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GENERALIZED FRUIT DIOPHANTINE EQUATION OVER NUMBER FIELDS

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Abstract

Let K be a number field and $\mathcal{O}_K$ be the ring of integers of K. In this article, we study the solutions of the generalized fruit Diophantine equation $ax^d - y^2 - z^2 + xyz - c = 0$ over K, where $d \geq 3$ is an integer and $a, c \in \mathcal{O}_K \setminus \{0\}$. Subsequently, we provide explicit values of square-free integers t such that the equation $ax^d - y^2 - z^2 + xyz - c = 0$ has no solution $(x_0, y_0, z_0) \in \mathcal{O}^3_{\mathbb{Q}(\sqrt{t})}$ with $2|x_0$, and demonstrate that the set of all such square-free integers t with $t \geq 2$ has density exactly $\frac{1}{6}$. As an application, we construct infinitely many elliptic curves E defined over number fields K having no integral point $(x_0, y_0) \in \mathcal{O}^2_K$ with $2|x_0$.

1. Introduction

Diophantine equations are among the most active and fascinating areas in number theory. In [18], Wiles proved the famous Fermat's Last Theorem using modularity and established that the Diophantine equation $x^n + y^n = z^n$ has no non-zero integer solutions for integers $n \geq 3$. Subsequently, significant progress has been achieved in the examination of the generalized Fermat equation $Ax^p + By^q = Cz^r$ over number fields (see [1], [2], [3], [4], [5], [6], [11], [13], [14] for more details).

In [8], Luca and Togbé first studied the integer solutions of the Diophantine equation $x^3 + by + 1 - xyz = 0$ with a fixed integer b. Later in [16], Togbé studied the integer solutions of the Diophantine equation $x^3 + by + 4 - xyz = 0$. In [9], Majumdar and Sury proved that the Diophantine equation $x^3 - y^2 - z^2 + xyz - 5 = 0$ has no integer solutions and named this equation the *fruit Diophantine equation*. As an application, in [9] Majumdar and Sury constructed infinitely many elliptic

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curves with no integral points. In [17], Vaishya and Sharma extended the work of [9] to the Diophantine equation $ax^3 - y^2 - z^2 + xyz - b = 0$ for integers a, b with $a \equiv 1 \pmod{12}$ and $b = 8a - 3$, and constructed infinitely many elliptic curves with torsion-free Mordell–Weil group over $\mathbb{Q}$. In [12], Prakash and Chakraborty generalized the work of [17] to the generalized fruit Diophantine equation $ax^d - y^2 - z^2 + xyz - b = 0$ for integers a, b with $a \equiv 1 \pmod{12}$ and $b = 2^d a - 3$, where d is an odd integer divisible by 3, and constructed infinitely many hyperelliptic curves with torsion-free Mordell–Weil group over $\mathbb{Q}$.

In this article, we study the solutions of the generalized fruit Diophantine equation $ax^d - y^2 - z^2 + xyz - c = 0$ over number fields K, where $d \geq 3$ is an integer and $a, c \in \mathcal{O}_K \setminus \{0\}$.

In Theorem 1, we show that for any $a, b \in \mathcal{O}_K \setminus \{0\}$ and $c = 2^d b - 3^r$ with integers $r \geq 2$ and $d \geq 3$ odd, the generalized fruit Diophantine equation $ax^d - y^2 - z^2 + xyz - c = 0$ has no solution $(x_0, y_0, z_0) \in \mathcal{O}_K^3$ with $2|x_0$. As an application of this result, for almost all algebraic integers $\alpha \in \mathcal{O}_K$, we construct elliptic curves E_α defined over K such that E_α has no integral point $(x_0, y_0) \in \mathcal{O}_K^2$ with $2|x_0$ (see Theorem 4). This generalizes the work of [17] where they constructed elliptic curves E_m defined over $\mathbb{Q}$ for integers m such that E_m has no point $(x_0, y_0) \in \mathbb{Z}^2$.

In Corollary 1, we provide explicit values of square-free integers t such that the hypothesis of Theorem 1 holds over $K = \mathbb{Q}(\sqrt{t})$. Finally, in Theorem 2, we show that the set of square-free integers $t \geq 2$ such that the hypothesis of Theorem 1 holds over $K = \mathbb{Q}(\sqrt{t})$ has density exactly $\frac{1}{6}$.

The structure of this article is as follows. In Section 2, we state the main results of the article, namely, Theorems 1 and 2. In Section 3, we prove Theorems 1 and 2. In Section 4, we state and prove Theorem 4.

2. Main Results for the Generalized Fruit Diophantine Equation over K

Throughout this article, K denotes a number field and $\mathcal{O}_K$ denotes the ring of integers of K. Let P_K denote the set of all prime ideals of $\mathcal{O}_K$. In this section, we study the solutions of the generalized fruit Diophantine equation, namely

$$ax^d - y^2 - z^2 + xyz - c = 0 \tag{1}$$

over K, where $d \geq 3$ is an integer and $a, c \in \mathcal{O}_K \setminus \{0\}$. Let $T_K := \{\mathfrak{P} \in P_K : \mathbf{e}(\mathfrak{P}|2) = 1 = \mathbf{f}(\mathfrak{P}|2)\}$, where $\mathbf{e}(\mathfrak{P}|2)$ and $\mathbf{f}(\mathfrak{P}|2)$ denote the ramification index and the inertia degree of the prime $\mathfrak{P}$ lying above 2, respectively.

2.1. Main Results

Theorem 1. *Let K be a number field with $T_K \neq \emptyset$. Let $a, b \in \mathcal{O}_K \setminus \{0\}$ and $c = 2^d b - 3^r$ with integers $r \geq 2$ and $d \geq 3$ odd. Then the Diophantine equation*

$ax^d - y^2 - z^2 + xyz - c = 0$ *has no solution* $(x_0, y_0, z_0) \in \mathcal{O}_K^3$ *with* $2|x_0$.

Remark 1. Note that, if 2 splits completely in the field K, then $T_K \neq \emptyset$. Hence, the conclusion of Theorem 1 holds over all the number fields K in which 2 splits completely.

The following corollary gives explicit values of square-free integers d such that the hypothesis of Theorem 1 holds over the quadratic field $K = \mathbb{Q}(\sqrt{d})$.

Corollary 1. *Let t be a square-free integer and let $K = \mathbb{Q}(\sqrt{t})$. Then the hypothesis of Theorem 1 holds over K if and only if $t \equiv 1 \pmod 8$.*

Proof. Note that, 2 splits completely in $K = \mathbb{Q}(\sqrt{t})$ if and only if $t \equiv 1 \pmod 8$ (see [10, Theorem 25]). By the definition of T_K, we conclude that 2 splits completely in $K = \mathbb{Q}(\sqrt{t})$ if and only if $T_K \neq \emptyset$. Hence, the proof of the corollary follows from Theorem 1 and Remark 1. □

Next, we will calculate the density of the set of all square-free integers $t \geq 2$ such that the hypothesis of Theorem 1 holds over $K = \mathbb{Q}(\sqrt{t})$. Let

$$\mathbb{N}^{\mathrm{sf}} := \{t \in \mathbb{Z}_{\geq 2} : t \text{ is a square-free integer}\}.$$

We shall now define the relative density of any subset $S \subseteq \mathbb{N}^{\mathrm{sf}}$.

Definition 1. For any subset $S \subseteq \mathbb{N}^{\mathrm{sf}}$, the *relative density* of S is defined by

$$\delta_{\mathrm{rel}}(S) := \lim_{X \to \infty} \frac{\#\{d \in S : d \leq X\}}{\#\{d \in \mathbb{N}^{\mathrm{sf}} : d \leq X\}},$$

if the above limit exists.

The following theorem computes the density of the set of all square-free integers $t \geq 2$ such that the hypothesis of Theorem 1 holds over $K = \mathbb{Q}(\sqrt{t})$.

Theorem 2. *Let $U := \{t \in \mathbb{N}^{sf} : t \equiv 1 \pmod 8\}$. Then $\delta_{rel}(U) = \frac{1}{6}$. In particular, if $t \in U$, then the hypothesis of Theorem 1 holds over $K = \mathbb{Q}(\sqrt{t})$.*

Combining Corollary 1 and Theorem 2, we conclude that the set of square-free integers $t \geq 2$ such that the hypothesis of Theorem 1 holds over $K = \mathbb{Q}(\sqrt{t})$ has density exactly $\frac{1}{6}$.

3. Proofs of Theorem 1 and Theorem 2

3.1. Proof of Theorem 1

To prove this theorem, we need the following lemma. Recall that P_K is the set of all prime ideals of $\mathcal{O}_K$ and $T_K = \{\mathfrak{P} \in P_K : \mathbf{e}(\mathfrak{P}|2) = 1 = \mathbf{f}(\mathfrak{P}|2)\}$.

Lemma 1. *Let K be a number field. For any prime ideal $\mathfrak{P} \in T_K$, we have $\mathcal{O}_K/\mathfrak{P}^n \simeq \mathbb{Z}/2^n\mathbb{Z}$ for all integers $n \geq 1$.*

Proof. Let $\mathfrak{P} \in T_K$. By the definition of T_K, we have $\mathbf{e}(\mathfrak{P}|2) = 1 = \mathbf{f}(\mathfrak{P}|2)$. Since $\mathbf{f}(\mathfrak{P}|2) = 1$, we have $\mathcal{O}_K/\mathfrak{P} \simeq \mathbb{Z}/2\mathbb{Z}$. Since $\mathcal{O}_K/\mathfrak{P} \simeq \mathfrak{P}^r/\mathfrak{P}^{r+1}$ for all $r \geq 1$, it follows that $|\mathcal{O}_K/\mathfrak{P}| = 2 = |\mathfrak{P}^r/\mathfrak{P}^{r+1}|$ for all $r \geq 1$, and hence $|\mathcal{O}_K/\mathfrak{P}^n| = 2^n$. Since $\mathbf{e}(\mathfrak{P}|2) = 1$, we get $v_{\mathfrak{P}}(2^{n-1}) = n-1$ and therefore $2^{n-1} \notin \mathfrak{P}^n$. Hence, 2^{n-1} is a non-zero element of the quotient ring $\mathcal{O}_K/\mathfrak{P}^n$. Since any ring of order 2^n in which $2^{n-1} \neq 0$ is isomorphic to $\mathbb{Z}/2^n\mathbb{Z}$, we conclude that $\mathcal{O}_K/\mathfrak{P}^n \simeq \mathbb{Z}/2^n\mathbb{Z}$. $\square$

Proof of Theorem 1. We will prove this theorem by contradiction. Suppose $(\alpha, \beta, \gamma) \in \mathcal{O}_K^3$ is a solution of Equation (1) with $2|\alpha$. Then $\alpha = 2\alpha_1$, for some $\alpha_1 \in \mathcal{O}_K$. So, we have $a(2\alpha_1)^d - \beta^2 - \gamma^2 + 2\alpha_1\beta\gamma - c = 0$, which reduces to the following equation:

$$\beta^2 - 2\alpha_1\beta\gamma + \gamma^2 = a(2\alpha_1)^d - c.$$

This gives $(\beta - \alpha_1\gamma)^2 - ({\alpha_1}^2 - 1)\gamma^2 = 2^d{\alpha_1}^d a - c$. Take $Y = \beta - \alpha_1\gamma$ and $Z = \gamma$. Then $Y, Z \in \mathcal{O}_K$, and we have

$$Y^2 - ({\alpha_1}^2 - 1)Z^2 = 2^d{\alpha_1}^d a - c. \tag{2}$$

By assumption $T_K \neq \emptyset$ and choose $\mathfrak{P} \in T_K$. By Lemma 1, we have $\mathcal{O}_K/\mathfrak{P} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{O}_K/\mathfrak{P}^2 \simeq \mathbb{Z}/4\mathbb{Z}$. We now consider two cases.

`Case 1`: $\mathfrak{P}|\alpha_1$. This gives $\alpha_1 \equiv 0 \pmod{\mathfrak{P}}$, and hence $\alpha_1^2 \equiv 0 \pmod{\mathfrak{P}^2}$. Since $c = 2^d b - 3^r$ with $d \geq 2$ is an integer, Equation (2) reduces to the following equation:

$$Y^2 + Z^2 \equiv 3^r \pmod{\mathfrak{P}^2}. \tag{3}$$

Since $\mathcal{O}_K/\mathfrak{P} \simeq \mathbb{Z}/2\mathbb{Z}$ and $Y \in \mathcal{O}_K$, we have either $Y \equiv 0 \pmod{\mathfrak{P}}$ or $Y \equiv 1 \pmod{\mathfrak{P}}$. Similarly for Z, we have either $Z \equiv 0 \pmod{\mathfrak{P}}$ or $Z \equiv 1 \pmod{\mathfrak{P}}$. If $Y \equiv 0 \pmod{\mathfrak{P}}$, then $Y^2 \equiv 0 \pmod{\mathfrak{P}^2}$. If $Y \equiv 1 \pmod{\mathfrak{P}}$, then $(Y-1)^2 \equiv 0 \pmod{\mathfrak{P}^2}$. Since $\mathcal{O}_K/\mathfrak{P}^2 \simeq \mathbb{Z}/4\mathbb{Z}$, we get $Y \equiv 1$ or $3 \pmod{\mathfrak{P}^2}$. So $(Y-1)^2 \equiv Y^2 - 1 \pmod{\mathfrak{P}^2}$. Since $(Y-1)^2 \equiv 0 \pmod{\mathfrak{P}^2}$, we get $Y^2 \equiv 1 \pmod{\mathfrak{P}^2}$. In both cases, we have either $Y^2 \equiv 0 \pmod{\mathfrak{P}^2}$ or $Y^2 \equiv 1 \pmod{\mathfrak{P}^2}$. Similarly, we have either $Z^2 \equiv 0 \pmod{\mathfrak{P}^2}$ or $Z^2 \equiv 1 \pmod{\mathfrak{P}^2}$.

Hence, $Y^2 + Z^2 \equiv 0$ or 1 or $2 \pmod{\mathfrak{P}^2}$. Since r is odd and $\mathcal{O}_K/\mathfrak{P}^2 \simeq \mathbb{Z}/4\mathbb{Z}$, we have $3^r \equiv 3 \pmod{\mathfrak{P}^2}$. By Equation (3), we get $Y^2 + Z^2 \equiv 3 \pmod{\mathfrak{P}^2}$, which is not possible.

`Case 2`: $\mathfrak{P} \nmid \alpha_1$. Since $\mathcal{O}_K/\mathfrak{P} \simeq \mathbb{Z}/2\mathbb{Z}$, we have $\alpha_1 \equiv 1 \pmod{\mathfrak{P}}$. Using the same argument as in the previous case, we get $\alpha_1^2 \equiv 1 \pmod{\mathfrak{P}^2}$. Since $c = 2^d b - 3^r$ and $d \geq 2$ is an integer, Equation (2) reduces to the following equation:

$$Y^2 \equiv 3^r \pmod{\mathfrak{P}^2}. \tag{4}$$

Since r is odd and $\mathcal{O}_K/\mathfrak{P}^2 \simeq \mathbb{Z}/4\mathbb{Z}$, we get $3^r \equiv 3 \pmod{\mathfrak{P}^2}$, and hence $Y^2 \equiv 3 \pmod{\mathfrak{P}^2}$. This is not possible since $Y^2 \equiv 0$ or $1 \pmod{\mathfrak{P}^2}$. This completes the proof of the theorem. □

3.2. Proof of Theorem 2

Recall that $\mathbb{N}^{\mathrm{sf}} = \{t \in \mathbb{Z}_{\geq 2} : t \text{ is a square-free integer}\}$. Before proving this theorem, we first recall the absolute density of any subset $S \subseteq \mathbb{N}$ (see [2, Section 7]).

Definition 2. For any subset $S \subseteq \mathbb{N}$ and a positive real number X, let $S(X) := \{d \in S : d \leq X\}$. Then the *absolute density* of S is defined by

$$\delta_{\mathrm{abs}}(S) := \lim_{X\to\infty} \frac{\#S(X)}{X},$$

if the above limit exists.

The following theorem is useful in the proof of Theorem 2 (see [2, Theorem 10]).

Theorem 3 ([2]). *For $r \in \mathbb{Z}$ and $N \in \mathbb{N}$, let $\mathbb{N}^{sf}_{r,N} := \{t \in \mathbb{N}^{sf} : t \equiv r \pmod{N}\}$. If $s := \gcd(r, N)$ is square-free, then $\#\mathbb{N}^{sf}_{r,N}(X) \sim \frac{\varphi(N)}{s\varphi(\frac{N}{s})N\prod_{q\mid N}(1-\frac{1}{q^2})} \times \frac{6}{\pi^2}X$, where φ denotes Euler's totient function and q varies over all the rational primes dividing N.*

We are now ready to prove Theorem 2.

Proof of Theorem 2. Since $\mathbb{N}^{\mathrm{sf}} = \mathbb{N}^{\mathrm{sf}}_{0,1}$, by Theorem 3, we have $\#\mathbb{N}^{\mathrm{sf}}(X) \sim \frac{6}{\pi^2}X$. By Definition 2, we have $\delta_{\mathrm{abs}}(\mathbb{N}^{\mathrm{sf}}) = \frac{6}{\pi^2}$ (see [7, page 635] for more details). Using Definitions 1 and 2, we conclude that for any subset $S \subseteq \mathbb{N}^{\mathrm{sf}}$, the absolute density $\delta_{\mathrm{abs}}(S)$ exists if and only if the relative density $\delta_{\mathrm{rel}}(S)$ exists. In particular, we have

$$\delta_{\mathrm{abs}}(S) = \delta_{\mathrm{abs}}(\mathbb{N}^{\mathrm{sf}}) \times \delta_{\mathrm{rel}}(S) = \frac{6}{\pi^2}\delta_{\mathrm{rel}}(S). \tag{5}$$

By Theorem 3, we have $\#\mathbb{N}^{\mathrm{sf}}_{1,8}(X) \sim \frac{1}{\pi^2}X$. Using Definition 2, we get $\delta_{\mathrm{abs}}(\mathbb{N}^{\mathrm{sf}}_{1,8}) = \frac{1}{\pi^2}$. Finally, by Equation (5), we have $\delta_{\mathrm{rel}}(\mathbb{N}^{\mathrm{sf}}_{1,8}) = \frac{\pi^2}{6} \times \frac{1}{\pi^2} = \frac{1}{6}$. This completes the proof of the theorem. □

4. Applications

In this section, we will give several applications of the first main result of this article, i.e., Theorem 1. For the first application, we construct infinitely many elliptic curves E defined over K such that E has no integral point $(x_0, y_0) \in \mathcal{O}_K^2$ with $2|x_0$.

Theorem 4. *Let K be a number field with $T_K \neq \emptyset$ and let $\alpha \in \mathcal{O}_K$ be an element not satisfying the polynomial $x^8 + 5x^6 - 432x^4 - 4320x^2 - 10800$. Let E_α/K be the elliptic curve defined over K given by the Weierstrass equation*

$$E_\alpha : y^2 - \alpha xy = x^3 - (\alpha^2 + 5). \tag{6}$$

Then E_α/K has no integral point $(x_0, y_0) \in \mathcal{O}_K^2$ with $2|x_0$.

Proof. Since the discriminant Δ_{E_α} of E_α is $\alpha^8 + 5\alpha^6 - 432\alpha^4 - 4320\alpha^2 - 10800$ and α is not a root of the polynomial $x^8 + 5x^6 - 432x^4 - 4320x^2 - 10800$, we get $\Delta_{E_\alpha} \neq 0$. Hence, E_α is an elliptic curve defined over K. Now, we will prove this theorem by contradiction.

Suppose E_α/K has an integral point $(x_1, y_1) \in \mathcal{O}_K^2$ with $2|x_1$. This gives

$$y_1^2 - \alpha x_1 y_1 = x_1^3 - (\alpha^2 + 5).$$

Hence, $x_1^3 - y_1^2 - \alpha^2 + \alpha x_1 y_1 - 5 = 0$. Therefore, $(x_1, y_1, \alpha) \in \mathcal{O}_K^3$ is an integral solution of Equation (1) with $a = b = 1$, $r = 1$ and $d = 3$. This contradicts Theorem 1 since $2|x_1$ and $T_K \neq \emptyset$. Hence, the proof of the theorem follows. □

Remark 2. Since the polynomial $x^8 + 5x^6 - 432x^4 - 4320x^2 - 10800$ has at most 8 solutions in K, the construction of elliptic curves E_α in Equation (6) holds for almost all $\alpha \in \mathcal{O}_K$. Note that in [17], Vaishya and Sharma constructed elliptic curves $E_m/\mathbb{Q}$ for integers m such that E_m has no point $(x_0, y_0) \in \mathbb{Z}^2$, while in Theorem 4 we construct elliptic curves E_α for almost all $\alpha \in \mathcal{O}_K$ such that E_α/K has no integral point $(x_0, y_0) \in \mathcal{O}_K^2$ with $2|x_0$.

To give the next application of Theorem 1, we first recall the following result (see [15, Chapter VIII, Theorem 7.1(a)]).

Theorem 5 ([15])**.** *Let K be a number field and let E/K be an elliptic curve given by the Weierstrass equation*

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where $a_i \in \mathcal{O}_K$ for all i. If $P = (x, y) \in E(K)$ is a torsion point of order $m \geq 2$ which is not a prime power, then $x, y \in \mathcal{O}_K$.

As a combination of Theorems 4 and 5, we construct infinitely many elliptic curves E defined over K having no torsion point $P \in E(K)$ of order $m \geq 2$ which is not a prime power.

Corollary 2. *Let α, E_α be as in Theorem 4. Then for each α, the elliptic curve E_α has no non-trivial torsion point $P = (x_0, y_0) \in E_\alpha(K)$ which is not a prime power order such that $2|x_0$ and $x_0 \in \mathcal{O}_K$.*

We now recall the Nagell–Lutz theorem (see [15, Chapter VIII, Corollary 7.2(a)])).

Theorem 6 ([15])**.** *Let $E/\mathbb{Q}$ be an elliptic curve defined over $\mathbb{Q}$ given by the Weierstrass equation*

$$E : y^2 = x^3 + Ax + B,$$

where $A, B \in \mathbb{Z}$. If $P = (x, y) \in E(\mathbb{Q})$ is a torsion point of order $m \geq 2$, then $x, y \in \mathbb{Z}$.

As a combination of Theorems 4 and 6, we construct infinitely many elliptic curves E defined over $\mathbb{Q}$ having no torsion point $P \in E(\mathbb{Q})$ of order $m \geq 2$.

Corollary 3. *Let $K = \mathbb{Q}$ and let α, E_α be as in Theorem 4. Then for each α, the elliptic curve E_α has no non-trivial torsion point $P = (x_0, y_0) \in E_\alpha(\mathbb{Q})$ with $2|\mathrm{Num}(x_0)$, where $\mathrm{Num}(x_0)$ denotes the numerator of the fraction x_0 in lowest form.*

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