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ANTI-RECURRENCE SEQUENCES

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Abstract

We extend the work of Kimberling and Moses, Zaslavsky, and Bosma et al. on anti-recurrence sequences. Kimberling and Moses formulated several questions about these sequences, which together suggest the meta-conjecture that every anti-recurrence sequence is the sum of a linear progression and an automatic sequence. We solve this conjecture under a restriction on the linear form that generates the anti-recurrence.

1. Introduction

In a linear recurrence sequence, each term is a linear combination of the ones that came before it. The study of such sequences is a topic in itself [3]. The first example that comes to mind is the Fibonacci sequence,

$$F_{n+1} = F_n + F_{n-1},$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$.

Recurrence sequences are defined by earlier terms in the sequence. In contrast to this, the *anti-recurrence* sequences, which we consider in this paper, are defined by earlier terms that are *not* in the sequence. The anti-Fibonacci numbers start with $A_0 = 0$. They extend by the rule that "the next anti-Fibonacci number is the sum of the two most recent non-members of the anti-Fibonacci sequence."

To see how the rule works, note that the first two non-members 1 and 2 add up to the anti-Fibonacci $A_1 = 3$. The next two non-members are 4 and 5, which add up to the anti-Fibonacci $A_2 = 9$, and so on. This sequence is listed under A075326

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in the On-Line Encyclopedia of Integer Sequences (OEIS):

$0, 3, 9, 13, 18, 23, 29, 33, 39, 43, 49, 53, 58, 63, 69, 73, 78, 83, 89, 93, 98, 103, 109, 113, \ldots.$

It was entered into the OEIS by Amarnath Murthy and was named anti-Fibonacci by Douglas Hofstadter in an unpublished note [6]. He observed that the first difference sequence

$$3, 6, 4, 5, 5, 6, 4, 6, 4, 6, 4, 5, 5, 6, 4, 5, 5, 6, 4, 5, 5, 6, 4, \ldots$$

consists of the two-letter words 64 and 55, apart from the initial letter 3. This is A249032 in the OEIS. All numbers with final digit 3 are anti-Fibonaccis, and the other anti-Fibonaccis either end with a 9 or an 8. Hofstadter observed, without giving a proof, that the pattern of 9's and 8's can be generated from a period-doubling substitution

$$9 \mapsto 98, \; 8 \mapsto 99.$$

The proof was supplied by Thomas Zaslavsky, in another unpublished note [13]. In particular, he gave an explicit equation for the anti-Fibonacci numbers:

$$\text{For all } n \geq 1, \quad A075326(n) - 5n + 2 = \texttt{PD}_{n-1}.$$

The period doubling sequence $\texttt{PD}_n$ consists of zeros and ones and is generated by

$$0 \mapsto 01, \; 1 \mapsto 00,$$

starting from $\texttt{PD}_0 = 0$. It is entry A096268 in the OEIS. Note that the indexing runs from 0 and not from 1. This is a convention. One needs to be aware that for automatic sequences such as PD, indexing starts at zero.

Clark Kimberling and Peter Moses studied the more general class of complementary sequences [7], for which anti-recurrence sequences are a special case. They observed some properties of anti-recurrence sequences, which Kimberling entered as conjectures under A265389, A299409, A304499, and A304502 in the OEIS. The conjectures for the first two sequences were verified by Bosma et al. [2] using Hamoon Mousavi's automatic theorem prover `Walnut` [9]. We settle the other two conjectures on A304499 and A304502 in this paper, again with the assistance of `Walnut`. These conjectures can be combined into a meta-conjecture (Conjecture 1), which is discussed in Section 6 of [7]. It was named the Clergyman's Conjecture in [2].

Conjecture 1. Every anti-recurrence sequence is a sum of a linear sequence and an automatic sequence.

The paper is organized as follows. In Section 1 we review the basic notions. Section 2 settles the conjectures of Kimberling for A304499 and A304502 using `Walnut`. In Section 3 we extend the results of Bosma et al. and solve the conjecture for anti-bonaccis. Our main result, Theorem 4 in Section 4, settles the conjecture under a restriction on the linear form that generates it. We are unable to settle the full conjecture.

2. Definitions, Notation, Preliminary Results

All numbers are natural numbers (positive integers) unless stated otherwise. We will write sequences in capitals as X_n. It denotes either the sequence or a number in the sequence, which should be clear from the context. The indexing starts at $n = 1$ for the sequences. We note that Hofstadter [6] reserves the index zero for the anti-recurrence number $A_0 = 0$. It plays no role in our considerations.

Two strictly increasing sequences A_n and B_n of natural numbers are *complementary* if every natural number belongs to exactly one of them. Let $\mathbf{a} = (a_1, \ldots, a_k)$ be a positive integral vector of dimension $k > 1$ and all $a_i > 0$. Let $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ be its associated linear form. We say that A_n is an *anti-recurrence sequence of order* k if

$$A_n = f(B_{(n-1)k+1}, B_{(n-1)k+2}, \ldots, B_{nk}) = \sum_{j=1}^{k} a_j B_{(n-1)k+j}.$$

The *trace* τ of the linear form is $\tau = \sum_{i=1}^{m} a_i \geq k$.

In a precise but elaborate naming convention, A_n is the anti-recurrence sequence and its complement B_n is the *non-anti-recurrence sequence*. We say that a set $\{B_{jk+1}, B_{jk+2}, \ldots, B_{(j+1)k}\}$ is the B-*block* that generates A_{j+1}. Note that we use X_n both for the sequence and the individual number and the context will make clear what we mean. If numbers $\{a, a+1, \ldots, b\}$ are consecutive, then we say that they form the interval $[a, b]$.

Lemma 1. *Successive anti-recurrence numbers satisfy*

$$A_{n+1} - A_n \geq k\tau,$$

for $n \geq 1$*, where* $\tau > 1$ *is the trace and* $k > 1$ *is the order of the sequence. The above is an equality if the* B*-blocks for both* A_{n+1} *and* A_n *are intervals such that their union is also an interval. In particular, the inequality is strict if one block is an interval and the other is not.*

Proof. Let $\mathbf{a} = (a_1, \ldots, a_k)$ and let B_n be the non-anti-recurrence sequence. We have that

$$A_n = \sum_{j=1}^{k} a_j B_{m+j}$$

for $m = k(n-1)$ and

$$A_{n+1} = \sum_{j=1}^{k} a_j B_{k+m+j}.$$

Now, $B_{k+m+j} - B_{m+j} \geq k$ since this sequence is increasing. If both B-blocks are intervals, then this is an equality. If one is an interval and the other is not, then there must be a j such that $B_{k+m+j} - B_{m+j} > k$. □

The *mex* or minimal excluded value of $S \subset \mathbb{N}$ is:

$$\operatorname{mex}(S) = \min\left(\mathbb{N} \setminus S\right).$$

It comes up naturally in anti-recurrence sequences, as observed by Kimberling and Moses.

Lemma 2. *A positive linear form* $\mathbf{a}$ *determines both the anti-recurrence sequence* A_n *and its complementary sequence* B_n.

Proof. We need to show that the complementary sequences A_n and B_n exist and are unique. Let $\mathcal{A}_n = \{A_i \colon i \leq n\}$ be the initial anti-recurrences and let $\mathcal{B}_{kn} = \{B_j \colon j \leq kn\}$ be the initial B-blocks. Assume inductively that

$$[1, B_{kn}] \subset \mathcal{A}_n \cup \mathcal{B}_{kn}.$$

Let

$$b = \operatorname{mex}\left(\mathcal{A}_n \cup \mathcal{B}_{kn}\right).$$

The interval $[b, b+k-1]$ has length k and by Lemma 1 can contain at most one number from $\mathcal{A}_n$. If it contains no such number, then $[b, b+k-1]$ must be the B-block from B_{kn+1} up to $B_{k(n+1)}$ since the sequence B_n consists of all numbers that are not in A_n. If one of the numbers in $[b, b+k-1]$ is in $\mathcal{A}_n$, then the B-block from B_{kn+1} up to $B_{k(n+1)}$ skip that number. In any case, the next B-block from B_{kn+1} up to $B_{k(n+1)}$ is uniquely determined by a mex-rule and generates A_{n+1}. □

The linear form $\mathbf{a} = (1, 1)$ gives the anti-Fibonacci numbers. Its complementary sequence A249031

$$1, 2, 4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 16, 17, 19, 20, 21, 22, \ldots.$$

is the *non-anti-Fibonacci sequence*. The proof of Lemma 2 shows that the B_n are defined blockwise by the mex. It is convenient to cut up these blocks into individual parts and define the k subsequences

$$B_n^j = B_{j+(n-1)k}$$

for $j = 1, \ldots, m$. For instance, the non-anti-Fibonacci sequence can be divided into A075325

$$1, 4, 6, 8, 11, 14, 16, 19, 21, \ldots$$

and A047215

$$2, 5, 7, 10, 12, 15, 17, 20, 22, \ldots.$$

We can generate the sequences B_n^j and A_n simultaneously, adding the mex to each sequence B_n^j from $j = 1$ to $j = m$, and then $A_n = \sum_j a_j B_n^j$. This is how Kimberling and Moses define anti-recurrence sequences.

A *deterministic finite state automaton with output*, or DFAO, is the simplest type of computing machine. It is able to read a finite input word and return an output. A DFAO is a 6-tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, \Gamma, \lambda)$, where:

- Q is a finite set of *states*,
- Σ is a finite *input alphabet*,
- $\delta : Q \times \Sigma \to Q$ is the *transition function*,
- $q_0 \in Q$ is the *initial state*,
- Γ is a finite *output alphabet*,
- $\lambda : Q \times \Sigma \to \Gamma$ is the *output function*.

For instance, there is a DFAO for the Period Doubling sequence that returns the digit $\mathtt{PD}_n$ upon input n in binary; see Figure 1. According to Cobham's little theorem [11], a DFAO corresponds to a substitution σ. To a state a, it assigns the word $\sigma(a)$ such that the j-th digit of $\sigma(a)$ corresponds to the transition from a under j. The DFAO in Figure 1 corresponds to the Period Doubling substitution $a \mapsto ab,\ b \mapsto aa$. Input of $\mathtt{PD}_n$ is in binary and starts at $n = 0$ instead of $n = 1$. The state a outputs 0 and state $\mathtt{b}$ outputs 1. For instance, $n = 9$ is expanded as 1001 in binary and has digit $\mathtt{PD}_9 = 1$.

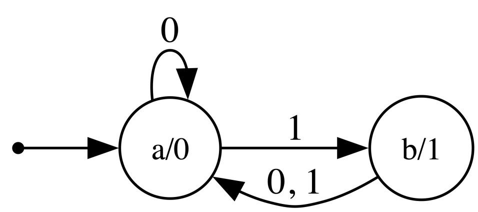


Figure 1: The automaton for the Period Doubling sequence $0100010101000100\cdots$.

A *k-DFAO* is an automaton with alphabet $\Sigma = \{0, 1, \ldots, k-1\}$. It reads numbers that are expanded in base k. Our DFAO for the Period Doubling word is a 2-DFAO. A sequence X_n is k-automatic if there exists a k-DFAO that gives output X_n on input n—another case where X_n is both a term and a sequence in one sentence.

We will use the automatic theorem prover `Walnut`. It has a transparent syntax that can be easily understood, even by readers that are unfamiliar with the software. We refer to Hamoon Mousavi's user manual [9] and Jeffrey Shallit's textbook [11] for more information. In the words of Jeffrey Shallit [12], `Walnut` serves as a telescope to view results that, at first, appear only distantly provable, and that is how we use it in this paper.

We now give a more precise statement of Conjecture 1.

Conjecture 2. Let $\mathbf{a} = (a_1, \ldots, a_k)$ be positive and integral of dimension $k > 1$. Let A_n be the anti-recurrence sequence generated by the linear form $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$. Then $A_n - \kappa n$ is τ-automatic for $\kappa = k\tau + 1$ and τ the trace of the linear form.

We prove this conjecture under a restriction on $\mathbf{a}$ in the final section of our paper. Thomas Zaslavsky [13] proved it for $\mathbf{a} = (1,1)$ and Bosma et al. [2] proved it for $\mathbf{a} = (1,1,1)$ and $\mathbf{a} = (1,1,1,1)$, naming it the Clergyman's Conjecture. A weak form of the conjecture says that the difference sequence $A_n - \kappa n$ is bounded. In fact, this is how Kimberling and Moses [7] formulate their conjectures, but they do provide conjectured substitutions that generate the specific difference sequences. We confirm these substitutions in Theorem 3.

3. The Anti-Pell and Anti-Jacobsthal Numbers

The recurrence $X_{n+1} = 2X_n + X_{n-1}$ generates the Pell numbers A000129 while $X_{n+1} = X_n + 2X_{n-1}$ generates the Jacobsthal numbers A001045. We consider their counterparts, the anti-recurrence sequences for $\mathbf{a} = (1,2)$ and $\mathbf{a} = (2,1)$. Kimberling conjectured on the OEIS that the difference sequence is bounded for these anti-recurrences.

For $\mathbf{a} = (1,2)$, we get the anti-Pell numbers A304502

$$5, 11, 20, 26, 34, 41, 47, 53, 61, 68, 74, 83, 89, 95, 103, 110, \ldots.$$

Here we ignore $A_0 = 0$. Observe that the subsequence $A_{3n+1} = 5 + 21n$ forms an arithmetic progression, in analogy with what we saw for the anti-Fibonacci sequence. The differences between consecutive numbers now show a period three:

$$6, 9, 6, 8, 7, 6, 6, 8, 7, 6, 9, 6, 6, 8, 7, \ldots,$$

with blocks 696, 876, and 687. This is in line with the meta-conjecture that the difference sequence must be 3-automatic. On the OEIS, Kimberling conjectures that:

$$0 \leq A_n - 7n + 3 \leq 2.$$

We apply the method of guessing an automaton as described in Shallit's 'book of `Walnut`' [11, p. 75] to guess a DFAO for the difference sequence $A_n - 7n$. We shift the index by one to comply with the convention that automatic sequences start at index 0 and we adjust the sequence to $A_{n+1} - 7n - 4$, to make the output alphabet $\Gamma = \{0, 1, 2\}$, as shown in Figure 2. Notice that all inputs $n \equiv 0 \pmod 3$ end in state a with output 1. This means that $A_n = 7n - 2$ if $n \equiv 1 \pmod 3$. For instance, if $n = 11$, then $A_{n+1} = A_{12}$. Now, feeding the base 3 expansion of n, i.e., 102, the

3-DFAO outputs 2 for ending in state c. Therefore, $A_{12} = 7 \cdot 11 + 6 = 83$. We call this automaton `a12` and we implement the anti-Pell numbers in `Walnut` by the command:

```
def a304502 "?msd_3 (n>0) & s=(7*n-3+a12[n-1])":
```

In particular, the variable s is equal to A_n.

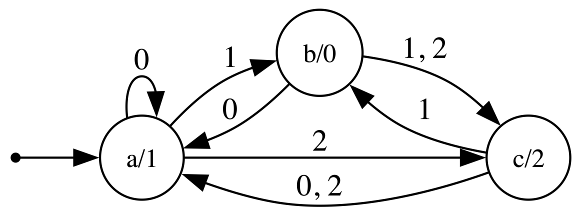


Figure 2: A 3-DFAO for the difference sequence $A_{n+1} - 7n - 4$, of the anti-Pell numbers.

To verify Kimberling's conjecture for anti-Pell numbers, we also need the two non-anti-Pell sequences B_n^1 and B_n^2. The full conjecture is that

$$\begin{aligned} 0 \le A_n - 7n + 3 \quad &\le 2, \\ 0 \le 3B_n^1 - 7n + 6 \quad &\le 3, \\ 0 \le 3B_n^2 - 7n + 2 \quad &\le 3. \end{aligned} \tag{1}$$

Observe that successive entries B_n^1, B_n^2 in a non-anti-recurrence sequence differ by one or two, because there can be at most one anti-recurrence number in between. Since $2B_n^2 + B_n^1 = A_n$, it follows that $3B_n^2 = A_n + i$ for $i \in \{1, 2\}$, and this can be used to obtain the third inequality from the first. In the OEIS, B_n^1 is A304500

$$1, 3, 6, 8, 10, 13, 15, 17, 19, 22, 24, 27, 29, 31, 33, 36, \ldots,$$

and B_n^2 is A304501

$$2, 4, 7, 9, 12, 14, 16, 18, 21, 23, 25, 28, 30, 32, 35, 37, \ldots.$$

It is possible to derive these sequences from the anti-Pell numbers. As already observed, B_n^2 is equal to $(A_n + 2)/3$ rounded down. If we have A_n and B_n^2 then we also have $B_n^1 = A_n - 2B_n^2$. The non-anti-recurrence sequences are implemented by the following commands:

```
def a304501 "?msd_3 Es $a304502(n,s) & t=(s+2)/3":
def a304500 "?msd_3 Es,t $a304502(n,s) & $a304501(n,t) & u+2*t=s":
```

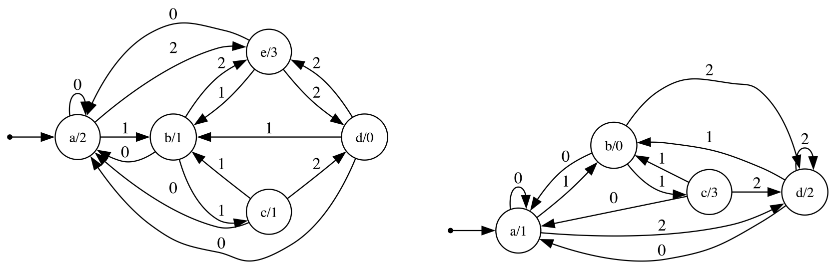


Figure 3: The 3-DFAO's for the non-anti-recurrences $3B^1_{n+1} - 7n - 1$ (left) and $3B^2_{n+1} - 7n - 5$ (right).

The automata for the difference sequences B^1_n and B^2_n are shown in Figure 3. For instance, B^2_8 can be computed from the input 7, which is 21 in base 3. It ends in state b with output 0, and therefore $B^2_8 = 18$. Observe that the final digit determines the output mod 3. These DFAO's are from [2], where they were given in lsd format. They were converted to msd by `Walnut`.

Theorem 1. *The anti-Pell numbers satisfy Kimberling's bounds in Equation (1).*

Proof. The outputs of the DFAO's in Figure 2 and Figure 3 are within Kimberling's bounds. Note that we shifted $A_n - 7n + 3$ to $A_{n+1} - 7(n+1) + 3$, and likewise for B^1_n and B^2_n, to comply with the rule that automatic sequences start at index 0. We need to verify that these DFAO's indeed correspond to the difference sequences for the anti-Pell numbers, which we do with the assistance of `Walnut`.

According to Lemma 2, the sequences are determined by their initial values and a mex rule. We first check that the initial values are $B^1_1 = 1,\ B^2_1 = 2, A_1 = 5$:

```
eval test "?msd_3 $a304500(1,1) & $a304501(1,2) & $a304502(1,5)":
```

`Walnut` evaluates the statement as `TRUE`.

We check the mex rule for B^1_n, which says that it is the least new number after the first n have been defined. In first-order logic, the statement is:

$$\forall n, s, t \in \mathbb{N}\ \left(t < s\ \wedge\ s = B^1_n\right) \implies \exists m < n\ (t = B^1_m\ \vee\ t = B^2_m\ \vee\ t = A_m).$$

In `Walnut` this statement becomes:

```
eval testB1 "?msd_3 An,s,t ($a304500(n,s) & t>0 & t<s) => (Em (m<n)
& ($a304500(m,t)|$a304501(m,t)|$a304502(m,t)))":
```

It is evaluated as `TRUE`. The mex condition requires B^2_n to be the first missing number after B^1_n:

$$\forall n, s \in \mathbb{N}\ \left(s > 1\ \wedge\ s = B^2_n\right) \implies \left(s - 1 = B^1_n\ \vee\ \exists\ m < n\ \ (s - 1 = A_m)\right).$$

In `Walnut` this statement is:

```
eval testB2 "?msd_3 An,s (s>1 & $a304501(n,s)) =>
($a304500(n,s-1)|(E m (m<n) & $a304502(m,s-1)))":
```

It is evaluated as `TRUE`. The final condition is that $A_n = B_n^1 + B_n^2$:

```
eval testA "?msd_3 An,s,t ($a304500(n,s) & $a304501(n,t)) =>
$a304502(n,s+2*t)":
```

It is evaluated as `TRUE`. This proves that these are indeed the anti-Pell sequence and its non-anti-Pell counterparts. Conjecture 2 holds for $\mathbf{a} = (1, 2)$ and Kimberling's bounds in Equation (1) apply. □

For $\mathbf{a} = (2, 1)$ we get the anti-Jacobsthal numbers A304499

$$4, 11, 19, 25, 32, 40, 46, 52, 61, 67, 74, 82, 88, 95, 103, 109, \ldots,$$

which resemble the anti-Pell numbers. If we divide the gaps $A_{n+1} - A_n$ between anti-recurrence numbers into blocks of three, we now find that there are more blocks: 786, 678, 966, 696, 669. Again, the sum of all blocks is the same and the subsequence A_{3n+1} is an arithmetic progression.

Kimberling's conjecture for this anti-recurrence is

$$\begin{aligned} 0 \le A_n - 7n + 4 &\quad \le 3, \\ 0 \le 3B_n^1 - 7n + 6 &\quad \le 4, \\ 0 \le 3B_n^2 - 7n + 2 &\quad \le 3. \end{aligned} \tag{2}$$

In this case, the second inequality follows from the first.

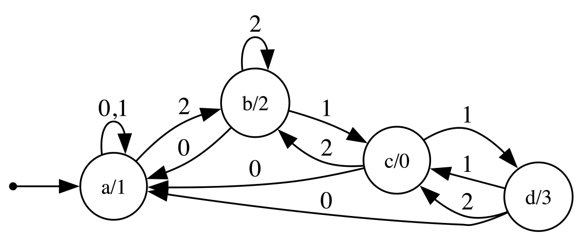


Figure 4: A 3-DFAO for the difference sequence $A_{n+1} - 7n - 3$ of the anti-Jacobsthals.

Our guessed automaton `a21` for the difference sequence of the anti-Jacobsthals is illustrated in Figure 4. All inputs $n \equiv 0 \pmod 3$ end in state a with output 1, which means that $A_n = 7n - 3$ if $n \equiv 1 \pmod 3$. We use this automaton to implement A304499 in `Walnut`:

```
def a304499 "?msd_3 (n>0) & s=(7*n-4+a21[n-1])":
```

We check that these numbers are not divisible by three:

```
eval test "?msd_3 As (En $a304499(n,s)) => (Et (s=3*t+1 |
s=3*t+2))":
```

which is TRUE. We can define the non-anti-recurrence sequences from A_n. The numbers B_n^1 are the rounded down $A_n/3$ and $B_n^2 = A_n - 2B_n^1$:

```
def a304497 "?msd_3 Er $a304499(n,r) => s=r/3":
def a304498 "?msd_3 Eq,r ($a304497(n,q) & $a304499(n,r)) =>
s=r-2*q":
```

We have defined our candidate sequences in `Walnut`. We still need to satisfy that our DFAO does indeed produce the right numbers.

Theorem 2. *The anti-Jacobsthal numbers satisfy Kimberling's bounds in Equation (2).*

Proof. We verify, in exactly the same way as for the anti-Pell numbers, that these sequences satisfy the criterion of Lemma 2, starting with the initial conditions:

```
eval test "?msd_3 $a304497(1,1) & $a304498(1,2) & $a304499(1,4)":
```

which is TRUE.

We test the `mex` conditions for the non-anti-recurrence sequences:

```
eval testB1 "?msd_3 An,s,t ($a304497(n,t) & s>0 & s<t) =>
(Em (m<n) & ($a304497(m,s)|$a304498(m,s)|$a304499(m,s)))":
eval testB2 "?msd_3 An,s (s>1 & $a304498(n,s)) =>
($a304497(n,s-1)|(E m (m<n) & $a304499(m,s-1)))":
```

and we test that the additive relation $A_n = 2B_n^1 + B_n^2$ holds:

```
eval testA "?msd_3 An,s,t
($a304497(n,s)&$a304498(n,t))=>$a304499(n,2*s+t)":
```

These are all TRUE, and therefore the anti-Jacobsthals satisfy Conjecture 2. To verify Kimberling's bounds in Equation (2), we only need to verify the first and third inequality:

```
eval testA  "?msd_3 An,s $a304499(n,s) => (7*n <= s+4 & s+4 <=
7*n+3)":
eval testB2 "?msd_3 An,s $a304498(n,s) => (7*n <= 3*s+2 & 3*s+2 <=
7*n+3)":
```

which is TRUE. □

This settles Kimberling's conjectures on A304499 and A304502.

4. The Anti-Bonacci Numbers

The recurrence relation for $\mathbf{a} = (1, 1, \ldots, 1)$ given by

$$X_n = X_{n-1} + \cdots + X_{n-k}$$

starting from the initial conditions $X_n = 0$ for $n \leq 0$ and $X_1 = 1$ produces the k-bonacci numbers. Apparently, they were first introduced in [8] under the name of k-generalized Fibonacci numbers. The most familiar cases are the Tribonacci numbers for $k = 3$ and the Tetrabonacci numbers for $k = 4$. Their anti-recurrent counterparts are the anti-Tribonacci sequence A265389

$$6, 16, 27, 36, 46, 57, 66, 75, 87, 96, 106, 117, 126, 136, 147, 156, \ldots,$$

and the anti-Tetrabonacci sequence A299409

$$10, 26, 45, 62, 78, 94, 114, 130, 146, 162, 180, 198, 214, 230, 248, \ldots.$$

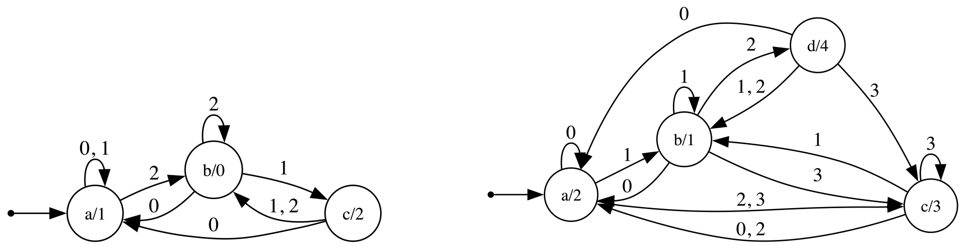
Kimberling conjectured bounds on these two sequences that were verified by Bosma et al. in [2] by using `Walnut`. In particular, Bosma et al. showed that the anti-k-bonacci sequence is a sum of a linear sequence and a k-automatic sequence for $k = 3$ and $k = 4$. However, the automata for the difference sequences in [2] are not that easy to interpret. The automaton for $k = 4$ has 10 states, for instance, and that is because these automata were given in lsd format. If we reverse them to msd format, we get much cleaner machines as shown in Figure 5a and 5b. For instance, in Figure (a), $A_{15} = 147$ and the input for the automaton is 112, with output 2. The numbers with output 1 for the DFAO in (a) correspond to the positions of 0 in Stewart's choral sequence A116178. The automaton in (a) was conjectured by Kimberling and Moses [7]. The automaton in (b) corresponds to the substitution $a \mapsto 21\bar{a}3$ where $\bar{a} = 5 - a$, by Cobham's little theorem. Both these DFAO's, as well as the DFAO for the anti-Fibonacci that we saw earlier, satisfy the following properties:

- The number of states is equal to k.
- The outputs are unique.
- All transitions on input 0 lead back to the initial state.

The third property is equivalent to the fact that the subsequence A_{kn+1} forms an arithmetic progression with increment $k\kappa = k^3 + k$.

The 5-bonacci numbers, or Pentanacci numbers, are A145029, but the anti-5-bonacci numbers have not yet been entered in the OEIS. The initial numbers are:

$$15, 40, 66, 95, 120, 145, 170, 197, 225, 250, 275, 300, 327, 355, 380, \ldots.$$



(a) The DFAO for $\mathbf{a}(1, 1, 1)$ (b) The DFAO for $\mathbf{a}(1, 1, 1, 1)$

Figure 5: The 3-DFAO for the difference sequence $A_{n+1} - 10n - 5$ for the anti-Tribonacci sequence in (a) and $A_{n+1} - 17n - 8$ for the anti-Tetrabonacci sequence in (b).

We have guessed the automaton `a11111` for the anti-5-bonacci sequence:

$$A_n - 26n + 13$$

as illustrated in Figure 6. It is possible to check with `Walnut` that this DFAO does indeed produce the anti-bonacci sequence for $k = 5$. The properties of the DFAO's that we observed for $k = 2, 3$, and 4 are again satisfied. We will now prove that these hold for all k-anti-bonaccis.

We fix k and denote the k-anti-bonacci numbers by A_n without including k in the notation. The difference sequence is $A_n - \kappa n$ with $\kappa = k^2 + 1$. In particular, A_1 is equal to the triangular number $t_k = 1 + 2 + \cdots + k = \binom{k+1}{2}$. We consider consecutive intervals of length $k^2 + 1$:

$$I_n = [(n-1)\kappa + 1, n\kappa].$$

Recall that the set $\{B_m^1, \ldots, B_m^k\}$ is the B-block that generates A_m. We will show that each I_n contains one anti-bonacci A_n and k such B-blocks. We can thus associate A_n to I_n, which generates k anti-bonaccis $A_{k(n-1)+j}$ for $j = 1, \ldots, k$. That gives a substitution rule for the anti-bonaccis. Since I_n contains only one anti-recurrence, at most one of the blocks is not an interval. Lemma 1 implies that at least $k - 2$ of the $A_{k(n-1)+j}$ are k^2 apart. Modulo κ, the next anti-recurrence decreases by 1 if this is the case. The first k numbers of I_n form the first B-block, which explains why the A_{kn+1} form an arithmetic progression. The following lemma makes this precise.

Lemma 3. *Let A_n be an anti-recurrence sequence of order $k > 1$. Let $i = \lfloor \frac{k}{2} \rfloor$ and $\kappa = k^2 + 1$ for $k > 2$. Then $A_n \in I_n$ and*

$$A_n \equiv ik + a_n \pmod{\kappa}$$

for some $1 \leq a_n \leq k$ if k is even, and $i + 1 \leq a_n \leq i + k$ if k is odd.

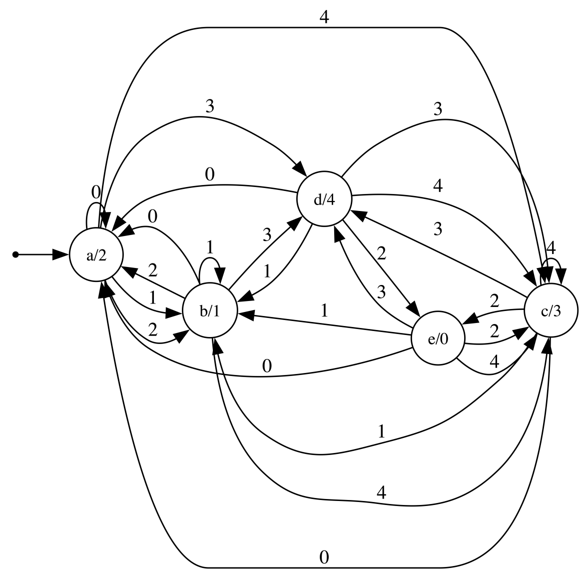


Figure 6: The 5-DFAO for the difference sequence $A_{n+1} - 26n - 13$ of the anti-5-bonacci numbers.

Proof. We have $A_1 = t_k \in [1, \kappa] = I_1$. If k is odd, then $t_k = ik + k$, and if it is even, then $t_k = ik + \frac{k}{2}$. So $a_1 = k$ if k is odd and $a_1 = \frac{k}{2}$ if it is even. The initial interval I_1 contains k B-blocks, all of which are intervals except for one. The possible exception is either the $(i+1)$-st block or the $(i+2)$-nd block. That is our inductive hypothesis.

A B-block determines an anti-bonacci, and therefore each I_j determines k anti-bonaccis. By Lemma 1, if blocks are consecutive intervals, then they determine anti-bonaccis that are $k^2 = \kappa - 1$ apart. Modulo κ, the next anti-bonacci decreases by 1. If they are not consecutive intervals, then the anti-bonaccis are further apart. Suppose that $A_j = ik + a_j$ for $1 \leq a_j \leq k$. If $a_j = 1$, then the $(i+1)$-st block in I_j is an interval that generates an anti-bonacci A that is equal to $k^2 + k$ plus the previous anti-bonacci. If $1 < a_j \leq k$ then the $(i+1)$-st block in I_j is not an interval. It generates an anti-bonacci A that is $k^2 + k + 1 - a_j$ plus the previous anti-bonacci. The next anti-bonacci is $k^2 + a_j - 1 + A$. If $1 \leq a_j \leq k$ then the interval I_j generates k anti-bonaccis with first differences

$$\overbrace{k^2 \cdots k^2}^{i-1}\ x(2k^2 + k - x)\ \overbrace{k^2 \cdots k^2}^{k-i-1} \tag{3}$$

with $x = k^2 + k + 1 - a_j$. For instance, if $k = 4$ then $i = 2$ and the differences are $16, x, 36 - x, 16$ for $x = 17, 18, 19$, and 20, respectively. The final difference is k^2, since the final block of I_j and the first block of I_{j+1} are consecutive intervals. Note that the total sum of the first differences is $k\kappa$, and therefore $A_{jk+1} \equiv A_{(j-1)k+1} \pmod{\kappa}$.

From these first differences we can compute $a_{(j-1)k+1}, \dots, a_{jk}$ for the anti-bonaccis that are generated from I_j. In Equation (3), the initial $i-1$ differences are $-1 \equiv \pmod{\kappa}$. These are followed by $k+1-a_j$ and $a_j - 2 \bmod \kappa$, followed by $k-i-1$ differences of -1. If $1 \le a_j \le k$ then

$$a_{(j-1)k+\ell} = \begin{cases} a_1 + 1 - \ell & \text{if } 1 \le \ell \le i, \\ a_1 + 1 - i + k - a_j & \text{if } \ell = i+1, \\ a_1 + 1 + k - \ell & \text{if } i+2 \le \ell \le k. \end{cases}$$

The $(i+1)$-st entry is the only one that depends on a_j. When k is even, we have $i = \frac{k}{2} = a_1$, and so

$$a_{(j-1)k+\ell} = \begin{cases} i + 1 - \ell & \text{if } 1 \le \ell \le i, \\ k + 1 - a_j & \text{if } \ell = i+1, \\ k + 1 - (\ell - i) & \text{if } i+2 \le \ell \le k. \end{cases} \tag{4}$$

These numbers are between 1 and k. Therefore, when k is even, our inductive hypothesis implies that the first kn values of a_j lie between 1 and k. This completes the case in which k is even.

When k is even, we have $a_1 = k/2$, whereas when k is odd we obtain the larger value $a_1 = k$. In the case that k is odd, the A's may lie in the $(i+1)$-st and $(i+2)$-nd blocks. In particular, this is the case if $a_j > k$. The first differences then are

$$\overbrace{k^2 \cdots k^2}^{i} \;\; x(2k^2 + k - x) \;\; \overbrace{k^2 \cdots k^2}^{k-i-2} \tag{5}$$

with $x = k^2 + 2k + 1 - a_j$. As before, we can compute the a's from these differences.

$$a_{(j-1)k+\ell} = \begin{cases} a_1 + 1 - \ell & \text{if } 1 \le \ell \le i+1, \\ a_1 - i + 2k - a_j & \text{if } \ell = i+2, \\ a_1 + k + 1 - \ell & \text{if } i+3 \le \ell \le k. \end{cases}$$

When k is odd, then $a_1 = k$ and $i = \frac{k-1}{2}$. If $a_j \le k$ then we get

$$a_{(j-1)k+\ell} = \begin{cases} k + 1 - \ell & \text{if } 1 \le \ell \le i, \\ k + i + 2 - a_j & \text{if } \ell = i+1, \\ 2k + 1 - \ell & \text{if } i+2 \le \ell \le k. \end{cases} \tag{6}$$

These numbers are in $[i+2, i+k]$. If $a_j > k$ then we get

$$a_{(j-1)k+\ell} = \begin{cases} k + 1 - \ell & \text{if } 1 \le \ell \le i+1, \\ 2k + i + 1 - a_j & \text{if } \ell = i+2, \\ 2k + 1 - \ell & \text{if } i+3 \le \ell \le k. \end{cases} \tag{7}$$

These numbers are in $[i+1, i+k]$. □

Equations (4), (6), and (7) for the remainders a_n allow us to extend Zaslavsky's formula from $k = 2$ to $k > 2$. The following result settles Conjecture 2 for anti-bonaccis.

Theorem 3. *Let A_n be an anti-recurrence sequence of order $k > 1$, and let $i = \lfloor \frac{k}{2} \rfloor$. There exists a k-uniform substitution σ on $\{1, 2, \ldots, k\}$ if k is even and $\{0, 1, \ldots, k-1\}$ if k is odd with unique fixed point $\omega = (i_n)$ such that*

$$A_n = \kappa(n-1) + t_k - i + i_{n-1}.$$

All $\sigma(j) = w_j$ have initial digit i, and therefore $\lim_{n\to\infty} \sigma^n(i) = \omega$.

Proof. By Lemma 3, $A_n = \kappa(n-1) + ik + a_n$. If k is even, then $ik = t_k - i$, and so we get $A_n = \kappa(n-1) + t_k - i + a_n$ for numbers $a_n \in [1, k]$. Equation (4) describes a k-substitution $a_j \mapsto w$ where the digits w_ℓ are given by $a_{(j-1)k+\ell}$, which is independent of a_j, except if $\ell = i + 1$, when the digit is $k + 1 - a_j$. The initial digit of each substitution word is equal to i. We write $i_{n-1} = a_n$ to comply with the rule that automatic sequences start with index 0.

If k is odd, then $ik = t_k - k$, and so we get that $A_n = \kappa(n-1) + t_k - k + a_n$ for numbers $a_n \in [i+1, i+k]$. If we write $i_{n-1} = a_n - k + i$ then we get that $A_n = \kappa(n-1) + t_k + i + i_{n-1}$. Since $k = 2i + 1$ if k is odd, $i_{n-1} \in [0, k-1]$. Equations (6) and (7) describe a k-substitution $a_j \mapsto w_j$ with initial digit a_1. Then we get that $A_n = \kappa(n-1) + t_k - i + i_{n-1}$ and the initial digit of the substitution words is i. This confirms our observations on the DFAO's for the anti-bonaccis. □

5. Rusty Numbers and Other Anti-Recurrences

The recurrence

$$X_{n+1} = dX_n + X_{n-1}$$

produces the so-called metallic or metallonacci numbers [1, 10]. It is impossible to resist the temptation to say that the A_n for the linear form $\mathbf{a} = (1, d)$ are the *rusty numbers*. The 3-rusty numbers are:

$$7, 15, 23, 35, 43, 51, 62, 71, 79, 87, 99, 107, 115, 123, 131, 142, 151, \ldots.$$

We can guess a 4-DFAO for its difference sequence, which is illustrated in Figure 7. The `Walnut` verification for the anti-Pell and anti-Jacobsthal sequences can also be applied to this DFAO, to check that it indeed produces the difference sequence. Note that all 0-transitions lead back to the initial state, which implies that the subsequence A_{4n+1} is an arithmetic progression. However, this does not apply to the 4-rusty numbers:

$$9, 19, 29, 39, 54, 64, 74, 84, 98, 109, 119, 129, 139, 154, 164, 174, 184, \ldots.$$

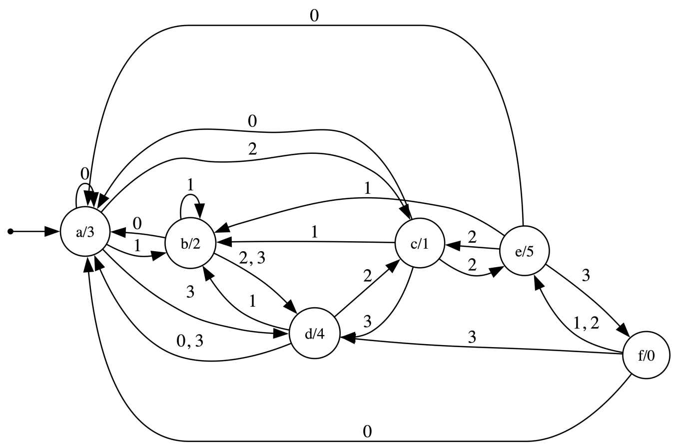


Figure 7: A 4-DFAO for the difference sequence $A_n - 9n - 4$ for the linear form $\mathbf{a} = (1, 3)$.

The subsequence A_{5n+1} is equal to the arithmetic progression $A_1 + 55n$ up until $n = 348$, when $A_{1741} \neq 9 + 55 \cdot 348$. The metallic numbers are well-studied and share many of the properties of the Fibonacci numbers. Surprisingly, proving or disproving the conjecture for the rusty numbers remains a challenge.

The general quadratic recurrence $X_{n+1} = pX_n + qX_{n-1}$ with arbitrary initial values produces the Horadam numbers [5]. We will show that the anti-Horadam numbers have an automatic difference sequence if $p \leq 2$.

Definition 1. A positive linear form $\mathbf{a}$ of dimension k and trace τ is A_1-*bounded* if $A_1 \leq (k-1)\tau + 2$, where A_1 is the first anti-recurrence number in the sequence generated by $\mathbf{a}$.

The linear form $\mathbf{a} = (a_1, a_2)$ of the anti-Horadam numbers is A_1-bounded if $a_2 \leq 2$.

Lemma 4. *Let $\mathbf{a}$ be a form of trace τ and dimension $k > 1$. We have $\tau + t_{k-1} \leq A_1$, and the inequality is strict if $k > 2$ and the sequence A_n is not anti-bonacci.*

Proof. The inequality follows from $A_1 = \sum_{j=1}^{k} j a_j = \tau + \sum_{j=1}^{k} (j-1) a_j \geq \tau + t_{k-1}$. This inequality is strict if $k > 2$ and if one of the a_j's is greater than 1. □

Lemma 5. *Let $\mathbf{a}$ be an A_1-bounded linear form. Then $A_n \in I_n = [\kappa(n-1)+1, \kappa n]$, with $\kappa = k\tau + 1$. The initial B-block of each I_n is an interval.*

Proof. By induction. For A_1 we need to prove that $k \leq A_1 \leq \kappa$. The left-hand inequality follows from $A_1 \geq \tau + t_{k-1} \geq k + t_{k-1}$. The right-hand inequality is immediate.

By our inductive assumption, each I_j contains one anti-recurrence and τ blocks. Therefore, each I_j generates τ anti-recurrence numbers, and from our inductive hypothesis we can generate $n\tau$ recurrence numbers. We only need to check A_{n+1}. The initial block of I_j is an interval which generates $A_{(j-1)\tau+1} = A_1 + (j-1)\kappa$. This gives the familiar arithmetic progression.

If B-blocks are consecutive intervals, then they generate A_h and A_{h+1} such that $A_{h+1} - A_h = \kappa - 1$ by Lemma 1. Modulo κ, the next number A_{h+1} reduces by one. Since there is only one anti-recurrence number, at least $\tau - 1$ of the blocks are intervals, and at least $\tau - 2$ of these are consecutive to a preceding block that is an interval. There is one B-block that is either not an interval, or not consecutive to a preceding block. The latter happens if the anti-recurrence number is between two B-blocks. In that case, $A_{h+1} - A_h = (k+1)\tau = \kappa + \tau - 1$. There are at most $\tau - 1$ reductions by one for the τ anti-recurrence numbers that are generated by I_j. There are at most 2 increases. The interval I_j generates τ anti-recurrences, which have $\tau - 1$ differences. If we include the first anti-recurrence of I_{j+1}, then we get τ differences. The total sum of these differences is zero, since $A_{j\tau+1} \equiv A_1 \pmod{\kappa}$. It follows that each anti-recurrence A_h that is generated by I_j is in the range $[A_1 - \tau + 1, A_1 + \tau - 1]$ modulo κ. By Lemma 4 we have that $A_1 - \tau + 1 \geq t_{k-1} + 1 \geq k$. By A_1-boundedness we have that $A_1 + \tau - 1 \leq k\tau + 1 = \kappa$. The numbers A_h that are generated by I_j are contained in I_h and are not in its initial interval of length k. In particular, A_{n+1} meets the required conditions. □

It follows from the proof of this lemma that the subsequence $A_{n\tau+1}$ is an arithmetic progression if $\mathbf{a}$ is A_1-bounded.

Theorem 4. *If* $\mathbf{a}$ *is* A_1*-bounded, then it generates an anti-recurrence sequence* A_n *such that* $A_n - \kappa n$ *is* τ*-automatic.*

Proof. In the proof of Lemma 5 we saw that I_j generates τ anti-recurrence numbers. This process depends only on the value of $A_j \pmod{\kappa}$. Furthermore, we found that the anti-recurrence numbers are all in $[A_1 - \tau + 1, A_1 + \tau - 1]$ if we compute modulo κ. Therefore, there are only $2\tau - 1$ possible values. We have a uniform substitution of length τ on an alphabet of size $2\tau - 1$. By Cobham's little theorem, the difference sequence $A_n - \kappa n$ is τ-automatic. It can be recognized by a DFAO with at most $2\tau - 1$ states. □

6. Final Remarks

We have shown that a specific class of anti-recurrence sequences are the sums of a linear sequence and an automatic sequence. Much remains to be explored, most notably extending Theorem 4 to general anti-recurrence sequences. Does the conjecture hold without the restriction of A_1-boundedness? Are the rusty numbers sums of linear sequences and automatic sequences? There is the more general class of complementary sequences that goes back to Fraenkel [4]. Is it possible to single out complementary sequences that are sums of linear sequences and automatic sequences?

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