DENSITY AND SYMMETRY IN THE GENERALIZED MOTZKIN NUMBERS MODULO p

Nadav Kohen
Department of Mathematics, Indiana University, Bloomington, Indiana
nkohen@iu.edu

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Abstract

We prove a new functional equation for the generalized Motzkin numbers modulo a prime: $M_{p-3-n}^{a,b} \equiv (b^2 - 4a^2)^{\frac{p-3}{2}-n} M_n^{a,b} \pmod{p}$. We also give a formula for the density of 0 in the sequence of generalized Motzkin numbers modulo a prime, p, in terms of the first p generalized central trinomial coefficients $T_n^{a,b} \bmod p$ (with $n < p$). We apply our method to various other sequences to obtain similar formulas. These formulas provide easy-to-compute tight lower bounds for the density of 0 in our sequences modulo primes. They also reveal an unexpected connection between the Riordan numbers and the number of directed animals of size n. Along the way, we provide a general characterization of the n such that $p \mid M_n^{a,b}$, which generalizes previous results of Deutsch and Sagan.

1. Introduction

The Motzkin numbers (A001006 of [7]), M_n, count the number of lattice paths from the origin to $(n, 0)$, which do not go below the x-axis, with steps UP $= (1, 1)$, LEVEL $= (1, 0)$, and DOWN $= (1, -1)$. See [3] for many other combinatorial settings in which the Motzkin numbers arise. Some work has been done to characterize M_n and similar sequences modulo various prime powers. For example, Deutsch and Sagan [2] characterized $M_n \bmod 3$. They also described all n such that $M_n \equiv 0 \pmod{p}$ for $p = 2, 4, 5$.

In Section 2, we prove the following new functional equation for Motzkin numbers modulo a prime (Theorem 2):

$$M_{p-3-k} \equiv (-3)^{\frac{p-3}{2}-k} M_k \pmod{p}.$$

This is accomplished by proving a similar functional equation for the central trinomial coefficients, which additionally functions as a central tool in Section 3.

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The central trinomial coefficients (A002426 of [7]), T_n, count the number of lattice paths from the origin to $(n, 0)$ using the same steps (UP, LEVEL, and DOWN) as Motzkin paths, but where there is no restriction that the paths not go below the x-axis (these are sometimes called *grand Motzkin paths*).

In Section 3, Theorem 3 answers the question, "What is the density of the (generalized) Motzkin numbers that are divisible by a prime p?" This is accomplished by reducing the study of the (generalized) Motzkin numbers modulo p to the study of the lesser-known (generalized) central trinomial coefficients modulo p, which have a simpler characterization provided by Proposition 4. This reduction allows us to provide a general characterization (in Corollary 3) of n such that $M_n \equiv 0 \pmod{p}$ in terms of the first p central trinomial coefficients. Theorem 3 follows from this characterization and tells us that

$$\begin{aligned} D_0 = & \frac{|\{n < p-2 \mid M_n \equiv 0 \pmod{p}\}|}{p} \\ & + \frac{2\,|\{m < p-1 \mid T_m \equiv T_{p-1}T_{m+1} \pmod{p}\}|}{(p-1)(p+1)} \\ & + \frac{2\,|\{m < p-1 \mid T_m \equiv T_{m+1} \pmod{p}\}|}{(p-1)p(p+1)} \end{aligned}$$

is the density of the Motzkin numbers divisible by p. This value, D_0, can be efficiently computed by checking fewer than $3p$ simple congruences.

For readers familiar with the Rowland-Zeilberger automaton [9], the intuition that the Motzkin numbers modulo p should be studied via the central trinomial coefficients comes from the observation that a random walk on the finite state machine for the Motzkin numbers quickly lands within, and never leaves, a subgraph corresponding to the finite state machine for the central trinomial coefficients, which have a simple algebraic description given by Proposition 4. The formal instantiation of this intuition, by rewriting $M_n \bmod p$ using combinations of central trinomial coefficients, is provided in Propositions 1 and 5. This approach of studying $M_n \bmod p$ via the central trinomial coefficients has previously been used by the author in [5] to show that $M_n \bmod p$ is uniformly recurrent if and only if $p \nmid T_n$ for all $n < p$.

In the final subsection of this paper, we demonstrate the general applicability of our approach by applying it to a few other sequences from the Online Encyclopedia of Integer Sequences [7]. This analysis yields a new result (Corollary 5): for every prime, p, the density of 0 in the Riordan numbers (A005043 of [7]) modulo p is equal to the density of 0 in the sequence A005773 of [7] modulo p.

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1.1. Notation and Conventions

The *generalized Motzkin numbers*, $M_n^{a,b}$, count the same lattice paths as the usual Motzkin numbers but where there are a distinct colors for UP and DOWN steps and b distinct colors for LEVEL steps, and two paths are equal only if they have the same steps and colors at each step. All of the results for the Motzkin numbers in this paper are proved in the generality of the generalized Motzkin numbers.

As with the generalized Motzkin numbers, the *generalized central trinomial coefficients*, $T_n^{a,b}$, have a colors associated with UP and DOWN steps and b colors associated with LEVEL steps. The central trinomial coefficients are so named because

$$T_n^{a,b} = \mathrm{ct}\left[(ax^{-1} + b + ax)^n\right],$$

where ct extracts the "constant term" of a Laurent polynomial. Under this paradigm, we note that

$$M_n^{a,b} = \mathrm{ct}\left[(ax^{-1} + b + ax)^n \cdot (1 - x^2)\right]$$

because the paths to $(n, 0)$ that do cross the x-axis can be bijected with arbitrary paths to $(n, 2)$.

Throughout this paper, the superscripts a and b are omitted from $M_n^{a,b}$ and $T_n^{a,b}$ when $a = b = 1$ (i.e., $M_n = M_n^{1,1}$ and $T_n = T_n^{1,1}$). If $Q(x)$ is a Laurent polynomial, then $\mathrm{ct}\,[Q(x)]$ denotes the constant term of $Q(x)$ (i.e., the coefficient of x^0), and $\deg Q(x)$ denotes the absolute value of the largest exponent of x (positive or negative) appearing in $Q(x)$ with non-zero coefficient. If Σ is a set, Σ^* denotes the set of words (i.e., strings) of any length whose characters are from Σ (including the empty word). If n is a non-negative integer, and p is a prime, then let $(n)_p \in \mathbb{F}_p^*$ be the word whose characters are the digits of n in base p. That is, if we let $(n)_p[i]$ denote the ith digit in the base-p expansion of n so that $n = \sum_{i \in \mathbb{Z}_{\geq 0}} (n)_p[i] p^i$, then $(n)_p = ((n)_p\,[\mathrm{length}(n_p) - 1]) \cdots ((n)_p[1])((n)_p[0])$. Note that when working with strings, exponents denote repetition. For example, $(p-1)^k \in \mathbb{F}_p^*$ denotes a run of k characters that are all the character $(p - 1)$. Also note that every statement made in this paper about $(n)_p$ should also hold for $0^k(n)_p$ for every k.

In this paper, we prove results dealing with the density of certain values within sequences:

Definition 1. The *asymptotic density*, or just *density*, of a subset, $S \subseteq \mathbb{N}$, is

$$\lim_{N\to\infty} \frac{S \cap \{n \in \mathbb{N} \mid n < N\}}{N}.$$

When it exists, this is equal to

$$\lim_{N\to\infty} \frac{S \cap \{n \in \mathbb{N} \mid n < p^N\}}{p^N},$$

which is the form we primarily use. Lastly, if p is a prime, we say that the density of a value $x \in \mathbb{F}_p$ in a sequence a_n over $\mathbb{F}_p$ is the usual density of $\{n \in \mathbb{N} \mid a_n = x\}$.

2. Symmetry

In this section, we provide an elementary proof of a congruence result relating $T_k^{a,b}$ and $T_{p-1-k}^{a,b}$, which was first proven by Noe [6] using Legendre polynomials. This symmetry is useful in proving the primary results of this paper in the next section. We also derive the new corresponding result for $M_k^{a,b}$ and $M_{p-3-k}^{a,b}$.

We follow the philosophy that the modulo p study of the generalized Motzkin numbers, $M_n^{a,b}$, is reducible to the study of their corresponding generalized central trinomial coefficients, $T_n^{a,b}$, which are well-structured and determined by their first p elements (see Proposition 4). Thus, we begin by stating an explicit rule for this reduction, which is itself a specific instance of a family of such rules for reducing to the study of $T_n^{a,b}$ for any sequence of the form $\mathrm{ct}\left[(ax^{-1}+b+ax)^n Q\right]$, where Q is some Laurent polynomial in x (see Section 4 of [5] for a discussion of this family of equations).

Proposition 1. *For all $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$,*

$$2a^2 M_n^{a,b} = (4a^2 - b^2)T_n^{a,b} + 2bT_{n+1}^{a,b} - T_{n+2}^{a,b},$$

and in particular,

$$2M_n = 3T_n + 2T_{n+1} - T_{n+2}.$$

Proof. Let $P = ax^{-1} + b + ax$ so that $T_n^{a,b} = \mathrm{ct}\left[P^n\right]$. Define

$$\begin{aligned} A_n &:= \mathrm{ct}\left[x \cdot P^n\right] = \mathrm{ct}\left[x^{-1} \cdot P^n\right] \text{ and} \\ B_n &:= \mathrm{ct}\left[x^2 \cdot P^n\right] = \mathrm{ct}\left[x^{-2} \cdot P^n\right]. \end{aligned}$$

This allows us to write $M_n^{a,b} = \mathrm{ct}\left[(1-x^2)\cdot P^n\right] = T_n^{a,b} - B_n$.

From $T_{n+1}^{a,b} = \mathrm{ct}\left[(ax^{-1}+b+ax)^{n+1}\right] = \mathrm{ct}\left[(ax^{-1}+b+ax)\cdot(ax^{-1}+b+ax)^n\right]$ we can conclude that $T_{n+1}^{a,b} = 2aA_n + bT_n^{a,b}$ so that $2aA_n = T_{n+1}^{a,b} - bT_n^{a,b}$. Similarly, from $A_{n+1} = aT_n^{a,b} + bA_n + aB_n$, we find that

$$\begin{aligned} T_{n+2}^{a,b} &= 2aA_{n+1} + bT_{n+1}^{a,b} \\ &= 2a(aT_n^{a,b} + bA_n + aB_n) + b(2aA_n + bT_n^{a,b}) \\ &= (b^2 + 2a^2)T_n^{a,b} + 2b(2aA_n) + 2a^2 B_n \\ &= (b^2 + 2a^2)T_n^{a,b} + 2b(T_{n+1}^{a,b} - bT_n^{a,b}) + 2a^2 B_n \end{aligned}$$

and thus $2a^2 B_n = T_{n+2}^{a,b} - 2bT_{n+1}^{a,b} + (b^2 - 2a^2)T_n^{a,b}$.

In conclusion, $2a^2 M_n^{a,b} = 2a^2 T_n^{a,b} - 2a^2 B_n = (4a^2 - b^2)T_n^{a,b} + 2bT_{n+1}^{a,b} - T_{n+2}^{a,b}$. □

Next, we give a mostly combinatorial proof of a general two-term recurrence for the sequences $T_n^{a,b}$, which is analogous to the well-known two-term recurrence for the Motzkin numbers (see Corollary 1).

Proposition 2. *For all $a, b, n \in \mathbb{Z}$ with $n \geq 2$,*

$$nT_n^{a,b} = b(2n-1)T_{n-1}^{a,b} - (b^2 - 4a^2)(n-1)T_{n-2}^{a,b},$$

and in particular,

$$nT_n = (2n-1)T_{n-1} + (3n-3)T_{n-2}.$$

Proof. Recall that $M_n^{a,b}$ counts the number of lattice paths from $(0,0)$ to $(n,0)$ using a distinct UP and DOWN steps, $(1,1)$ and $(1,-1)$, respectively, and b distinct LEVEL steps, $(1,0)$, which do not go below the x-axis. Additionally, $T_n^{a,b}$ simply counts the number of such paths without the x-axis restriction, and $A_n = \text{ct}\left[x \cdot P^n\right]$ (as in Proposition 1) does the same but counting paths to $(n,1)$.

Let γ_n count the number of paths from the origin to $(n,1)$ such that the only intersection with the x-axis is the starting point. Then $\gamma_n = aM_{n-1}^{a,b}$ since adding an UP step to the beginning of a path (there are a ways to do this) counted by the $M_{n-1}^{a,b}$ gives a unique path counted by γ_n. Furthermore, $n\gamma_n = A_n$ since there are $n\gamma_n$ paths from the origin to $(n,1)$ that never return to the x-axis and which have a special vertex labeled. This labeled vertex partitions the path into a front and a back (consisting of UP, DOWN, and LEVEL steps) $P = FB$. Now the path $Q = BF$ is an arbitrary path to $(n,1)$, counted by A_n, whose rightmost-lowest point is the labeled vertex (making the process reversible, and hence a bijection). Putting these together, we get $A_n = n\gamma_n = anM_{n-1}^{a,b}$.

We now combine the fact that $T_n^{a,b} = 2aA_{n-1} + bT_{n-1}^{a,b}$ (as discussed in the proof of Proposition 1), the equation $A_{n-1} = a(n-1)M_{n-2}^{a,b}$, and Proposition 1 to get

$$\begin{aligned}
T_n^{a,b} &= 2aA_{n-1} + bT_{n-1}^{a,b} \\
&= (n-1)(2a^2M_{n-2}^{a,b}) + bT_{n-1}^{a,b} \\
&= (n-1)\left((4a^2 - b^2)T_{n-2}^{a,b} + 2bT_{n-1}^{a,b} - T_n^{a,b}\right) + bT_{n-1}^{a,b} \\
&= -(n-1)T_n^{a,b} + (2bn - 2b + b)T_{n-1}^{a,b} - (b^2 - 4a^2)(n-1)T_{n-2}^{a,b},
\end{aligned}$$

from which we can conclude $nT_n^{a,b} = b(2n-1)T_{n-1}^{a,b} - (b^2 - 4a^2)(n-1)T_{n-2}^{a,b}$. □

In fact, the already-known analogous recurrence for the generalized Motzkin numbers can be derived directly from the previous two results by a tedious algebraic manipulation. An independent proof is also given by Woan in [10].

Corollary 1. *For all $a, b, n \in \mathbb{Z}$ with $n \geq 2$,*

$$(n+2)M_n^{a,b} = b(2n+1)M_{n-1}^{a,b} - (b^2 - 4a^2)(n-1)M_{n-2}^{a,b},$$

and in particular,

$$(n+2)M_n = (2n+1)M_{n-1} + (3n-3)M_{n-2}.$$

□

As it turns out, Proposition 2 is sufficient to prove our symmetry directly! See equation 14 of [6] for an alternative proof using Legendre polynomials.

Theorem 1. *For all* $a, b \in \mathbb{Z}$, $p > 2$, *and* $0 \le k \le \frac{p-1}{2}$,

$$T^{a,b}_{p-1-k} \equiv (b^2 - 4a^2)^{\frac{p-1}{2}-k} T^{a,b}_k \pmod{p},$$

and in particular,

$$T_{p-1-k} \equiv (-3)^{\frac{p-1}{2}-k} T_k \pmod{p}.$$

Proof. We use induction on the values $k \le \frac{p-1}{2}$. For $k = \frac{p-1}{2}$, we have that

$$T^{a,b}_{p-1-\frac{p-1}{2}} = T^{a,b}_{\frac{p-1}{2}} = (b^2 - 4a^2)^0 T^{a,b}_{\frac{p-1}{2}}.$$

For $k = \frac{p-3}{2}$, we wish to show that $T^{a,b}_{\frac{p+1}{2}} \equiv (b^2 - 4a^2) T^{a,b}_{\frac{p-3}{2}} \pmod{p}$, which follows from Proposition 2 since

$$\begin{aligned} 2^{-1} T^{a,b}_{\frac{p+1}{2}} &\equiv \frac{p+1}{2} T^{a,b}_{\frac{p+1}{2}} \\ &= bp T^{a,b}_{\frac{p-1}{2}} - (b^2 - 4a^2)\left(\frac{p-1}{2}\right) T^{a,b}_{\frac{p-3}{2}} \\ &\equiv 2^{-1}(b^2 - 4a^2) T^{a,b}_{\frac{p-3}{2}} \pmod{p}. \end{aligned}$$

Now let $0 \le k < \frac{p-3}{2}$. Then using Proposition 2 and induction we have that $(p-1-k)T^{a,b}_{p-1-k}$ is equal to

$$\begin{aligned} & b(2(p-1-k)-1)T^{a,b}_{p-2-k} - (b^2-4a^2)(p-2-k)T^{a,b}_{p-3-k} \\ &\equiv -b(2k+3)T^{a,b}_{p-2-k} + (b^2-4a^2)(2+k)T^{a,b}_{p-3-k} \\ &\equiv -b(2k+3)(b^2-4a^2)^{\frac{p-3}{2}-k}T^{a,b}_{k+1} + (2+k)(b^2-4a^2)^{\frac{p-3}{2}-k}T^{a,b}_{k+2} \\ &= (b^2-4a^2)^{\frac{p-3}{2}-k}\left((k+2)T^{a,b}_{k+2} - b(2k+3)T^{a,b}_{k+1}\right) \\ &= (b^2-4a^2)^{\frac{p-3}{2}-k}\left(b(2k+3)T^{a,b}_{k+1} - (b^2-4a^2)(k+1)T^{a,b}_k - b(2k+3)T^{a,b}_{k+1}\right) \\ &= (-k-1)(b^2-4a^2)^{\frac{p-1}{2}-k}T^{a,b}_k \\ &\equiv (p-1-k)(b^2-4a^2)^{\frac{p-1}{2}-k}T^{a,b}_k \pmod{p}. \end{aligned}$$

□

In the case that $p \nmid b^2 - 4a^2$, we have that $b^2 - 4a^2$ is invertible so that we can also conclude that $T^{a,b}_k \equiv (b^2 - 4a^2)^{k-\frac{p-1}{2}} T^{a,b}_{p-1-k} \pmod{p}$. Of course, in the case that $p \mid b^2 - 4a^2$, we instead conclude that $T^{a,b}_k \equiv 0 \pmod{p}$ for all $\frac{p-1}{2} < k < p$.

Before moving on, we quickly mention an easy and useful Corollary of Theorem 1. In [6], this is a corollary of Theorem 8.8.

Corollary 2. *For all $a, b \in \mathbb{Z}$, and all primes, p, such that $p \nmid b^2 - 4a^2$,*

$$\left(T_{p-1}^{a,b}\right)^{-1} \equiv T_{p-1}^{a,b} \pmod{p}.$$

Equivalently, $\left(T_{p-1}^{a,b}\right)^2 \equiv 1 \pmod{p}$.

Proof. By Theorem 1, $T_{p-1}^{a,b} \equiv (b^2 - 4a^2)^{\frac{p-1}{2}} \pmod{p}$ so that, assuming $p \nmid b^2 - 4a^2$, we have $(T_{p-1}^{a,b})^2$ is congruent to 1 modulo p by Fermat's little theorem. □

Remark 1. Theorem 1 seems to help explain the observed density of the sequence of primes that do not divide any central trinomial coefficient (A113305 of OEIS [7]). For primes less than 10^6 the density of these primes is near 0.6075. As a corollary of Proposition 4 of the next section, p is in this sequence if and only if it does not divide any of the first p central trinomial coefficients. But the author was originally surprised by how high this density seems to be, since if we pretend that on average T_n (with $n < p$) has a $\frac{1}{p}$ chance of being congruent to 0 modulo p, then we expect the density to be $\lim_{p\to\infty} \left(1 - \frac{1}{p}\right)^p = e^{-1} \approx 0.3679$. But in view of Theorem 1, we now know that p does not divide any central trinomial coefficient if and only if it does not divide the first $\frac{p+1}{2}$ of them! Now if we make the same assumption we instead get the density estimate $\lim_{p\to\infty} \left(1 - \frac{1}{p}\right)^{\frac{p+1}{2}} = e^{-\frac{1}{2}} \approx 0.6065$, which is much closer to the experimental value.

The analogous result to Theorem 1 for the generalized Motzkin numbers can be proven directly from their two-term recurrence in a very similar fashion to the proof of Theorem 1. However, we instead provide a proof using Proposition 1 and Theorem 1.

Theorem 2. *For all $a, b \in \mathbb{Z}$, $p > 3$, and $0 \le k \le \frac{p-3}{2}$,*

$$M_{p-3-k}^{a,b} \equiv (b^2 - 4a^2)^{\frac{p-3}{2}-k} M_k^{a,b} \pmod{p},$$

and in particular,

$$M_{p-3-k} \equiv (-3)^{\frac{p-3}{2}-k} M_k \pmod{p}.$$

Proof. By Proposition 1, $2a^2 M_{p-3-k}^{a,b}$ is equal to

$$\begin{aligned}
&(4a^2 - b^2)T_{p-3-k}^{a,b} + 2bT_{p-2-k}^{a,b} - T_{p-1-k}^{a,b} \\
&\quad \equiv -(b^2 - 4a^2)^{\frac{p-3}{2}-k} T_{k+2}^{a,b} + 2b(b^2 - 4a^2)^{\frac{p-3}{2}-k} T_{k+1}^{a,b} - (b^2 - 4a^2)^{\frac{p-1}{2}-k} T_k^{a,b} \\
&\quad = (b^2 - 4a^2)^{\frac{p-3}{2}-k} \left(-T_{k+2}^{a,b} + 2bT_{k+1}^{a,b} + (4a^2 - b^2)T_k^{a,b}\right) \\
&\quad = (b^2 - 4a^2)^{\frac{p-3}{2}-k} (2a^2 M_k^{a,b}) \pmod{p}.
\end{aligned}$$

This gives us the theorem when a is not a multiple of p, and otherwise

$$M_n^{a,b} = \mathrm{ct}\left[(ax^{-1}+b+ax)^n(1-x^2)\right] \equiv \mathrm{ct}\left[b^n(1-x^2)\right] = b^n \pmod p$$

in which case $(b^2-4a^2)^{\frac{p-3}{2}-k}M_k^{a,b} \equiv b^{p-3-2k}b^k = b^{p-3-k} \equiv M_{p-3-k}^{a,b} \pmod p$, as desired. □

As an additional application of Theorem 1 we prove the following conjecture (Batalov 2022) from the OEIS page for the Motzkin numbers (A001006 of [7]), which can also be inferred from Tables 4 and 5 of [1]: If p is a prime of the form $6m+1$, then M_{p-2} is divisible by p. In fact, we can prove a stronger result:

Proposition 3. *A prime p divides M_{p-2} if and only if $p \equiv 1 \pmod 3$.*

Proof. For $p=2$, we know that $2 \nmid M_0$ and $2 \not\equiv 1 \pmod 3$, so assume $p>2$. Note that $T_p \equiv 1 \pmod p$ (see Proposition 4). We prove the proposition by reducing to central trinomial coefficients:

$$\begin{aligned} 2M_{p-2} &= 3T_{p-2} + 2T_{p-1} - T_p \\ &\equiv 3(-3)^{\frac{p-3}{2}} + 2(-3)^{\frac{p-1}{2}} - 1 \\ &= -(-3)^{\frac{p-1}{2}} + 2(-3)^{\frac{p-1}{2}} - 1 \\ &= (-3)^{\frac{p-1}{2}} - 1 \pmod p \end{aligned}$$

and therefore, by quadratic reciprocity, $p \mid M_{p-2}$ if and only if $(-3)^{\frac{p-1}{2}} \equiv 1 \pmod p$ if and only if $p \equiv 1 \pmod 3$. □

3. Density

In this section, we answer the question: "What is the density of the Motzkin numbers that are divisible by p?" In Subsection 3.1, we do the same for some additional sequences. In all cases, answering this question is accomplished by further elaborating the consequences of Proposition 1 (or analogous results) and Theorem 1 alongside the following simple description of $T_n^{a,b} \bmod p$.

Proposition 4. *For every prime p and $n \in \mathbb{N}$, if $a_n = \mathrm{ct}\,[P^n]$ where P is a Laurent polynomial with $\deg P = 1$, then $a_n \equiv \prod a_{(n)_p[i]} \pmod p$. In particular, $T_n^{a,b} \equiv \prod T_{(n)_p[i]}^{a,b} \pmod p$.*

Proof. We induct on the number of digits in $(n)_p$. Certainly if $n = (n)_p[0] < p$, then $a_n = a_{(n)_p[0]}$. Otherwise, if $n = qp + (n)_p[0]$, then

$$\begin{aligned}
a_n &= \mathrm{ct}\left[P(x)^{qp+(n)_p[0]}\right] \\
&\equiv \mathrm{ct}\left[P(x^p)^q P(x)^{(n)_p[0]}\right] && (P(x)^p \equiv P(x^p) \pmod{p}) \\
&= \mathrm{ct}\left[P(x^p)^q\right] \mathrm{ct}\left[P(x)^{(n)_p[0]}\right] && ((n)_p[0] < p \text{ so there is no cancellation}) \\
&= \mathrm{ct}\left[P(x)^q\right] \mathrm{ct}\left[P(x)^{(n)_p[0]}\right] && \left(\mathrm{ct}\left[P(x^k)^n\right] = \mathrm{ct}\left[P(x)^n\right]\right) \\
&= a_q a_{(n)_p[0]} \\
&= \prod a_{(n)_p[i]} \pmod{p} && \text{(by induction).}
\end{aligned}$$

□

Note that this proposition shows that the sequence $T_n^{a,b} \bmod p$ can be described by its first p terms, making Theorem 1 more important than it may initially seem. Proposition 4 is Corollary 3.1 of [4], where these congruences are referred to as Lucas congruences.

Given the base-p digit-multiplicative nature of $T_n^{a,b}$, every combination of offsets $T_{n+i}^{a,b}$ can be understood by "factoring out" common digits between the $(n+i)_p$ for various i. For example, if $(n)_p = qn_0$ and $n_0 \neq p-1$, then

$$\alpha T_n^{a,b} + \beta T_{n+1}^{a,b} \equiv \alpha T_q^{a,b} T_{n_0}^{a,b} + \beta T_q^{a,b} T_{n_0+1}^{a,b} = T_q^{a,b}\left(\alpha T_{n_0}^{a,b} + \beta T_{n_0+1}^{a,b}\right) \pmod{p}.$$

A version of this idea, sufficient for our purposes, is encapsulated in the following formalism where we bound the maximum distance between indices by p. This bound ensures that the following only has two summands. This is sufficient for our purposes since the maximum distance in index, h, in the expansion of M_n in terms of T_n from Proposition 1 is 2.

Lemma 1. *If $a_n = \mathrm{ct}[P^n]$, where $\deg P = 1$, $b_n = \sum_{i=0}^{h} \alpha_i a_{n+i}$, $p > h$, $C_{n_0} = \min(h, p-n-1)$, and $(n)_p = qm(p-1)^k n_0$ with $q \in \mathbb{F}_p^*, m, n_0 \in \mathbb{F}_p$, $m \neq p-1$ and $k \geq 0$, then*

$$b_n \equiv a_q \left(a_m a_{p-1}^k \sum_{i=0}^{C_{n_0}} \alpha_i a_{n_0+i} + a_{m+1} \sum_{i=p-n_0}^{h} \alpha_i a_{i-(p-n_0)}\right) \pmod{p}.$$

In particular, if $n_0 < p-h$, or equivalently, $h < p-n_0$, then

$$b_n \equiv a_q \left(a_m a_{p-1}^k \sum_{i=0}^{h} \alpha_i a_{n_0+i}\right) \equiv a_{\frac{n-n_0}{p}} b_{n_0} \pmod{p}.$$

Proof. If $n_0 + i \le p-1$, then $(n+i)_p = qm(p-1)^k(n_0+i)$, while if $n_0+i \ge p$, then $(n_0+i)_p = q(m+1)0^k(n_0+i-p)$. Thus, noting that $a_0 = 1$,

$$\begin{aligned} b_n &= \sum_{i=0}^{h} \alpha_i a_{n+i} \\ &= \sum_{i=0}^{C_{n_0}} \alpha_i a_{n+i} + \sum_{i=p-n_0}^{h} \alpha_i a_{n+i} \\ &\equiv \sum_{i=0}^{C_{n_0}} \alpha_i (a_q a_m a_{p-1}^k a_{n_0+i}) + \sum_{i=p-n_0}^{h} \alpha_i (a_q a_{m+1} a_{n_0+i-p}) \\ &= a_q \left(a_m a_{p-1}^k \sum_{i=0}^{C_{n_0}} \alpha_i a_{n_0+i} + a_{m+1} \sum_{i=p-n_0}^{h} \alpha_i a_{n_0+i-p} \right) \pmod{p} \end{aligned}$$

□

We now apply this lemma's factorization to $2a^2 M_n^{a,b}$ via Proposition 1. Additionally, we make use of Theorem 1 multiple times to produce a more elegant final form. Note that the third case in the formula for $2a^2 M_n^{a,b}$ below (where $n_0 = p-1$) is more elegant after our application of Theorem 1 because we intend to set $M_n^{a,b}$ congruent to 0 in what follows.

Proposition 5. *For all $n \in \mathbb{N}$, if $p \nmid b^2 - 4a^2$ and if we let $\ell = p-2-m$, then modulo $p > 2$, $2a^2 M_n^{a,b}$ is congruent to*

$$\begin{cases} 2a^2 T_q^{a,b} M_{n_0}^{a,b} & (n)_p = qn_0, n_0 < p-2 \\ T_q^{a,b}\left(bT_m^{a,b}(T_{p-1}^{a,b})^{k+1} - T_{m+1}^{a,b}\right) & (n)_p = qm(p-1)^k(p-2) \\ (b^2-4a^2)^{m+1-\frac{p-1}{2}} T_q^{a,b}\left(bT_\ell^{a,b} - (T_{p-1}^{a,b})^{k+1} T_{\ell+1}^{a,b}\right) & (n)_p = qm(p-1)^k(p-1). \end{cases}$$

In particular, when $a = b = 1$, we get that, modulo p,

$$2M_n \equiv \begin{cases} 2T_q M_{n_0} & (n)_p = qn_0, n_0 < p-2 \\ T_q\left(T_m T_{p-1}^{k+1} - T_{m+1}\right) & (n)_p = qm(p-1)^k(p-2) \\ (-3)^{m+1-\frac{p-1}{2}} T_q\left(T_\ell - T_{p-1}^{k+1} T_{\ell+1}\right) & (n)_p = qm(p-1)^k(p-1). \end{cases}$$

Proof. Since $2a^2 M_n^{a,b} = (4a^2 - b^2) T_n^{a,b} + 2bT_{n+1}^{a,b} - T_{n+2}^{a,b}$ by Proposition 1, Lemma 1 tells us that if $(n)_p = qn_0$ with $n_0 < p-2$, then $2a^2 M_n^{a,b} \equiv T_q^{a,b}\left(2a^2 M_{n_0}^{a,b}\right) \pmod{p}$.

The lemma also tells us that if $(n)_p = qm(p-1)^k(p-2)$, then modulo p,

$$2a^2 M_n^{a,b} \equiv T_q^{a,b}\left(T_m^{a,b}(T_{p-1}^{a,b})^k((4a^2-b^2)T_{p-2}^{a,b} + 2bT_{p-1}^{a,b}) + T_{m+1}^{a,b}(-T_0^{a,b})\right).$$

In this case we can use Theorem 1 (and $T_1^{a,b} = b$) to see that $T_{p-1}^{a,b} \equiv (b^2-4a^2)^{\frac{p-1}{2}}$ (mod p) and $T_{p-2}^{a,b} \equiv (b^2-4a^2)^{\frac{p-1}{2}-1}T_1^{a,b} = (b^2-4a^2)^{-1}T_{p-1}^{a,b}b \pmod p$ (noting that $p \nmid b^2-4a^2$), thus $(4a^2-b^2)T_{p-2}^{a,b}+2bT_{p-1}^{a,b} \equiv bT_{p-1}^{a,b} \pmod p$. This yields our desired form,

$$2a^2M_n^{a,b} \equiv T_q^{a,b}\left(bT_m^{a,b}(T_{p-1}^{a,b})^{k+1} - T_{m+1}^{a,b}\right) \pmod p.$$

Lastly, the lemma (alongside Theorem 1) tells us that if $(n)_p = qm(p-1)^k(p-1)$, then $2a^2M_n^{a,b}$ is congruent to

$$\begin{aligned}
&T_q^{a,b}\left(T_m^{a,b}(T_{p-1}^{a,b})^k((4a^2-b^2)T_{p-1}^{a,b}) + T_{m+1}^{a,b}(2bT_0^{a,b} - T_1^{a,b})\right)\\
&\quad = T_q^{a,b}\left(-(b^2-4a^2)T_m^{a,b}(T_{p-1}^{a,b})^{k+1} + bT_{m+1}^{a,b}\right)\\
&\quad \equiv T_q^{a,b}\left(-(b^2-4a^2)^{m+1-\frac{p-1}{2}}T_{p-1-m}^{a,b}(T_{p-1}^{a,b})^{k+1} + b(b^2-4a^2)^{m+1-\frac{p-1}{2}}T_{p-2-m}^{a,b}\right)\\
&\quad = (b^2-4a^2)^{m+1-\frac{p-1}{2}}T_q^{a,b}\left(bT_\ell^{a,b} - T_{\ell+1}^{a,b}(T_{p-1}^{a,b})^{k+1}\right) \pmod p.
\end{aligned}$$

□

We now set both sides of the above equality congruent to 0 so that computing the density of Motzkin numbers divisible by p becomes a simple matter of bookkeeping. Note that in order to obtain a nice form below, we apply Corollary 2 to move $T_{p-1}^{a,b}$ to the right-hand side of all equivalences.

Corollary 3. *If $p \nmid b^2-4a^2$, $\ell = p-2-m$, $p > 2$, and if $T_n^{a,b} \not\equiv 0 \pmod p$ for all natural numbers $n < p$, then modulo p,*

$$M_n^{a,b} \equiv 0 \textit{ if and only if } \begin{cases} M_{n_0}^{a,b} \equiv 0 & (n)_p = qn_0, n_0 < p-2 \\ bT_m^{a,b} \equiv (T_{p-1}^{a,b})^{k+1}T_{m+1}^{a,b} & (n)_p = qm(p-1)^k(p-2) \\ bT_\ell^{a,b} \equiv (T_{p-1}^{a,b})^{k+1}T_{\ell+1}^{a,b} & (n)_p = qm(p-1)^k(p-1), \end{cases}$$

where $0 \le m < p-1$ implies that $0 \le \ell < p-1$. In particular, when $a = b = 1$, then modulo p,

$$M_n \equiv 0 \textit{ if and only if } \begin{cases} M_{n_0} \equiv 0 & (n)_p = qn_0, n_0 < p-2 \\ T_m \equiv T_{p-1}^{k+1}T_{m+1} & (n)_p = qm(p-1)^k(p-2) \\ T_\ell \equiv T_{p-1}^{k+1}T_{\ell+1} & (n)_p = qm(p-1)^k(p-1). \end{cases}$$

□

Corollary 3 generalizes existing results about the divisibility of the Motzkin numbers modulo small primes. For example, $p = 5$ has treatments in [2] (Theorem 5.4) and [8] (Theorem 3) that we can procedurally extract from Corollary 3 by writing down the first 3 Motzkin numbers $(1, 1, 2)$ and the first 5 central trinomial coefficients $(1, 1, 3, 7, 19 \equiv 1, 1, 3, 2, 4 \pmod 5)$ and checking finitely many congruences to find all forms of $(n)_p$ for which $M_n \equiv 0 \pmod p$.

Note that we require $T_n^{a,b} \not\equiv 0 \pmod{p}$ for every $n < p$ since this implies the same statement holds for all n by Proposition 4. Thus, we do not need to consider when $T_q^{a,b} \equiv 0 \pmod{p}$ in setting the right-hand side of Proposition 5 congruent to 0. If $T_n^{a,b} \equiv 0 \pmod{p}$ for some n (below p), then Proposition 6 of [5] tells us that 0 has density 1 in $M_n^{a,b}$ (i.e., the set of indices $\{i \in \mathbb{N} \mid M_i^{a,b} \equiv 0 \pmod{p}\}$ has density one). Hence, including this assumption does not hinder us in our goal.

We now perform the bookkeeping of computing the density of 0. We begin by stating a result in the generality of Lemma 1 so as to encapsulate the ideas of this method independently of the particulars discovered in Proposition 5 by application of Theorem 1.

Lemma 2. *Let $a_n = \mathrm{ct}\,[P^n]$, where P is symmetric, $\deg P = 1$, $b_n = \sum_{i=0}^{h} \alpha_i a_{n+i}$, $C_n = \min(h, p-n-1)$, $p > h$, and $I = \{p-h, p-h+1, \ldots, p-1\} \times \{0, 1, \ldots, p-2\}$. If $a_n \equiv 0 \pmod{p}$ for some natural number n (n may be taken less than p), then the asymptotic density of 0 in $b_n \bmod p$ is 1, and otherwise it is equal to*

$$\frac{|\{0 \le n < p-h \mid b_n \equiv 0 \pmod{p}\}|}{p}$$
$$+ \frac{\left|\left\{(n,m) \in I \mid a_m \sum_{i=0}^{C_n} \alpha_i a_{n+i} \equiv -a_{m+1} \sum_{i=C_n+1}^{h} \alpha_i a_{i-(p-n)} \pmod{p}\right\}\right|}{(p-1)(p+1)}$$
$$+ \frac{\left|\left\{(n,m) \in I \mid a_m \sum_{i=0}^{C_n} \alpha_i a_{n+i} \equiv -a_{p-1} a_{m+1} \sum_{i=C_n+1}^{h} \alpha_i a_{i-(p-n)} \pmod{p}\right\}\right|}{(p-1)p(p+1)}.$$

Proof. The case in which there exists n such that $a_n \equiv 0 \pmod{p}$ is treated in Proposition 6 of [5].

By Lemma 1, $b_n \equiv 0 \pmod{p}$ if and only if either $n_0 < p-h$ and $b_{n_0} \equiv 0 \pmod{p}$ or $n_0 \ge p-h$ and $a_m a_{p-1}^k \sum_{i=0}^{C_{n_0}} \alpha_i a_{n_0+i} \equiv -a_{m+1} \sum_{i=p-n_0}^{h} \alpha_i a_{i-(p-n_0)} \pmod{p}$. Note that, by its definition, a_n is a generalized central trinomial coefficient sequence so that by Corollary 2, we have that $a_{p-1}^2 \equiv 1 \pmod{p}$. Thus, a_{p-1}^k is congruent to a_{p-1} or 1 modulo p depending on the parity of k. In the set of strings, $(n)_p = qm(p-1)^k n_0$, over $\mathbb{F}_p$, the distribution on k is geometric (with success probability of a trial being $\frac{1}{p}$) so that it is even $\frac{p}{p+1}$ of the time and odd $\frac{1}{p+1}$ of the time.

Putting this all together, each element of the set $\{0 \le n < p-h \mid b_n \equiv 0 \pmod{p}\}$ contributes $\frac{1}{p}$ to the density of 0 in $b_n \bmod p$ since $\frac{1}{p}$ of the strings over $\mathbb{F}_p$ end with such n. Likewise each element of the set

$$\left\{p-h \le n < p, 0 \le m < p-1 \mid a_m \sum_{i=0}^{C_n} \alpha_i a_{n+i} \equiv -a_{m+1} \sum_{i=C_n+1}^{h} \alpha_i a_{i-(p-n)}\right\}$$

(with congruences modulo p) corresponds to k being even and thus contributes

$\frac{1}{p} \cdot \frac{1}{p-1} \cdot \frac{p}{p+1} = \frac{1}{(p-1)(p+1)}$ to the density of 0, while

$$\left\{ p - h \leq n < p, 0 \leq m < p - 1 \mid a_m \sum_{i=0}^{C_n} \alpha_i a_{n+i} \equiv -a_{p-1} a_{m+1} \sum_{i=C_n+1}^{h} \alpha_i a_{i-(p-n)} \right\}$$

(with congruences modulo p) corresponds to k being odd and thus contributes $\frac{1}{p} \cdot \frac{1}{p-1} \cdot \frac{1}{p+1} = \frac{1}{(p-1)p(p+1)}$ to the density. □

We now apply Lemma 2 to the generalized Motzkin numbers using the equations from Corollary 3 in place of those from Lemma 1. Furthermore, reasoning about the densities in $T_n^{a,b} \bmod p$ further yields a description of the density of all other values in $M_n^{a,b} \bmod p$ in terms of the density of 0, D_0.

Theorem 3. *If $T_n^{a,b} \equiv 0 \pmod p$ for some natural number $n < p$, then the asymptotic density of 0 in $M_n^{a,b}$ is 1. Otherwise, for all $p > 2$, it is equal to*

$$\begin{aligned} D_0 = & \frac{\left|\left\{ n < p-2 \mid M_n^{a,b} \equiv 0 \pmod p \right\}\right|}{p} \\ & + \frac{2\left|\left\{ m < p-1 \mid bT_m^{a,b} \equiv T_{p-1}^{a,b} T_{m+1}^{a,b} \pmod p \right\}\right|}{(p-1)(p+1)} \\ & + \frac{2\left|\left\{ m < p-1 \mid bT_m^{a,b} \equiv T_{m+1}^{a,b} \pmod p \right\}\right|}{(p-1)p(p+1)}. \end{aligned}$$

Furthermore, if $\left\{ T_n^{a,b} \mid n \in \mathbb{F}_p \right\}$ generates $\mathbb{F}_p^\times$ as a multiplicative group, then the asymptotic density of any fixed $k > 0$ in $M_n^{a,b} \bmod p$ is $\frac{1-D_0}{p-1}$.

Proof. The value of D_0 is given by combining Lemma 2 with Corollaries 2 and 3 (we may assume $p \nmid b^2 - 4a^2$, as otherwise $T_{p-1}^{a,b} \equiv 0 \pmod p$). Note that the factors of 2 come from the fact that n takes two values when $p - 2 \leq n < p$ (namely $p - 2$ and $p - 1$) and Corollary 3 shows that in both cases we count the same set of elements.

To prove the last claim, we model $T_n^{a,b}$ (as a function from $\mathbb{N}$ to $\mathbb{F}_p^\times$) by a directed multigraph whose states are the elements of $\mathbb{F}_p^\times$, and whose p outgoing transitions are labeled by $k \in \mathbb{F}_p$, where a transition from i labeled k goes to state $j = i \cdot T_k^{a,b} \bmod p$. By Proposition 4, we can read the value of $T_n^{a,b}$ off of this graph by starting at the state 1 and following the transitions corresponding to the digits, $(n)_p$. The state we end at is the value of $T_n^{a,b} \bmod p$. As it happens, each state also has exactly p incoming transitions, since the number of transitions from i to j is equal to the number of transitions from 1 to $i^{-1}j$. Thus, counting the number of these transitions from all i (to a fixed j) is equal to counting all transitions leaving 1, of which there are p. The assumption that $\left\{ T_n^{a,b} \mid n \in \mathbb{F}_p \right\}$ generates $\mathbb{F}_p^\times$ as a multiplicative group is equivalent to the statement that this directed graph is connected. Treating this graph as a Markov process (where every transition has probability

$\frac{1}{p}$), our assumption implies that the process is irreducible. Each state having a self-loop (labeled by 0) makes the process aperiodic and irreducible so that every state converges over time to the stable state. All in-degrees and out-degrees of our graph being p implies that the stable state is the uniform state $\left(\frac{1}{p-1} \quad \cdots \quad \frac{1}{p-1}\right)$. This implies that the density of each element of $\mathbb{F}_p^\times$ in the sequence $T_n^{a,b}$ is $\frac{1}{p-1}$.

Finally, from Proposition 5 we can then see that each value of $\mathbb{F}_p^\times$ must have equal density in $M_n^{a,b} \bmod p$ (because of the factor of $T_q^{a,b}$ in each case). □

There is also an alternative direct argument for the final statement of Theorem 3 that does not appeal to Proposition 5. We only sketch the argument here to avoid introducing unnecessary background. The argument goes that with density 1, a random walk on the Rowland-Zeilberger automaton (see [9]) of $M_n^{a,b} \bmod p$ reaches either the 0 state or else the subgraph corresponding to $T_n^{a,b} \bmod p$ (which is exactly the graph described two paragraphs ago) and then cannot escape in either case. In fact, one interpretation of this paper is as describing what proportion of random walks on this automaton that reach the 0 state. For example, under this interpretation, the first summand in D_0 above corresponds to the probability that walks go to the 0 state on their very first step.

Conjecture 1. For all $a, b \in \mathbb{Z}$, if p does not divide any member of the set $S_p = \left\{T_n^{a,b} \mid n \in \mathbb{F}_p\right\}$, then S_p generates $\mathbb{F}_p^\times$ as a multiplicative group. Equivalently, $\left\{T_n^{a,b} \bmod p \mid n \in \mathbb{N}\right\} = \mathbb{F}_p^\times$. Consequently, all values of $\mathbb{F}_p^\times$ appear in $(T_n^{a,b} \bmod p)_{n\in\mathbb{N}}$ and $(M_n^{a,b} \bmod p)_{n\in\mathbb{N}}$, and furthermore these values appear with equal density.

In order to complete the full picture, note that when $p = 2$, we get that

$$D_0 = \begin{cases} 1 & \text{if } b \equiv 0 \pmod 2 \\ 0 & \text{if } a \equiv 0 \text{ and } b \equiv 1 \pmod 2 \\ \frac{1}{3} & \text{if } a \equiv 1 \text{ and } b \equiv 1 \pmod 2 \end{cases}.$$

When $b = T_1^{a,b} \equiv 0 \pmod 2$, this is because the argument that $D_0 = 1$ when $p \mid T_n^{a,b}$ for some n (including $n = 1$) does not require $p > 2$. Otherwise, when $a \equiv 0 \pmod 2$ we have $M_n^{a,b} \bmod 2 = \mathrm{ct}\left[1^n(1-x^2)\right] = 1$ for all n, and when $a \equiv 1 \pmod 2$ we have $M_n^{a,b} \bmod 2 \equiv M_n$ and it is not hard to manually derive that the density of 0 in $M_n \bmod 2$ is $\frac{1}{3}$ (for example, treat Figure 1 of [8] as a Markov process and compute the probability of ending at each sink).

Finally, when $a = b = 1$ above, we get a simple formula for the density of Motzkin numbers divisible by p. Furthermore, applying Theorem 1 to this simple formula yields the lower bound of $D_0 \geq \frac{2}{p(p-1)}$, which has appeared in the conclusion of [1].

Corollary 4. *The asymptotic density of* 0 *in* $M_n \bmod p$ *is* 1 *if* $p \mid T_n$ *for some*

$n < p$, and otherwise, for $p > 2$, it is

$$\begin{aligned} D_0 =& \frac{|\{n < p-2 \mid M_n \equiv 0 \pmod p\}|}{p} \\ &+ \frac{2\,|\{m < p-1 \mid T_m \equiv T_{p-1}T_{m+1} \pmod p\}|}{(p-1)(p+1)} \\ &+ \frac{2\,|\{m < p-1 \mid T_m \equiv T_{m+1} \pmod p\}|}{(p-1)p(p+1)}. \end{aligned}$$

Additionally, if Conjecture 1 is true for $a = b = 1$, then all other elements of $\mathbb{F}_p$ have density $\frac{1-D_0}{p-1}$. Lastly, $D_0 \geq \frac{2}{p(p-1)}$ is a lower bound.

Proof. Since we know that $T_0 = T_1 = 1$, that $T_2 = 3$, that

$$T_{p-3} \equiv (-3)^{\frac{p-1}{2}-2} T_2 = -(-3)^{\frac{p-1}{2}-1} \equiv -T_{p-2} \pmod p,$$

and that $T_{p-1} \equiv (-3)^{\frac{p-1}{2}} \equiv \pm 1 \pmod p$, we are guaranteed that each of the second and third sets in D_0 above are non-empty (since $T_0 \equiv T_1$ and $T_{p-3} \equiv (-1)T_{p-2} \pmod p$) giving us a lower bound of

$$D_0 \geq \frac{2 \cdot 1}{(p-1)(p+1)} + \frac{2 \cdot 1}{(p-1)p(p+1)} = \frac{2}{p(p-1)}.$$

□

3.1. Application to Other Sequences

To further demonstrate the fruitfulness of this approach, we quickly derive analogous results for a few additional sequences.

Proposition 6. *Let $s_n = \mathrm{ct}\left[(x^{-1} + 1 + x)^n x\right]$, which is the sequence A005717 of [7]. Then the asymptotic density of 0 in $s_n \bmod p$ is 1 if $p \mid T_n$ for some $n < p$ and otherwise, for $p > 2$, it is*

$$D_0 = \frac{|\{m < p-1 \mid T_m \equiv T_{p-1}T_{m+1}\}| + p\,|\{m < p-1 \mid T_m \equiv T_{m+1}\}|}{(p-1)(p+1)}$$

(with congruences modulo p). Additionally, $D_0 \geq \frac{1}{p-1}$ is a lower bound.

Proof. First note that $2s_n = T_{n+1} - T_n$. Consequently, Lemma 2 tells us that

$$\begin{aligned} D_0 =& \frac{|\{0 \leq n < p-1 \mid s_n \equiv 0 \pmod p\}|}{p} \\ &+ \frac{|\{0 \leq m < p-1 \mid T_m \equiv T_{p-1}T_{m+1} \pmod p\}|}{(p-1)(p+1)} \\ &+ \frac{|\{0 \leq m < p-1 \mid T_m \equiv T_{m+1} \pmod p\}|}{(p-1)p(p+1)}, \end{aligned}$$

and so our formula for D_0 follows from the observation that $s_n \equiv 0 \pmod p$ if and only if $T_n \equiv T_{n+1} \pmod p$. Lastly, the lower bound follows from the fact that $T_0 = T_1 = 1$, that $T_{p-1} \equiv (-3)^{\frac{p-1}{2}} \equiv \pm 1 \pmod p$, and that, also by Theorem 1,

$$T_{p-3} \equiv (-3)^{\frac{p-1}{2}-2} \cdot 3 = -(-3)^{\frac{p-1}{2}-1} \equiv -T_{p-2} \pmod p.$$

Therefore, $D_0 \geq \frac{1+p\cdot 1}{(p-1)(p+1)} = \frac{1}{p-1}$. □

Because $D_0 = \frac{1}{p-1}$ for $p = 5, 11, 13$ and others, this bound is tight. We turn now to the Riordan numbers, $R_n = \operatorname{ct}\left[(x^{-1}+1+x)^n(1-x)\right]$, which is the sequence A005043 of [7].

Proposition 7. *The asymptotic density of* 0 *in* $R_n \bmod p$ *is* 1 *if* $p \mid T_n$ *for some* $n < p$ *and otherwise, for* $p > 2$*, it is*

$$D_0 = \frac{|\{m < p-1 \mid 3T_m \equiv T_{p-1}T_{m+1}\}| + p\,|\{m < p-1 \mid 3T_m \equiv T_{m+1}\}|}{(p-1)(p+1)}$$

(with congruences modulo p*). Additionally,* $D_0 \geq \frac{1}{p-1}$ *is a lower bound.*

Proof. First note that $R_n = -T_{n+1} + 3T_n$. Consequently, Lemma 2 tells us that

$$\begin{aligned} D_0 =& \frac{|\{0 \leq n < p-1 \mid R_n \equiv 0 \pmod p\}|}{p} \\ &+ \frac{|\{0 \leq m < p-1 \mid 3T_m \equiv T_{p-1}T_{m+1} \pmod p\}|}{(p-1)(p+1)} \\ &+ \frac{|\{0 \leq m < p-1 \mid 3T_m \equiv T_{m+1} \pmod p\}|}{(p-1)p(p+1)}, \end{aligned}$$

and so our formula for D_0 follows from the observation that $R_n \equiv 0 \pmod p$ if and only if $3T_n \equiv T_{n+1} \pmod p$. Lastly, the lower bound follows from the fact that $3T_1 = T_2 = 1$ and that, by Theorem 1,

$$-T_{p-1} \equiv -(-3)^{\frac{p-1}{2}} \cdot T_0 = 3(-3)^{\frac{p-1}{2}-1} \equiv 3T_{p-2} \pmod p.$$

Therefore, $D_0 \geq \frac{1+p\cdot 1}{(p-1)(p+1)} = \frac{1}{p-1}$. □

Because $D_0 = \frac{1}{p-1}$ for $p = 5, 11, 23, 31$ and others, this bound is tight. We get the same result for A005773 of [7], which is no coincidence.

Proposition 8. *Let* $s_n = \operatorname{ct}\left[(x^{-1}+1+x)^n(1+x)\right]$*, which is the sequence A005773 of [7]. Then the asymptotic density of* 0 *in* $s_n \bmod p$ *is* 1 *if* $p \mid T_n$ *for some* $n < p$ *and otherwise, for* $p > 2$*, it is*

$$D_0 = \frac{|\{m < p-1 \mid T_m \equiv -T_{p-1}T_{m+1}\}| + p\,|\{m < p-1 \mid T_m \equiv -T_{m+1}\}|}{(p-1)(p+1)}$$

(with congruences modulo p*). Additionally,* $D_0 \geq \frac{1}{p-1}$ *is a lower bound.*

Proof. First note that $2s_n = T_{n+1} + T_n$. Consequently, Lemma 2 tells us that

$$D_0 = \frac{|\{0 \le n < p-1 \mid s_n \equiv 0 \pmod{p}\}|}{p} + \frac{|\{0 \le m < p-1 \mid T_m \equiv -T_{p-1}T_{m+1} \pmod{p}\}|}{(p-1)(p+1)} + \frac{|\{0 \le m < p-1 \mid T_m \equiv -T_{m+1} \pmod{p}\}|}{(p-1)p(p+1)},$$

and so our formula for D_0 follows from the observation that $s_n \equiv 0 \pmod{p}$ if and only if $T_n \equiv -T_{n+1} \pmod{p}$. Lastly, the lower bound follows from the fact that $T_0 = T_1 = 1$ and that, by Theorem 1,

$$T_{p-3} \equiv (-3)^{\frac{p-1}{2}-2} \cdot 3 = -(-3)^{\frac{p-1}{2}-1} \equiv -T_{p-2} \pmod{p}.$$

Therefore, $D_0 \ge \frac{1+p\cdot 1}{(p-1)(p+1)} = \frac{1}{p-1}$. □

Because $D_0 = \frac{1}{p-1}$ for $p = 5, 11, 23, 31$ and others, this bound is tight. In fact, these formulas show an unexpected connection between these last two sequences.

Corollary 5. *Let* $s_n = \mathrm{ct}\left[(x^{-1}+1+x)^n(1+x)\right]$*, which is A005773 of [7], and let* $R_n = \mathrm{ct}\left[(x^{-1}+1+x)^n(1-x)\right]$*, which is A005043 of [7]. Then for every prime,* p*, the densities of* 0 *in the sequences* $(s_n \bmod p)_{n\in\mathbb{N}}$ *and* $(R_n \bmod p)_{n\in\mathbb{N}}$ *are equal.*

Proof. This result follows from Propositions 7 and 8 for these densities along with a simple application of Theorem 1: Since

$$3T_m \equiv 3(-3)^{m-\frac{p-1}{2}}T_{p-1-m} = -(-3)^{m+1-\frac{p-1}{2}}T_{p-1-m} \pmod{p}$$

and $T_{m+1} \equiv (-3)^{m+1-\frac{p-1}{2}}T_{p-2-m} \pmod{p}$, we have that $3T_m \equiv T_{m+1} \pmod{p}$ if and only if $-T_{p-1-m} \equiv T_{p-2-m} \pmod{p}$. Therefore,

$$|\{m < p-1 \mid 3T_m \equiv T_{m+1} \pmod{p}\}| = |\{m < p-1 \mid T_m \equiv -T_{m+1} \pmod{p}\}|$$

because these sets biject via $m \mapsto p-2-m$. Likewise, Theorem 1 tells us that

$$3T_m \equiv -(-3)^{m+1-\frac{p-1}{2}}T_{p-1-m} \equiv -T_{p-1}(-3)^{m+1}T_{p-1-m} \pmod{p}$$

and that

$$T_{p-1}T_{m+1} \equiv (-3)^{\frac{p-1}{2}}(-3)^{m+1-\frac{p-1}{2}}T_{p-2-m} = (-3)^{m+1}T_{p-2-m} \pmod{p},$$

so $3T_m \equiv T_{p-1}T_{m+1} \pmod{p}$ if and only if $-T_{p-1}T_{p-1-m} \equiv T_{p-2-m} \pmod{p}$. Therefore,

$$|\{m < p-1 \mid 3T_m \equiv T_{p-1}T_{m+1}\}| = |\{m < p-1 \mid T_m \equiv -T_{p-1}T_{m+1}\}|$$

(with congruences modulo p) because these sets biject via $m \mapsto p-2-m$. □

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