into Equation (13), we have the desired result. □

Corollary 1. *For $n \in \mathbb{N}$ fixed and for $|x| < 4$, we have*

$$\begin{aligned}
\sum_{k=n+1}^{\infty}\frac{4^k x^k}{(k-n)\binom{2k}{k}} &= xT_{n-2}(x) + \sqrt{\frac{x}{1-x}}\arcsin(\sqrt{x})S_n(x)\\
&\quad + \sum_{k=0}^{n-1}\frac{1}{k-n}\left(1-\frac{4^k}{\binom{2k}{k}}\right)x^k - \left(\frac{1}{2}\sum_{k=1}^{n}\frac{4^k}{k\binom{2k}{k}}s_{n-k}\right)x^n,
\end{aligned}$$

where the polynomials $T_{n-2}(x)$ and $S_n(x)$ are as in Definitions 1 and 2, and s_{n-k} is the $(n-k)$-th coefficient of S_n.

For ease of discussion, from now on we denote by $t_j^{(i)}$ the coefficient of x^j in the polynomial $T_i(x)$, as defined in Definition 2. Similarly, we denote by $s_j^{(i)}, m_j^{(i)}, n_j^{(i)}$ the coefficients of x^j in the polynomials $S_i(x), M_i(x), N_i(x)$, as defined in Definitions 1, 3, and 4, respectively.

Theorem 2. *For any fixed integers $m \geq 1, n \geq 1$ and for $|x| < 4^{m-1}$, we have that*

$$\begin{aligned}
&\sum_{k=n+1}^{\infty}\frac{4^k x^k}{(k-n)^2\binom{2mk}{mk}}\\
&= \frac{m2^{2n+1}x^n}{\binom{2mn}{mn}}\sum_{j=0}^{m-1}(\arcsin(\sqrt{z}))^2 + \sum_{k=1}^{n-1} r_k x^k + qx^n\\
&\quad - 4^{-n(m-1)}mx^n\sum_{j=0}^{m-1}\left(\frac{\arcsin(\sqrt{z})\sqrt{z(1-z)}}{z^{mn}}\cdot\sum_{k=0}^{mn-1}s_k^{(mn)}M_{mn-k-1}(z)z^k\right)
\end{aligned}$$

where $z = 4^{1/m-1}w^j x^{1/m}$ depends on j, $w = e^{2\pi i/m}$ is the m-th root of unity, the *polynomial* $M_{mn-k-1}(z)$ *is as in Definition 3, and*

$$r_k := m4^{k(1-m)}\left(\frac{t_{mk-1}^{(mn-2)}}{(k-n)} + \frac{1}{2}\sum_{l=0}^{mk-1}\frac{s_l^{(mn)}m_{mk-l-1}^{(mn-l-1)}}{k-n}\right) + \frac{4^k}{(k-n)^2}\left(\frac{1}{4^{km}} - \frac{1}{\binom{2mk}{mk}}\right)$$

and

$$q := m^2 4^{-n(m-1)} \cdot \left(-\frac{2^{2mn+1}}{\binom{2mn}{mn}} + \sum_{l=0}^{mn-2} s_l^{(mn)}\left(\frac{4^{mn-l}}{(mn-l)\binom{2(mn-l)}{mn-l}} - \frac{1}{2}\sum_{p=1}^{mn-l-1}\frac{4^p m_{mn-l-1-p}^{(mn-l-1)}}{p(2p+1)\binom{2p}{p}}\right)\right).$$

Proof. By multiplying both sides of the equation in Theorem 1 by x^{-n-1}, integrating and then again multiplying by x^n, we have

$$\begin{aligned}
&\sum_{k=n+1}^{\infty}\frac{4^k x^k}{(k-n)^2\binom{2mk}{mk}} \\
&= 4^{1/m-1}x^n\sum_{j=0}^{mn-1}w^j\int x^{1/m-n-1}T_{mn-2}(z)\,dx \\
&\quad + x^n\sum_{j=0}^{m-1}\int x^{-n-1}\sqrt{\frac{z}{1-z}}\arcsin(\sqrt{z})S_{mn}(z)\,dx - \sum_{k=0}^{n-1}\frac{4^k x^k}{(k-n)^2\binom{2mk}{mk}} \\
&\quad + \sum_{k=0}^{n-1}\frac{x^k}{(k-n)^2\binom{2mk}{mk}} - \left(\frac{4^{n(1-m)}m}{2}\sum_{k=1}^{mn}\frac{4^k s_{mn-k}^{(mn)}}{k\binom{2k}{k}}\right)x^n\log(x) + Cx^n,
\end{aligned} \tag{15}$$

where C is a constant to be determined.

On the one hand, in Equation (15) we have

$$\begin{aligned}
\int x^{1/m-n-1}T_{mn-2}(z)\,dx &= m4^{(m-1)(1/m-n)}w^{-j}\int z^{-mn}T_{mn-2}(z)\,dz \\
&= m4^{(m-1)(1/m-n)}w^{-j}\sum_{k=0}^{mn-2}t_k^{(mn-2)}\int z^{k-mn}\,dz \\
&= m4^{(m-1)(1/m-n)}\sum_{k=0}^{mn-2}\frac{t_k^{(mn-2)}z^{k-mn+1}}{k-mn+1} \\
&= x^{-n+1/m}\sum_{k=0}^{mn-2}\frac{4^{k(1/m-1)}t_k^{(mn-2)}w^{kj}x^{k/m}}{k-mn+1}.
\end{aligned}$$

Then the first term of the right-hand side of (15) equals

$$m4^{1/m-1}x^{1/m}\sum_{k=0}^{mn-2}\left(\frac{4^{k(1/m-1)}t_k^{(mn-2)}}{k-mn+1}x^{k/m}\sum_{j=0}^{m-1}w^{j(k+1)}\right)$$
$$= m^2\sum_{k=1}^{n-1}\frac{4^{k(1-m)}t_{mk-1}^{(mn-2)}x^k}{m(k-n)}.$$

By Lemma 2, and expanding the second term of the right-hand side of Equation (15),

$$x^n\sum_{j=0}^{m-1}\int x^{-n-1}\sqrt{\frac{z}{1-z}}\arcsin(\sqrt{z})S_{mn}(z)\,dx$$
$$= mx^n4^{-n(m-1)}\sum_{j=0}^{m-1}\int\frac{z^{-mn-1/2}\arcsin(\sqrt{z})}{\sqrt{1-z}}S_{mn}(z)\,dz$$
$$= mx^n4^{-n(m-1)}$$
$$\cdot\sum_{j=0}^{m-1}\left(\sum_{k=0}^{mn-1}s_k^{(mn)}\int z^{-(mn-k)-1/2}\frac{\arcsin(\sqrt{z})}{\sqrt{1-z}}\,dz+s_{mn}^{(mn)}\arcsin^2(\sqrt{z})\right) \quad (16)$$
$$= mx^n4^{-n(m-1)}$$
$$\cdot\sum_{j=0}^{m-1}\left(\sum_{k=0}^{mn-1}s_k^{(mn)}\left(d_{mn-k}\log(z)-\frac{\arcsin(\sqrt{z})\sqrt{z(1-z)}}{z^{mn-k}}M_{mn-k-1}(z)\right.\right.$$
$$\left.\left.+\frac{N_{mn-k-2}(z)}{z^{mn-k-1}}\right)+s_{mn}^{(mn)}(\arcsin(\sqrt{z}))^2\right).$$

Let us expand one at a time the terms of this last expression. On the one hand, since $d_n=\frac{4^n}{2n\binom{2n}{n}}$ for all $n\in\mathbb{N}$, we have

$$mx^n4^{-n(m-1)}\sum_{j=0}^{m-1}\sum_{k=0}^{mn-1}s_k^{(mn)}d_{mn-k}\log(z)$$
$$= mx^n4^{-n(m-1)}\Big((1-m)\log(4)+\log(x)+i\pi(m-1)\Big)\left(\sum_{k=0}^{mn-1}s_k^{(mn)}d_{mn-k}\right)$$
$$= mx^n4^{-n(m-1)}\Big((1-m)\log(4)+\log(x)+i\pi(m-1)\Big)\cdot\sum_{k=1}^{mn}\frac{4^ks_{mn-k}^{(mn)}}{2k\binom{2k}{k}}.$$

On the other hand, we have

$$
\begin{aligned}
&mx^n 4^{-n(m-1)} \sum_{j=0}^{m-1}\sum_{k=0}^{mn-1} s_k^{(mn)} \frac{N_{mn-k-2}(z)}{z^{mn-k-1}} \\
&= mx^n 4^{-n(m-1)} \sum_{j=0}^{m-1} \left(\sum_{k=0}^{mn-2} \Big(\frac{s_k^{(mn)}}{z^{mn}} \sum_{l=0}^{mn-k-2} n_l^{(mn-k-l)} z^{l+k+1} \Big) \right) \\
&= mx^n 4^{-n(m-1)} \sum_{j=0}^{m-1} \left(\sum_{s=0}^{mn-2} \frac{z^{s+1}}{z^{mn}} \sum_{k=0}^{s} s_k^{(mn)} n_{s-k}^{(mn-k-2)} \right) \\
&= mx^n 4^{-n(m-1)} \\
&\quad \cdot \sum_{s=0}^{mn-2} \left(4^{(1/m-1)(s+1-mn)} x^{1/m(s+1-mn)} \Big(\sum_{j=0}^{m-1} w^{j(s+1-mn)} \Big) \sum_{k=0}^{s} s_k^{(mn)} n_{s-k}^{(mn-k-2)} \right) \\
&= m^2 x^n 4^{-n(m-1)} \sum_{p=1}^{n-1} \left(4^{(1/m-1)(-mp)} x^{-p} \sum_{k=0}^{m(n-p)-1} s_k^{(mn)} n_{m(n-p)-k-1}^{(mn-k-2)} \right) \\
&= m^2 4^{-n(m-1)} \sum_{k=1}^{n-1} \left(4^{k(1-m)} \sum_{l=0}^{mk-1} s_l^{(mn)} n_{mk-l-1}^{(mn-l-2)} \right) x^k \\
&= m^2 4^{-n(m-1)} \sum_{k=1}^{n-1} \left(\frac{4^{k(1-m)}}{2} \sum_{l=0}^{mk-1} \frac{s_l^{(mn)} m_{mk-l-1}^{(mn-l-1)}}{m(k-n)} \right) x^k .
\end{aligned}
$$

The latter equality above is justified by the relation between the coefficients of polynomials $M(x)$ and $N(x)$, as shown in Definition 4.

We still need to determine the value of C in Equation (15). By comparing coefficients of x^n in that equation, along with the calculations done up to this point, we have

$$
\begin{aligned}
0 = C &+ m4^{-n(m-1)} \Big((1-m)\log(4) + \pi i (m-1) \Big) \cdot \sum_{k=1}^{mn} \frac{4^k s_{mn-k}^{(mn)}}{2k\binom{2k}{k}} \\
&+ m4^{-n(m-1)} \operatorname{Coef}_{x^n} \Big(x^n \sum_{j=0}^{m-1} s_{mn}^{(mn)} \arcsin^2(\sqrt{z}) \Big) \\
&- m4^{-n(m-1)} \operatorname{Coef}_{x^n} \left(x^n \sum_{j=0}^{m-1} \frac{\arcsin(\sqrt{z})\sqrt{z(1-z)} \sum_{k=0}^{mn-1} s_k^{(mn)} M_{mn-n-1}(z) z^k}{z^{mn}} \right)
\end{aligned}
$$

Sprugnoli [17] showed that $\arcsin^2(\sqrt{z}) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{4^k z^k}{k^2\binom{2k}{k}}$, so we have

$$
\begin{aligned}
\operatorname{Coef}_{x^n} \left(x^n s_{mn}^{(mn)} \sum_{j=0}^{m-1} \arcsin^2(\sqrt{z}) \right) &= \operatorname{Coef}_{x^n} \left(\frac{x^n s_{mn}^{(mn)}}{2} \sum_{j=0}^{m-1} \sum_{k=1}^{\infty} \frac{4^k (4^{1/m-1} w^j)^k x^{k/m}}{k^2 \binom{2k}{k}} \right) \\
&= 0.
\end{aligned}
$$

On the other hand, by Equation (14), we obtain

$$\begin{aligned}\arcsin\left(\sqrt{z}\right)\sqrt{z(1-z)} &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{4^k z^k}{k\binom{2k}{k}} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{4^k z^{k+1}}{k\binom{2k}{k}} \\ &= z - \frac{1}{2}\sum_{k=1}^{\infty}\frac{4^k z^{k+1}}{k(2k+1)\binom{2k}{k}}.\end{aligned}$$

Then

$$\begin{aligned}&\mathrm{Coef}_{x^n}\left(x^n\sum_{j=0}^{m-1}\frac{\arcsin(\sqrt{z})\sqrt{z(1-z)}}{z^{mn}}\sum_{k=0}^{mn-1}s_k^{(mn)}M_{mn-k-1}(z)z^k\right) \\ &= \mathrm{Coef}_{x^n}\left(\sum_{j=0}^{m-1}\frac{\sum_{k=0}^{mn-1}\left(s_k^{(mn)}\left(z-\frac{1}{2}\sum_{l=1}^{\infty}\frac{4^l z^{l+1}}{l(2l+1)\binom{2l}{l}}\right)\sum_{p=0}^{mn-k-1}m_p^{(mn-k-1)}z^{p+k}\right)}{(4^{1/m-1}w^j)^{mn}}\right) \\ &= \mathrm{Coef}_{x^n}\left(\sum_{j=0}^{m-1}\frac{z^{mn}\sum_{k=0}^{mn-1}s_k^{(mn)}\left(m_{mn-k-1}^{(mn-k-1)}-\frac{1}{2}\sum_{l=1}^{mn-k-1}\frac{4^l m_{mn-k-l-1}^{(mn-k-1)}}{l(2l+1)\binom{2l}{l}}\right)}{(4^{1/m-1}w^j)^{mn}}\right) \\ &= m\left(\sum_{l=0}^{mn-1}s_l^{(mn)}m_{mn-l-1}^{(mn-l-1)} - \frac{1}{2}\sum_{l=0}^{mn-2}\sum_{p=1}^{mn-l-1}\frac{s_l^{(mn)}4^p}{p(2p+1)\binom{2p}{p}}m_{mn-l-p-1}^{(mn-l-1)}\right).\end{aligned}$$

The proof is then complete by Remarks 1 and 3. □

Corollary 2. *For every fixed integer $n \geq 1$ and for all $|x| < 1$, we have*

$$\begin{aligned}&\sum_{k=n+1}^{\infty}\frac{4^k x^k}{(k-n)^2\binom{2k}{k}} \\ &\quad = \frac{2^{2n+1}x^n}{\binom{2n}{n}}(\arcsin\left(\sqrt{x}\right))^2 - \arcsin\left(\sqrt{x}\right)\sqrt{x(1-x)}\left(\sum_{k=0}^{n-1}s_k^{(n)}M_{n-k-1}(x)x^k\right) \\ &\quad\quad + \sum_{k=1}^{n-1}\overline{r_k}x^k + \overline{q}x^n,\end{aligned}$$

where

$$\begin{aligned}\overline{r_k} &:= \left(\frac{t_{k-1}^{(n-2)}}{k-n} + \frac{1}{2}\sum_{l=0}^{k-1}\frac{s_l^{(n)}m_{k-l-1}^{(n-l-1)}}{k-n}\right) + \frac{4^k}{(k-n)^2}\left(\frac{1}{4^k} - \frac{1}{\binom{2k}{k}}\right); \\ \overline{q} &:= \left(-\frac{2^{2n+1}}{\binom{2n}{n}} + \sum_{l=0}^{n-2}s_l^{(n)}\left(\frac{4^{n-l}}{(n-l)\binom{2(n-l)}{n-l}} - \frac{1}{2}\sum_{p=1}^{n-l-1}\frac{4^p}{p(2p+1)\binom{2p}{p}}m_{n-l-1-p}^{(n-l-1)}\right)\right),\end{aligned}$$

and the polynomial $M_{n-k-1}(x)$ is as in Definition 3.

5. Applications

We present here some new results concerning sums in which the generalized central binomial coefficient appears in the denominator, using the theory developed in this work.

Let $x = \pm\frac{1}{4}$. By varying the values of m, n in Theorem 1, we have

(a) $$\sum_{k=3}^{\infty} \frac{1}{(k-2)\binom{2k}{k}} = \frac{13}{36} - \frac{\pi\sqrt{3}}{18},$$

(b) $$\sum_{k=4}^{\infty} \frac{(-1)^k}{(k-3)\binom{2k}{k}} = -\frac{19}{200} + \frac{\sqrt{5}}{10}\operatorname{arcsinh}(1/2),$$

(c) $$\sum_{k=2}^{\infty} \frac{1}{(k-1)\binom{4k}{2k}} = \frac{7}{18} - \frac{\pi\sqrt{3}}{18} - \frac{\sqrt{5}}{15}\operatorname{arcsinh}(1/2),$$

(d) $$\sum_{k=3}^{\infty} \frac{(-1)^k}{(k-2)\binom{4k}{2k}}$$
$$= -\frac{219}{4900} - \frac{\sqrt{34}}{595}\left(\left(2\sqrt{\sqrt{17}+1} - 7\sqrt{\sqrt{17}-1}\right)\arcsin\left(\frac{\sqrt{5-2\sqrt{2}} - \sqrt{5+2\sqrt{2}}}{4}\right)\right.$$
$$\left. + \left(2\sqrt{\sqrt{17}-1} + 7\sqrt{\sqrt{17}+1}\right)\log\left(\frac{\sqrt{2(\sqrt{17}-3)} + \sqrt{5-2\sqrt{2}} + \sqrt{5+2\sqrt{2}}}{4}\right)\right),$$

(e) $$\sum_{k=2}^{\infty} \frac{1}{(k-1)\binom{6k}{3k}} = \frac{37}{200} - \frac{7\pi\sqrt{3}}{180} + \frac{1}{210}\left(\sqrt{42(\sqrt{21}+3)}\log\left(\frac{\sqrt{2(\sqrt{21}-3)}+\sqrt{7}+\sqrt{3}}{4}\right)\right.$$
$$\left. - \sqrt{42(\sqrt{21}-3)}\arcsin\left(\frac{\sqrt{7}-\sqrt{3}}{4}\right)\right).$$

Let $x = \frac{1}{4}$. By varying the values of m, n in Theorem 2, we have

(a) $$\sum_{k=3}^{\infty} \frac{1}{(k-2)^2\binom{2k}{k}} = -\frac{53}{108} + \frac{\pi\sqrt{3}}{12} + \frac{\pi^2}{108},$$

(b) $$\sum_{k=2}^{\infty} \frac{1}{(k-1)^2\binom{4k}{2k}} = -\frac{41}{27} + \frac{\pi\sqrt{3}}{6} + \frac{\pi^2}{54} + \frac{5\sqrt{5}\operatorname{arcsinh}(1/2)}{9} - \frac{2}{3}\operatorname{arcsinh}^2(1/2),$$

(c) $$\sum_{k=4}^{\infty} \frac{1}{(k-3)^2\binom{4k}{2k}} = -\frac{406705027}{11093751900} + \frac{1823\sqrt{3}\pi}{237160} + \frac{\pi^2}{8316} - \frac{\operatorname{arcsinh}^2(1/2)}{231}$$
$$- \frac{3137\sqrt{5}\operatorname{arcsinh}(1/2)}{640332},$$

(d) $$\sum_{k=2}^{\infty}\frac{1}{(k-1)^2\binom{6k}{3k}}$$
$$= -\frac{134}{125}+\frac{21\pi\sqrt{3}}{200}+\frac{\pi^2}{120}-\frac{12}{5}\log^2(2)+\frac{3}{5}\arcsin^2\Big(\frac{\sqrt{7}-\sqrt{3}}{4}\Big)$$
$$+\frac{1}{200}\arcsin\Big(\frac{\sqrt{7}-\sqrt{3}}{4}\Big)\left(23\sqrt{3(2\sqrt{21}+3)}-27\sqrt{2\sqrt{21}-3}\right)$$
$$-\frac{\log(2)}{100}\left(23\sqrt{3(2\sqrt{21}-3)}+27\sqrt{2\sqrt{21}+3}\right)-\frac{3\log^2\left(\sqrt{2(\sqrt{21}-3)}+\sqrt{7}+\sqrt{3}\right)}{5}$$
$$+\log\left(\sqrt{2(\sqrt{21}-3)}+\sqrt{7}+\sqrt{3}\right)\Big(\frac{12\log(2)}{5}+\frac{23\sqrt{3(2\sqrt{21}-3)}+27\sqrt{2\sqrt{21}+3}}{200}\Big).$$

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