#A20 INTEGERS 26 (2026)

ON COLORED PARTITIONS AND EULER-TYPE IDENTITIES

Beimar J. Naranjo
Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
bnaranjom@unal.edu.co

José L. Ramírez
Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
jlramirezr@unal.edu.co

Received: 6/22/25, Revised: 9/29/25, Accepted: 12/29/25, Published: 1/19/26

Abstract

We study colored versions of integer partitions and compositions in which each part is assigned a color given by a composition or partition of the same size as the part. We introduce several families of such objects and establish bijections with permutations, set partitions, and multisets. As a consequence, we recover classical sequences such as Bell numbers and factorials. In the second part, we define the notion of admissible sets of compositions and prove a generalization of Euler's partition theorem in this context. These results extend recent work of Goyal and are established through bijective arguments.

1. Introduction

A *partition* of a positive integer n is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of positive integers such that $\sum_{i=1}^{\ell} \lambda_i = n$. For example, the partitions of 4 are (4), $(3,1)$, $(2,2)$, $(2,1,1)$, and $(1,1,1,1)$, so that $p(4) = 5$, where $p(n)$ denotes the number of partitions of n. A *composition* of n is an ordered sequence of positive integers that sum to n. For example, the compositions of 3 are (3), $(2,1)$, $(1,2)$, and $(1,1,1)$. The number of compositions of n is $c(n) = 2^{n-1}$ for $n \geq 1$, with $c(0) = 1$ (see [14, p. 18]).

Euler's partition theorem asserts that the number of partitions of n into distinct parts equals the number of partitions into odd parts (cf. [4]). A similar identity holds for compositions: the number of compositions of n into odd parts equals the number of compositions into parts greater than one [8].

DOI: 10.5281/zenodo.18305059

A classical variation of integer partitions allows each part of size m to appear in up to m distinct copies. These are known as *colored partitions*. In this context, each part of size m may occur in m different colors, denoted by m_i, where $1 \leq i \leq m$ indicates the color. Colored partitions appeared implicitly in the work of MacMahon [10, Chapters 11–12], and were later recognized in the study of the hard hexagon model—specifically in Regime III—as noted by Andrews and Paule [5]. The first explicit study of colored partitions was carried out by Agarwal and Andrews [2, 3]. For example, the colored partitions of 3 are

$$(3_1), \quad (3_2), \quad (3_3), \quad (2_1, 1_1), \quad (2_2, 1_1), \quad (1_1, 1_1, 1_1).$$

The corresponding generating function is (cf. [2, 3])

$$\sum_{n=0}^{\infty} p_c(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n},$$

where $p_c(n)$ denotes the number of colored partitions of n. This generating function also counts the number of plane partitions (cf. [11]).

Similarly, a *colored composition* of n is a composition in which each summand of size m can appear in one of m different colors. Let $c_c(n)$ denote the number of colored compositions of n. For instance, $c_c(3) = 8$, with colored compositions:

$$(3_1), \quad (3_2), \quad (3_3), \quad (2_1, 1_1), \quad (2_2, 1_1), \quad (1_1, 2_1), \quad (1_1, 2_2), \quad (1_1, 1_1, 1_1).$$

This notion was introduced by Agarwal [1], who showed that the number of colored compositions of n is the Fibonacci number F_{2n}.

Recently, Goyal [7] introduced the notion of split n-color partitions, where each part is assigned a color obtained by splitting its index into at most two summands. She showed, among other results, that the number of such partitions of an integer into distinct parts coincides with the number of split n-color partitions in which even parts are not allowed to be colored by a split whose summands are both odd.

In this paper, we study new variations of colored partitions and compositions in which the colors themselves are given by integer compositions or partitions. We consider families where parts are colored by compositions with a bounded number of summands, by partitions into distinct or restricted parts, or more generally, by admissible sets (defined in Section 3) specified through parity-based closure properties. These constructions allow us to extend classical results such as Euler's partition theorem in rich new directions.

In the first part of the paper, we explore structural properties of these colored partitions, including several bijections to well-known combinatorial objects. For instance, we show that certain classes of colored partitions are in bijection with permutations, set partitions, and multisets. In particular, we recover Bell numbers and factorial numbers through natural encodings of coloring constraints.

In the second part, we introduce a general framework based on admissible sets, which unifies many earlier results, including the extension proposed by Goyal [7], and leads to a general colored version of Euler's theorem.

2. Colored Partitions and Compositions

2.1. Colored Partitions with Restrictions

A colored partition *has different colors* if no two parts are assigned the same color. Let $pdc_c(n)$ denote the number of such colored partitions of n. For example, $pdc_c(5) = 12$, and the corresponding partitions are

$$(5_1),\quad (5_2),\quad (5_3),\quad (5_4),\quad (5_5),\quad (4_2,1_1),\quad (4_3,1_1),$$
$$(4_4,1_1),\quad (3_2,2_1),\quad (3_3,2_1),\quad (3_1,2_2),\quad (3_3,2_2).$$

The number $pdc_c(n)$ equals the number of colored compositions of n in which the sequence of colors is strictly increasing. This set was enumerated by Bala [6], and its generating function is

$$\sum_{n\geq 0} pdc_c(n)\, q^n = \prod_{n=1}^{\infty} \frac{1-q+q^n}{1-q}.$$

The following table shows the first few values of $pdc_c(n)$:

n	1	2	3	4	5	6	7	8	9	10
$pdc_c(n)$	1	2	4	7	12	20	33	53	84	131

This sequence corresponds to A126348 in the OEIS [13]. We now define the *size* of a colored part ρ_j to be ρ. For example, the size of the part 12_8 is 12. Let $P_{\leq n}$ be the set of colored partitions in which each part has a distinct color and size at most n. For example,

$$\begin{aligned} P_{\leq 3} = \{&(1_1),\ (2_1),\ (2_2),\ (3_1),\ (3_2),\ (3_3),\ (2_2,1_1),\ (3_2,1_1),\ (3_3,1_1),(2_2,2_1),\\ &(3_1,2_2),\ (3_2,2_1),\ (3_3,2_1),\ (3_3,2_2),\ (3_2,3_1),\ (3_3,3_1),\ (3_3,3_2),(3_3,2_2,1_1),\\ &(3_3,3_2,1_1),\ (3_3,2_2,2_1),\ (3_3,3_1,2_2),\ (3_3,3_2,2_1),\ (3_3,3_2,3_1)\}. \end{aligned}$$

Theorem 1. *The number of colored partitions with parts of size at most n and different colors is $(n+1)!-1$.*

Proof. We construct a bijection between the set $P_{\leq n}$ and the set of all permutations of $[n+1] = \{1,2,\ldots,n+1\}$, excluding the identity.

Given a colored partition in $P_{\leq n}$, first order the parts so that the sequence of colors is strictly increasing, and then add 1 to each part size. This yields a sequence of the form $(\rho(1)_{\mu_1}, \rho(2)_{\mu_2}, \ldots, \rho(k)_{\mu_k})$, where $2 \leq \rho(i) \leq n+1$ and $\mu_1 < \mu_2 < \cdots < \mu_k$. Starting from the identity permutation $12\cdots(n+1)$, we perform a sequence of transpositions: at each step, we swap the entries in positions $\rho(i)$ and μ_i. Since $\rho(i) > \mu_i$, each swap moves a larger index to an earlier position, eventually transforming the identity into a nontrivial permutation. For example, the colored partition $(4_4, 3_2, 1_1)$ becomes $(2_1, 4_2, 5_4)$ after increasing the part sizes. Applying transpositions to the identity permutation 12345:

- Swap positions 2 and 1: 21345,
- Swap positions 4 and 2: 24315,
- Swap positions 5 and 4: 24351.

The resulting permutation is 24351.

Conversely, given any permutation $\pi = \pi(1)\pi(2)\cdots\pi(n+1)$ of $[n+1]$, define σ by $\sigma^{-1}(i) = \pi(i)$. Let $j_1 < j_2 < \cdots < j_k$ be the positions at which σ differs from the identity. Starting from $12\cdots(n+1)$, we recover the partition by reversing the transpositions: each pair $(j_i, \sigma^{-1}(j_i))$ corresponds to the colored part $(\sigma^{-1}(j_i)-1)_{j_i}$, provided that $\sigma^{-1}(j_i) > j_i$. In our example, with $\pi = 24351$, we find $\sigma = 51324$. The non-fixed positions are $j_1 = 1$, $j_2 = 2$, $j_3 = 4$, and $j_4 = 5$. We recover:

- $(2-1)_1 = 1_1$, giving 21345,
- $(4-1)_2 = 3_2$, giving 24315,
- $(5-1)_4 = 4_4$, giving 24351,

which corresponds to the colored partition $(4_4, 3_2, 1_1)$. □

Example 1. For $n = 2$, we consider permutations of the set $[3] = \{1, 2, 3\}$, excluding the identity. There are $3! - 1 = 5$ such permutations: 213, 132, 231, 312, 321. Each corresponds uniquely to a colored partition in $P_{\leq 2}$:

Permutation	Colored partition
213	(1_1)
132	(2_1)
231	(2_2)
312	$(2_2,\ 1_1)$
321	$(2_2,\ 2_1)$

Each part has size at most 2, and all colors are distinct. The bijection follows the transposition rule described in the proof.

A colored partition has *fully different parts* if no two parts share the same size or the same color. For example, 3_2 and 2_1 are fully different. Let $pf_c(n)$ denote the number of colored partitions of n with fully different parts. For instance, when $n = 5$, we have $pf_c(5) = 12$, with the following partitions:

$$(5_1),\quad (5_2),\quad (5_3),\quad (5_4),\quad (5_5),\quad (4_2, 1_1),\quad (4_3, 1_1),$$
$$(4_4, 1_1),\quad (3_1, 2_2),\quad (3_2, 2_1),\quad (3_3, 2_1),\quad (3_3, 2_2).$$

Theorem 2. *For all $n \geq 1$,*

$$pf_c(n) = \sum_{\substack{p_1+\cdots+p_k=n \\ p_1>\cdots>p_k}} p_k(p_{k-1} - 1)\cdots(p_1 - (k-1)).$$

Proof. Let $(p_1, \ldots, p_k)$ be a partition of n into k distinct parts with $p_1 > \cdots > p_k$. To form a colored partition with fully different parts, we assign to each p_i a color not used for any smaller part. The smallest part p_k has p_k available colors. The next part p_{k-1} must avoid the color used by p_k, so it has $p_{k-1} - 1$ choices. More generally, p_i has $p_i - (k - i)$ available colors. The total number of colorings is therefore $p_k(p_{k-1} - 1)\cdots(p_1 - (k - 1))$. Summing over all partitions of n into k distinct parts yields the result. □

The following table shows the values of $pf_c(n)$ for $1 \leq n \leq 10$:

n	1	2	3	4	5	6	7	8	9	10
$pf_c(n)$	1	2	4	6	12	17	31	43	77	105

This sequence does not appear to be listed in the OEIS.

Let $Q_{\leq n}$ be the set of colored partitions with fully different parts and parts of size at most n. For example,

$$Q_{\leq 3} = \{(1_1),\ (2_1),\ (2_2),\ (3_1),\ (3_2),\ (3_3),\ (2_2, 1_1),\ (3_2, 1_1),\ (3_3, 1_1),$$
$$(3_1, 2_2),\ (3_2, 2_1),\ (3_3, 2_1),\ (3_3, 2_2),\ (3_3, 2_2, 1_1)\}.$$

The next result gives a relation between the cardinality of $Q_{\leq n}$ and the Bell numbers. Let B_n denote the nth *Bell number* (sequence A000110 in the OEIS), which counts the number of set partitions of $[n]$; see [12] for background. The first few values are

$$1,\quad 1,\quad 2,\quad 5,\quad 15,\quad 52,\quad 203,\quad 877,\quad 4140,\ \ldots$$

Theorem 3. *The number of colored partitions into fully different parts with parts of size at most n is equal to $B_{n+1} - 1$.*

Proof. We construct a bijection between the set $Q_{\leq n}$ of colored partitions with fully different parts (of size at most n) and the set of set partitions of $[n + 1]$, excluding the singleton partition $\{[n + 1]\}$.

Let $\lambda = (\rho(1)_{\mu_1}, \ldots, \rho(m)_{\mu_m}) \in Q_{\leq n}$, where the sizes satisfy $\rho(1) < \cdots < \rho(m)$ and the colors $\mu_1, \ldots, \mu_m$ are all distinct. Process the parts from left to right. Start by forming the first block $B_1 = \{\rho(1), \mu_1\}$. For each subsequent part $\rho(i)_{\mu_i}$:

- If μ_i already appears in a block, add $\rho(i)$ to that block.
- Otherwise, create a new block with $\rho(i)$ and μ_i.

After all parts have been processed, insert the elements of $[n+1]$ not yet included (in particular, $n+1$) into a final block. This yields a partition of $[n+1]$ into at least two blocks.

Conversely, given a set partition of $[n+1]$ with at least two blocks, remove the block containing $n+1$. For each remaining block:

- If the block is a singleton $\{\rho\}$, include the part ρ_ρ.
- If the block has more than one element, list its elements in increasing order $\rho_1 < \rho_2 < \cdots < \rho_k$ and include the parts $(\rho_k)_{\rho_{k-1}},\ (\rho_{k-1})_{\rho_{k-2}},\ \ldots,\ (\rho_2)_{\rho_1}$.

The result is a colored partition in $Q_{\leq n}$. This process defines a bijection, completing the proof. □

Example 2. The case $n = 3$ illustrates the bijection between colored partitions in $Q_{\leq 3}$ and set partitions of $[4] = \{1, 2, 3, 4\}$ with at least two blocks. The correspondences are shown below:

$$
\begin{array}{rlrl}
(1_1) \leftrightarrow & \{\{1\},\{2,3,4\}\} & (3_2,1_1) \leftrightarrow & \{\{1\},\{2,3\},\{4\}\} \\
(2_1) \leftrightarrow & \{\{1,2\},\{3,4\}\} & (3_3,1_1) \leftrightarrow & \{\{1\},\{3\},\{2,4\}\} \\
(2_2) \leftrightarrow & \{\{2\},\{1,3,4\}\} & (3_2,2_1) \leftrightarrow & \{\{1,2,3\},\{4\}\} \\
(3_1) \leftrightarrow & \{\{1,3\},\{2,4\}\} & (3_3,2_1) \leftrightarrow & \{\{1,2\},\{3\},\{4\}\} \\
(3_2) \leftrightarrow & \{\{2,3\},\{1,4\}\} & (3_3,2_2) \leftrightarrow & \{\{2\},\{3\},\{1,4\}\} \\
(3_3) \leftrightarrow & \{\{3\},\{1,2,4\}\} & (3_1,2_2) \leftrightarrow & \{\{1,3\},\{2\},\{4\}\} \\
(2_2,1_1) \leftrightarrow & \{\{1\},\{2\},\{3,4\}\} & (3_3,2_2,1_1) \leftrightarrow & \{\{1\},\{2\},\{3\},\{4\}\}
\end{array}
$$

2.2. Double Compositions

In 1979, Kaneiwa [9] introduced the concept of *double partitions*, where each part m is assigned a partition of m as its color. Inspired by this idea, we define *double compositions* as compositions in which each summand m is assigned a partition of m. That is, each part carries a partition-valued color. For example, the double compositions of 3 are

$$
\begin{array}{llll}
(1_1,\ 1_1,\ 1_1), & (2_{1+1},\ 1_1), & (2_2,\ 1_1), & (1_1,\ 2_{1+1}), \\
(1_1,\ 2_2), & (3_{1+1+1}), & (3_{2+1}), & (3_3).
\end{array}
$$

Let $d_c(n)$ denote the number of double compositions of n. By conditioning on the size and color of the first part, one can derive the recurrence:

$$d_c(n) = \sum_{k=1}^{n} d_c(n-k)\, p(k), \quad \text{with } d_c(0) = 1,$$

where $p(k)$ is the number of integer partitions of k. The corresponding generating function is

$$\sum_{n\geq 0} d_c(n)\, q^n = \left(2 - \prod_{n=1}^{\infty} \frac{1}{1-q^n}\right)^{-1}.$$

The following table shows the values of $d_c(n)$, the number of double compositions of n:

n	1	2	3	4	5	6	7	8	9	10
$d_c(n)$	1	1	3	8	22	59	160	431	1164	3140

This sequence corresponds to A055887 in the OEIS [13]. This sequence also counts the number of ways to partition a multiset of length n whose elements form an initial interval of positive integers, and in which each block consists of repeated copies of a single element (that is, a constant multiset). For example, when $n = 3$, there are four such multisets $\{1,1,1\}$, $\{1,1,2\}$, $\{1,2,2\}$, $\{1,2,3\}$. Among these, there are exactly 8 partitions into constant multisets:

$$\{\{1\},\{1\},\{1\}\}, \quad \{\{1,1\},\{2\}\}, \quad \{\{1\},\{2,2\}\}, \quad \{\{1\},\{2\},\{3\}\},$$
$$\{\{1,1,1\}\}, \quad \{\{1\},\{1\},\{2\}\}, \quad \{\{1\},\{2\},\{2\}\}, \quad \{\{1,1\},\{1\}\}.$$

Theorem 4. *The number of partitions of a multiset of length n, whose elements form an initial interval of positive integers and whose blocks are constant multisets, is equal to $d_c(n)$.*

Proof. We construct a bijection between these multiset partitions and double compositions of n.

Given a double composition $(\rho(1)_{\lambda_1}, \ldots, \rho(m)_{\lambda_m})$, where each λ_i is a partition of $\rho(i)$, we proceed as follows: for each i, create a block containing $\rho(i)$ copies of the integer i. Then subdivide this block according to $\lambda_i = \lambda_{i,1} + \cdots + \lambda_{i,j_i}$, placing each sub-block of size $\lambda_{i,s}$ into the partition as a constant multiset.

Conversely, given a multiset of length n with the stated properties, sort its blocks by the integer they contain (and secondarily by block size, if necessary). Let $\rho(i)$ be the total number of times the integer i appears in the multiset. If there are j_i constant blocks consisting of copies of i with respective sizes $\lambda_{i,1}, \ldots, \lambda_{i,j_i}$, then $\lambda_i = \lambda_{i,1} + \cdots + \lambda_{i,j_i}$ is a partition of $\rho(i)$, and the corresponding term in the double composition is $\rho(i)_{\lambda_i}$. This establishes a bijection between the two structures. □

Example 3. The following correspondence illustrates the bijection described in the proof above:

$$(3_{2+1},\ 2_{1+1},\ 3_3,\ 1_1,\ 2_{1+1}) \ \leftrightarrow\ \{\{1,1\},\ \{1\},\ \{2\},\ \{2\},\ \{3,3,3\},\ \{4\},\ \{5\},\ \{5\}\}.$$

Each part of the double composition becomes a group of identical elements, and its coloring determines how those elements are split into constant multisets.

If we restrict our attention to colored compositions in which each partition used as a color consists of *distinct parts*, we obtain the following corollary.

Corollary 1. *The number of partitions of a multiset of length n, whose elements form an initial interval of positive integers and whose blocks are distinct constant multisets, is equal to the number of colored compositions of n with partitions into distinct parts.*

Example 4. For $n = 3$, there are exactly five such multiset partitions and corresponding double compositions:

$$\begin{aligned}\{\{1,1,1\}\} &\leftrightarrow (3_3),\\ \{\{1\},\{1,1\}\} &\leftrightarrow (3_{2+1}),\\ \{\{1,1\},\{2\}\} &\leftrightarrow (2_2,1_1),\\ \{\{1\},\{2,2\}\} &\leftrightarrow (1_1,2_2),\\ \{\{1\},\{2\},\{3\}\} &\leftrightarrow (1_1,1_1,1_1).\end{aligned}$$

In each case, the blocks of the multiset partition are distinct constant multisets, and each summand in the colored composition is assigned a partition into distinct parts.

2.3. Colored Partitions with Compositions

A *partition colored with compositions* is a partition in which each part of size m is assigned a composition of m as its color. For example, the following are partitions of 4 colored with compositions:

$$(1_1,1_1,1_1),\quad (2_{1+1},1_1),\quad (2_2,1_1),\quad (3_3),\quad (3_{1+1+1}),\quad (3_{2+1}),\quad (3_{1+2}).$$

Let $p_{cc}(n)$ denote the number of such coloured partitions of n. This sequence satisfies the following properties, which are list in A034691 in the OEIS [13] and can be proved by classical methods. The generating function is given by

$$\sum_{n\geq 0} p_{cc}(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^{2^{n-1}}} = 1 + q + 3q^2 + 7q^3 + 18q^4 + 42q^5 + \cdots.$$

Moreover, $p_{cc}(n)$ satisfies the recurrence

$$p_{cc}(n) = \frac{1}{n} \sum_{m=1}^{n} \left(\sum_{d|m} d \cdot 2^{d-1} \cdot p_{cc}(n-m) \right), \quad \text{with } p_{cc}(0) = 1.$$

An explicit formula for $p_{cc}(n)$ is given by

$$p_{cc}(n) = \sum_{s_1+2s_2+\cdots+ns_n=n} \prod_{i=1}^{n} \binom{2^{i-1}+s_i-1}{s_i},$$

where $s_1, s_2, \ldots s_n$ are nonnegative integers.

3. Variations of Euler's Partition Theorem for Admissible Compositions

In 2019, Goyal [7] proposed a generalization of Euler's partition theorem involving colored partitions in which the available colors for each part are compositions with at most two summands and total size at most n. In this section, we show how such generalizations fit into a broader unified framework.

We begin by introducing a structural condition on sets of compositions that guarantees the existence of Euler-type identities. A subset S of the set of all compositions is said to be *even-admissible* if, for every composition $c_1 + \cdots + c_k$ in S, the composition $2c_1 + \cdots + 2c_k$ is also in S. Furthermore, if all c_i are even, then the composition $c_1/2 + \cdots + c_k/2$ must also lie in S. Likewise, S is called *odd-admissible* if, for every composition $c_1 + \cdots + c_k$ in S, the composition $(2c_1 - 1) + \cdots + (2c_k - 1)$ is in S. Additionally, if all c_i are odd, then the composition $(c_1+1)/2 + \cdots + (c_k+1)/2$ must also belong to S. We say that S is *admissible* if it is both even-admissible and odd-admissible.

Our goal is to show that Euler-type results hold whenever parts in a partition are colored using compositions drawn from an even-admissible or odd-admissible set S. In particular, Goyal's original construction corresponds to a specific admissible set, and thus becomes a special case of our more general theory.

Given such a set S and a nonnegative integer n, we define:

$$S_n = \{\sigma \in S : |\sigma| = n\}, \qquad S_{\leq n} = \{\sigma \in S : |\sigma| \leq n\}.$$

In this notation,

$$S = \bigcup_{n \geq 0} S_n = \bigcup_{n \geq 0} S_{\leq n}.$$

Theorem 5 (Generalized Euler's Partition Theorem)**.** *Let S be an even-admissible set of compositions, and let $p_S(n)$ denote the number of partitions of n in which each part of size k is colored by an element of S_k (resp. $S_{\leq k}$). Then the number of* *such partitions into distinct parts equals the number of such partitions in which no even part is assigned a color corresponding to a composition consisting entirely of even integers.*

Proof. We define a bijection between the set of partitions of n into distinct parts (colored using S) and the set of partitions of n in which even parts are not assigned colors that are fully even.

Starting with a colored partition into distinct parts, we scan the parts from left to right. Whenever we encounter an even part ρ with a color of the form $\rho_{\ell_1+\cdots+\ell_m}$ where all ℓ_i are even, we replace it with two identical parts:

$$\left(\frac{\rho}{2}\right)_{\ell_1/2+\cdots+\ell_m/2} + \left(\frac{\rho}{2}\right)_{\ell_1/2+\cdots+\ell_m/2}.$$

This operation preserves the total sum and creates a valid colored partition of n in which that particular even part no longer carries an all-even composition. We repeat this operation recursively until no even part in the partition is colored with an all-even composition.

Conversely, given a colored partition of n in which no even part has an all-even color, we again scan from left to right. Whenever we find two identical parts of the form $\rho_{\ell_1+\cdots+\ell_m} + \rho_{\ell_1+\cdots+\ell_m}$, we merge them into a single part $(2\rho)_{2\ell_1+\cdots+2\ell_m}$. Since S is even-admissible, the resulting color remains in S. Repeating this process in reverse constructs a colored partition into distinct parts. Thus, the two sets of partitions are in bijection. □

Observe that the structure of this bijection closely mirrors the classical proof of Euler's partition theorem, where parts are doubled or halved depending on their parity.

Corollary 2. *Let S be an odd-admissible set of compositions. Then the number of partitions of n into distinct parts, where each part of size k is colored by elements of S_k (resp. $S_{\leq k}$), is equal to the number of such partitions in which no even part is colored by a composition consisting entirely of odd summands.*

Note that if S is taken to be the set of integer partitions, then S is admissible. Thus, both the theorem and the corollary apply when parts are colored by partitions instead of compositions. Similarly, if S is empty, we recover the Euler's partition theorem.

3.1. Some Applications

A colored partition *with restricted ℓ-compositions* of n is a colored partition of n in which each part m is assigned a composition of m with at most ℓ summands. The compositions with at most ℓ summands are called *ℓ-restricted compositions.* Let $c_{\leq \ell}(n)$ denote the number of ℓ-restricted compositions of n.

Remark 1. The sequence $c_{\leq \ell}(n)$ can be obtained by summing over the possible number of summands in traditional compositions. Therefore,

$$c_{\leq \ell}(n) = \sum_{j=1}^{\ell} \binom{n-1}{j-1}.$$

Moreover, the number of compositions of integers up to n and at most ℓ summands is

$$\sum_{k=1}^{n} \sum_{j=1}^{\ell} \binom{k-1}{j-1}.$$

Although this expression is straightforward to derive, we seek a simplified form in certain cases. Let us first consider a concrete example. The number of compositions of integers less than or equal to n, each having at most two parts, is

$$\sum_{k=1}^{n} c_{\leq 2}(k) = \sum_{k=1}^{n} \sum_{j=1}^{2} \binom{k-1}{j-1} = \binom{n+1}{2}.$$

Indeed, consider placing two dividers (marks) among the $n+1$ gaps between the 1's in the string 1^n. For example, when $n = 4$, there are $n+1 = 5$ positions in which to place two dividers (including positions at the ends). Each placement determines a composition of $k \leq n$ into at most two parts. For instance, placing dividers as follows:

$$\boxtimes 1 \square 1 \boxtimes 1 \square 1 \square$$

corresponds to the composition (2), while

$$\square 1 \boxtimes 1 \square 1 \boxtimes 1 \square$$

corresponds to the composition $(1, 2)$. The number of ways to place two dividers in $n+1$ positions is $\binom{n+1}{2}$.

We can generalize the previous argument and obtain the following identities:

1. If ℓ is even, say $\ell = 2m$ with $m \geq 1$, then

$$\sum_{k=1}^{n} c_{\leq 2m}(k) = \sum_{i=1}^{n} \sum_{j=1}^{\ell} \binom{i-1}{j-1} = \sum_{j=1}^{m} \binom{n+1}{2j}.$$

2. If ℓ is odd, say $\ell = 2m+1$ with $m \geq 0$, then

$$\sum_{k=1}^{n} c_{\leq 2m+1}(k) = \sum_{i=1}^{n} \sum_{j=1}^{\ell} \binom{i-1}{j-1} = \sum_{j=0}^{m} \binom{n+1}{2j+1} - 1.$$

For the family of colored partitions with restricted ℓ-compositions, Euler's partition theorem remains valid. In fact, all of these results are special cases of Theorem 5. Moreover, the approach based on restricted ℓ-compositions provides a more general framework than the one introduced by Goyal [7], which originally inspired the formulation of this result.

Corollary 3. *The number of partitions with distinct parts, in which each part of size k is colored by a restricted ℓ-composition of k, equals the number of such partitions in which no even part is colored by a restricted ℓ-composition consisting entirely of even summands.*

Proof. This result follows from the fact that the family of restricted ℓ-compositions is admissible. □

A colored partition has *different sizes* if no two parts share the same size. Let $pdn_c(n)$ denote the number of colored partitions of n with different sizes. For example, $pdn_c(5) = 15$, and the corresponding partitions are:

$$(5_1),\quad (5_2),\quad (5_3),\quad (5_4),\quad (5_5),\quad (4_1,1_1),\quad (4_2,1_1),\quad (4_3,1_1),\quad (4_4,1_1),$$
$$(3_1,2_1),\quad (3_1,2_2),\quad (3_2,2_1),\quad (3_2,2_2),\quad (3_3,2_1),\quad (3_3,2_2)$$

This sequence appears as A022629 in the OEIS [13]. Since each part of size n may be colored in n distinct ways, the generating function is given by

$$\sum_{n=0}^{\infty} pdn_c(n)\, q^n = \prod_{n=1}^{\infty}(1 + nq^n).$$

A colored partition has *different parts* if any two parts differ either in size or in color. For instance, 2_1 and 2_2 are considered different parts. Let $pd_c(n)$ denote the number of colored partitions of n with different parts. For example, $pd_c(5) = 16$, with the following partitions:

$$(5_1),\quad (5_2),\quad (5_3),\quad (5_4),\quad (5_5),\quad (4_1,1_1),\quad (4_2,1_1),\quad (4_3,1_1),\quad (4_4,1_1),$$
$$(3_1,2_1),\quad (3_1,2_2),\quad (3_2,2_1),\quad (3_2,2_2),\quad (3_3,2_1),\quad (3_3,2_2),\quad (2_2,2_1,1_1).$$

Corollary 4. *The number of colored partitions of n with different parts is equal to the number of colored partitions of n in which even parts are not colored with even (resp. odd) colors.*

Proof. This result follows from the fact that the set of positive integers is an admissible set, and in this case the colors in a colored partition are identified with the positive integers themselves. □

Remark 2. It remains an open question whether one can obtain proofs of Theorems 1 and 3 by means of generating functions.

Acknowledgement. We thank the referee for the careful reading of the manuscript and for the valuable comments that have significantly improved the presentation of the paper.

References

[1] A. K. Agarwal, n-colour compositions, *Indian J. Pure Appl. Math.* **31** (2000), 1421–1427.

[2] A. K. Agarwal, n-colour partitions, in A. K. Agarwal et al. (eds.), *Number Theory and Discrete Mathematics*, Trends in Mathematics, Birkhäuse, 2002, 301–314.

[3] A. K. Agarwal and G. E. Andrews, Rogers-Rarnanujam identities for partitions with "N copies of N", *J. Combin. Theory Ser. A* **45** (1987), 40–49.

[4] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, Cambridge, 1998.

[5] G. E. Andrews and P. Paule, MacMahon's partition analysis XIV: partitions with n copies of n, *J. Combin. Theory Ser. A* **203** (2024), 105836.

[6] P. Bala, Colored compositions, https://oeis.org/A253829/a253829.pdf.

[7] M. Goyal, An analogue of Euler's identity and split perfect partitions, *Bull. Aust. Math. Soc.* **99** (2019), 353–361.

[8] J. Huang, Compositions with restricted parts, *Discrete Math.* **343** (7) (2020), 111875.

[9] R. Kaneiwa, An asymptotic formula for Cayley's double partition function $p(2; n)$, *Tokyo J. Math.* **2** (1) (1979), 175–183.

[10] P. A. MacMahon, *Collected Papers*, Vol. I, Mathematicians of Our Time, MIT Press, Cambridge, MA and London, 1978.

[11] M. Merca and I.-I. Radu, Plane partitions as sums over partitions, *Symmetry* **15** (2023), 1820.

[12] I. Mező, *Combinatorics and Number Theory of Counting Sequences*, CRC Press, 2019.

[13] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, http://oeis.org/.

[14] R. Stanley, *Enumerative Combinatorics*, 2nd Edition, Cambridge, 2012.