#A23 INTEGERS 26 (2026)

ON THE LENGTH OVER WHICH k-GÖBEL SEQUENCES REMAIN INTEGERS

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Received: 5/19/25, Revised: 12/1/25, Accepted: 1/8/26, Published: 1/19/26

Abstract

We prove that the sequence $(N_k)_k$, where each N_k is defined as the smallest positive integer n for which the nth term $g_{k,n}$ of the k-Göbel sequence is not an integer, is unbounded.

1. Introduction

For $k \geq 2$, the *k-Göbel sequence* $(g_{k,n})_n$ is defined by the initial value $g_{k,1} = 2$ and the recursion $ng_{k,n} = (n-1)g_{k,n-1} + g_{k,n-1}^k$. Let $N_k := \inf\{n \geq 1 \mid g_{k,n} \notin \mathbb{Z}\}$. The 2-Göbel sequence with $N_2 = 43$, which is Göbel's original sequence [7, A003504], has attracted interest as an example of *the strong law of small numbers* ([2]). (The growth of $g_{k,n}$ is very fast. In fact, the value $g_{2,43} \approx 5.4 \times 10^{178485291567}$ is very large.)

The behavior of the sequence $(N_k)_k$ ([7, A108394]) is not yet understood well and remains mysterious. In [6], Matsuhira, Matsusaka, and Tsuchida proved that $\min_{k\geq 2} N_k = 19$. As mentioned in [6, Section 3] and [4, Episode 3], the following three questions are fundamental problems about the sequence $(N_k)_k$:

1. Why is N_k a prime number for most values of k? Or rather, in what cases does N_k become a composite number?

2. Does N_k always take a finite value for any $k \geq 2$?

3. Is the sequence $(N_k)_k$ unbounded?

DOI: 10.5281/zenodo.18305125
[1]This research was supported by JSPS KAKENHI Grant Number JP22K13960.
[2]This research was supported by JSPS KAKENHI Grant Number JP21K13762.

For Problems (1) and (2), only numerical data has been obtained. In [5], it is shown that N_k is prime for 86.5% of values up to $k \le 10^7$, and that N_k is finite for $k \le 10^{14}$.

In this short note, however, we report that Problem (3) can be solved in a very elementary way within the framework of [6]. Let $m\#$ denote the primorial of m or, in other words, the radical of $m!$.

Theorem. *Let m be a positive integer. If $k \ge 2$ satisfies $k \equiv 1 \pmod{m!/m\#}$, then $N_k > m$. In particular, $\sup_{k\ge 2} N_k = \infty$.*

2. Preliminaries

Let $k \ge 2$, $r \ge 1$ be integers and p a prime. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at (p) and $\nu_p(n)$ be the p-adic valuation of n. For any positive integer n with $\nu_p(n!) \le r$, we define $g_{k,p,r}(n) \in \mathbb{Z}/p^{r-\nu_p(n!)}\mathbb{Z} \cup \{\mathsf{F}\}$ as in [6, Definition 9]: for $n = 1$, $g_{k,p,r}(1) = 2 \bmod p^r$. For $n \ge 2$, when $g_{k,p,r}(n-1) = \mathsf{F}$, $g_{k,p,r}(n) = \mathsf{F}$. When $g_{k,p,r}(n-1) = a \bmod p^{r-\nu_p((n-1)!)}$,

$$g_{k,p,r}(n) = \begin{cases} \mathsf{F} & \text{if } (n-1)a + a^k \not\equiv 0 \pmod{p^{\nu_p(n)}}, \\ \dfrac{(n-1)a + a^k}{p^{\nu_p(n)}} \cdot c \bmod p^{r-\nu_p(n!)} & \text{otherwise}, \end{cases}$$

where c is an integer such that $c \cdot (n/p^{\nu_p(n)}) \equiv 1 \pmod{p^{r-\nu_p(n!)}}$. As Lemma 2 below shows, one can determine whether $g_{k,n}$ is p-integral from the value of $g_{k,p,r}(n)$, which can be calculated recursively. The symbol F is an arbitrary object that is distinct from every element of $\mathbb{Z}/p^{r-\nu_p(n!)}\mathbb{Z}$; we simply adopt the notation of [6], where the initial letter of the word "false" is used to indicate a failure to be p-integral.

Let $\varphi(n)$ denote Euler's totient function. We utilize the following three results of Matsuhira, Matsusaka, and Tsuchida in our proof.

Lemma 1 ([6, Lemma 4])**.** *For all $1 \le n < p$, we have $g_{k,n} \in \mathbb{Z}_{(p)}$.*

Lemma 2 ([6, Lemma 10])**.** *Let n be a positive integer with $\nu_p(n!) \le r$. Then, $g_{k,p,r}(n) = \mathsf{F}$ if and only if $g_{k,n} \notin \mathbb{Z}_{(p)}$.*

Lemma 3 ([6, Proposition 12])**.** *Let k and l be integers satisfying $r + 1 \le k \le l$ and $k \equiv l \pmod{\varphi(p^r)}$. Then for any positive integer n satisfying $\nu_p(n!) \le r$, we have $g_{k,p,r}(n) = g_{l,p,r}(n)$.*

3. Proof

Proof of Theorem. Let $k \ge 2$ and m be positive integers. Since $g_{k,1} = 2$, $g_{k,2} = 1 + 2^{k-1}$, and $3g_{k,3} = 2 + 2^k + (1 + 2^{k-1})^k \equiv 0 \pmod 3$ by definition, we have

$N_k > 3$ and may assume $m \geq 4$. Assume that $k \equiv 1 \pmod{m!/m\#}$. For each prime $p \leq m$, set $r_p := \nu_p(m!)$ and $k_p := \varphi(p^{r_p}) + 1$. It is clear that $k_p > r_p \geq 1$. Let us temporarily suppose that for some prime $p \leq m$, we have $g_{k_p,p,r_p}(m) \neq \mathsf{F}$. Since $\varphi(p^{r_p})$ divides $m!/m\#$ (as $m \geq 4$), we see that $k \equiv k_p \pmod{\varphi(p^{r_p})}$, and thus, by Lemma 3, it follows that $g_{k,p,r_p}(m) = g_{k_p,p,r_p}(m) \neq \mathsf{F}$. Therefore, by Lemma 2, we conclude that $g_{k,n} \in \mathbb{Z}_{(p)}$ for all $1 \leq n \leq m$. In order to prove $N_k > m$, it suffices to show that $g_{k_p,p,r_p}(m) \neq \mathsf{F}$ for each prime $p \leq m$. Indeed, once this has been established, the case $p \leq m$ is covered by the above argument and the case $p > m$ by Lemma 1, so that for every prime p we have $g_{k,n} \in \mathbb{Z}_{(p)}$ for all $1 \leq n \leq m$. We then obtain $g_{k,n} \in \bigcap_p \mathbb{Z}_{(p)} = \mathbb{Z}$ for all $1 \leq n \leq m$, which implies that $N_k > m$.

Since $k_2 > r_2$, we see that $2 + 2^{k_2} \equiv 2 \pmod{2^{r_2}}$. Hence, we have

$$g_{k_2,2,r_2}(2) = 1 \bmod 2^{r_2-1}.$$

It is clear that, subsequently,

$$g_{k_2,2,r_2}(n) = 1 \bmod 2^{r_2-\nu_2(n!)}$$

for $2 \leq n \leq m$. In particular, $g_{k_2,2,r_2}(m) \neq \mathsf{F}$.

Let $p \leq m$ be an odd prime. For any $1 \leq n \leq m$, we have

$$g_{k_p,p,r_p}(n) = 2 \bmod p^{r_p-\nu_p(n!)}.$$

In fact, if $g_{k_p,p,r_p}(n-1) = 2 \bmod p^{r_p-\nu_p((n-1)!)}$, then since

$$(n-1)2 + 2^{k_p} \equiv 2n \pmod{p^{r_p-\nu_p((n-1)!)}}$$

by Euler's theorem, we have $g_{k_p,p,r_p}(n) = 2 \bmod p^{r_p-\nu_p(n!)}$. In particular, $g_{k_p,p,r_p}(m) \neq \mathsf{F}$. □

4. Remark

By replacing the initial value in the definition of the k-Göbel sequence with $g_{k,1} = l$, we define the (k,l)-*Göbel sequence*, which has been investigated in [1, 3, 5]. Generalizing our previous arguments as follows, the theorem holds in the same form for (k,l)-Göbel sequences as well.

Fix l and a prime $p \leq m$, and in the definition of $g_{k,p,r}$, replace the initial condition with $g_{k,p,r}(1) = l \bmod p^r$. We use the notation r_p and k_p as in the previous section. We can easily check that by induction on n, for $n \leq p^{\nu_p(l)}$,

$$g_{k_p,p,r_p}(n) = (l/p^{\nu_p(n)}) \cdot c_{n,p} \bmod p^{r_p-\nu_p(n!)}$$

holds, while for $n \geq p^{\nu_p(l)}$, we have

$$g_{k_p,p,r_p}(n) = l/p^{\nu_p(l)} \bmod p^{r_p-\nu_p(n!)}.$$

Here, $c_{n,p} \in \mathbb{Z}$ satisfies

$$c_{n,p} \cdot (n/p^{\nu_p(n)}) \equiv 1 \pmod{p^{r_p-\nu_p(n!)}}.$$

Therefore, the same proof works for a general l.

Acknowledgement. We thank the anonymous referee for helpful suggestions that improved the clarity of our manuscript.

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