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ON GENERALIZING THE VAN DER WAERDEN THEOREM TO SOME SYMMETRIC FUNCTIONS

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Abstract

Let n, m be positive integers and $c \in \mathbb{Z}_n$, where $\mathbb{Z}_n$ is the ring of integers modulo n. We address the following problem, partially solved by N. Alon. Does an infinite sequence over $\mathbb{Z}_n$ contain m same-length consecutive blocks $B_1, \ldots, B_m$ such that $\sum B_j + c \prod B_j = 0$ for every $j = 1, \ldots, m$ (where $\sum B$ and $\prod B$ denote, respectively, the sum and the product of the elements in block B)? In the case of $c = 0$, this problem is equivalent to the van der Waerden theorem. We provide an almost complete answer to the above problem, excluding only the case of square-free n and $c = -1$. After investigating $B \mapsto \sum B + c \prod B$, we provide related examples of generalizing the Van der Waerden theorem to symmetric functions.

1. Background

In 1927, van der Waerden proved a seminal theorem stating that any finite coloring of the integers contains a monochromatic arithmetic progression of arbitrary length [15]. Subsequent generalizations established that this phenomenon persists in far broader contexts, extending to the fields of number theory, logic, algebra, analysis, and computer science. Below we list a few well-known generalizations of the van der Waerden theorem.

- Hales and Jewett showed that any finite coloring of a sufficiently high-dimensional combinatorial cube contains monochromatic combinational lines [9].

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- Gallai and Witt extended the van der Waerden theorem by showing that any finite coloring of the integer lattice $\mathbb{Z}^d$ contains a monochromatic affine copy of every finite point configuration [16, 12].
- Rado characterized all linear equations that have monochromatic solutions in every finite coloring [11].
- In the density setting, Szemerédi proved that any subset of the integers with positive upper density must contain arbitrarily long arithmetic progressions [13], a result that answered the Erdös–Turán conjecture and can be seen as a density version of van der Waerden's theorem. Katznelson and Furstenberg provided an additional proof of the Szemerédi theorem using ergodic theory [7].
- Erdös and Graham later established the canonical van der Waerden theorem by proving that any sufficiently large finite coloring of $\mathbb{N}$ yields either a monochromatic or a rainbow arithmetic progression of arbitrary length [6].
- Bergelson and Leibman proved a polynomial generalization of the van der Waerden theorem by replacing a linear polynomial that defines an arithmetic progression with a polynomial of arbitrary degree [2].

In addition, a dispersed collection of other generalizations recently appeared in Chapters 2–7 of [10], too varied to enumerate here.

2. Introduction

This work considers the zero-sum formulation of the van der Waerden theorem and generalizes it to various symmetric functions. The problems addressed in our paper lie on the intersection of combinatorics on words and Ramsey theory and stem from two primary sources.

- The first one is the classical work of Thue [14],[3] who proved the existence of an infinite sequence over a 3-letter alphabet with no identical consecutive blocks. This seminal paper has developed into a broad theory of combinatorics on words. One notable direction in this field involves assuming an algebraic structure on the alphabet, particularly the ring of integers modulo n [1],[5].
- The second source is the zero-sum Ramsey theory on the integers, addressed in Chapter 10 of [10]. In essence, the colors in the traditional Ramsey theory are replaced with the elements of $\mathbb{Z}_n$, and the notion of monochromatic is replaced with the notion of zero-sum. As will be seen below, the problems addressed in our paper relate to the classical van der Waerden theorem [8].

Let n be a positive integer and let $\mathbb{Z}_n$ be the ring of integers modulo n. Consider an arbitrary sequence $A = \{a_k\}_{k=1}^{\infty}$ over $\mathbb{Z}_n$. A *block* of length l consists of $l > 1$ consecutive elements from A, relabeled and reindexed as b_k's and denoted by $B = (b_{s+1}, \ldots, b_{s+l}) \in \mathbb{Z}_n^l$. Furthermore, we say that blocks $B_1, \ldots, B_m$ are *consecutive* if the first element of block B_{j+1} follows the last element of block B_j for all $j = 1, \ldots, m-1$. Next, consider a family $\mathcal{F} = \{f^{(l)}\}_{l=2}^{\infty}$ where each $f^{(l)} : \mathbb{Z}_n^l \to \mathbb{Z}_n$ is a function in l variables. For a positive integer m, we say that $\mathcal{F}$ is m-vanishing if for all sequence $A = \{a_k\}_{k=1}^{\infty}$ over $\mathbb{Z}_n$ there exist integer $l > 1$ and m consecutive blocks $B_1, \ldots, B_m$ each of length l such that $f^{(l)}(B_1) = \ldots = f^{(l)}(B_m) = 0$. Finally, we say that $\mathcal{F}$ is *vanishing* if it is m-vanishing for all positive integer m.

For the sake of simplicity, we will denote the sum and the product over $\mathbb{Z}_n$ of all the elements in a block B as, respectively, $\sum B$ and $\prod B$. Furthermore, we will use $\mathbb{Z}_n$ arithmetic throughout the paper unless stated otherwise.

In this paper, we investigate the vanishing property of the family

$$\mathcal{F}_c = \{(b_1, \ldots, b_l) \mapsto \sum_{i=1}^{l} b_i + c \prod_{i=1}^{l} b_i : l = 2, 3, \ldots\}$$

for all $c \in \mathbb{Z}_n$. It has been motivated by the following Theorem 1, which is equivalent to the van der Waerden theorem (consider an auxiliary sequence $A' = \{a'_k\}_{k=1}^{\infty}$ with $a'_k = \sum_{i=1}^{k} a_i$ and realize that a monochromatic arithmetic progression of length m in A' is equivalent to $m-1$ consecutive zero-sum blocks in A, see, e.g., Theorem 4 in [1]).

Theorem 1. *Let n be a positive integer. Then the family*

$$\mathcal{F} = \{(b_1, \ldots, b_l) \mapsto \sum_{i=1}^{l} b_i : l = 2, 3, \ldots\}$$

is vanishing.

Considering $\mathcal{F}_c$ is a particular case of a wider area of investigation of elementary symmetric polynomials appearing in [4]. As it can be seen from Theorem 2 below, we only need to investigate $\mathcal{F}_c$ for $c \in \{1, -1\}$.

Theorem 2. *If $n > 1$, then $\mathcal{F}_c$ is not m-vanishing for any $c \in \mathbb{Z}_n \setminus \{0, 1, -1\}$ and $m > 0$.*

Proof. Consider sequence $A = -1, 1, -1, 1, \ldots$ over $\mathbb{Z}_n$ and an arbitrary block B within it. Trivially, $\sum B \in \{0, 1, -1\}$ and $\prod B \in \{1, -1\}$. Then $\sum B + c \prod B = 0$ implies either $0 = (-1)^k c$ or $(-1)^k = c$, and either way we arrive at a contradiction. □

In Sections 3 and 4 we consider the cases of $c = 1$ and $c = -1$, respectively. In Section 5 we provide additional examples of generalizing Theorem 1, i.e., the zero-sum formulation of the van der Waerden theorem.

3. The Case of $c = 1$

The main result of this section is Theorem 3 below which gives a complete classification of the m-vanishing property of $\mathcal{F}_1$. The proof follows from several propositions provided further below, and Theorems 4 and 5 from Section 4.

Theorem 3. *Let $n > 1$.*

(a) If $n \notin \{2, 3, 4, 6, 8\}$, then $\mathcal{F}_1$ is not m-vanishing for any $m > 0$.

(b) If $n \in \{2, 3, 4, 8\}$, then $\mathcal{F}_1$ is vanishing.

(c) If $n = 6$, then $\mathcal{F}_1$ is 1-vanishing but not m-vanishing for any $m > 1$.

Proof. Part (c) is a result of combining Propositions 3 and 4, and part (b) follows from Theorem 4 for the case of $n = 2$, Proposition 1 for the case of $n = 3$, and Proposition 2 for the case of $n \in \{4, 8\}$.

Let $n = p_1 \dots p_k$ be the decomposition of n into prime factors (not necessarily distinct). If $p_i > 3$ for some $i = 1, \dots, k$, then either by Proposition 5 or Proposition 7 there exists an infinite sequence A over $\mathbb{Z}_{p_i}$ that does not contain any block B satisfying $\sum B + \prod B \equiv 0 \bmod p_i$. Viewing A as a sequence over $\mathbb{Z}_n$ then implies that it cannot contain a block B satisfying $\sum B + \prod B \equiv 0 \bmod n$.

To complete the proof of part (a), it remains to consider the case of $n = 2^h 3^{k-h} > 8$ for $0 \leq h \leq k$. It follows that $k \geq 2$, and the subcases of $h = 0$ and $h = k$ are handled by Propositions 9 and 8, respectively. Finally, the proof of the subcase of $0 < h < k$, which implies $k \geq 3$, is given by Theorem 5. □

Lemma 1. *If $n = 3$, then $\mathcal{F}_1$ is vanishing.*

Proof. We will follow the idea of Noga Alon (see Theorem 3.5 (a) in [4]). By the van der Waerden theorem, there exists a positive integer w such that every 3^{m+1}-coloring of $1, 2, \dots, w$ has a monochromatic arithmetic progression of length $m+1$. We consider two cases.

`Case 1:` every block of length w contains a 0. Because the van der Waerden theorem guarantees the existence of m consecutive zero-sum blocks $B_1, \dots, B_m$ each of length $l \geq w$, it must be that $0 = \prod B_j = \sum B_j$ for all $j = 1, \dots, m$.

`Case 2:` there exists a block of length w with no 0's, $B = (b_1, \dots, b_w) \in \{1, 2\}^w$. Consider a 3^{m+1}-coloring

$$\chi(k) = \left(\sum_{i=1}^{k} b_i, \prod_{i=1}^{k} b_i, \begin{cases} b_{k-1} & \text{if } k > 1 \\ 0 & \text{otherwise} \end{cases}, \dots, \begin{cases} b_{k-m+1} & \text{if } k > m-1 \\ 0 & \text{otherwise} \end{cases} \right)$$

for all $k = 1, \dots, w$, induced by B. Let $\chi(s) = \chi(s+l) = \ldots = \chi(s+ml)$ describe its monochromatic arithmetic progression of length $m+1$, so in particular the blocks

$B_j = (b_{s+(j-1)l+1}, \ldots, b_{s+jl})$ satisfy $\sum B_j = 0$ and $\prod B_j = 1$ for all $j = 1, \ldots, m$. Because shifting the j-th block left by $j-1$ for $j = 1, \ldots, m$ does not change its elements due to the identity

$$b_{s+jl-t} = b_{s+(j-1)l-t} \text{ for all } t = 1, \ldots, j-1,$$

the blocks $\overline{B}_j = (b_{s+(j-1)l-j+2}, \ldots, b_{s+jl-j+1})$ each of length l must also each sum to 0 and multiply to 1. Note that this entails $l \geq 3$, and observe that every non-zero residue in $\mathbb{Z}_3$ is its own multiplicative inverse. It follows that the consecutive blocks $\overline{B}_j^- = (b_{s+(j-1)l-j+2}, \ldots, b_{s+jl-j})$ each of length $l-1$, obtained by removing the right-most element from $\overline{B}_j$, satisfy $\sum \overline{B}_j^- = -\prod \overline{B}_j^-$ for all $j = 1, \ldots, m$. □

Lemma 2. *If $n \in \{4, 8\}$, then $\mathcal{F}_1$ is vanishing.*

Proof. By the van der Waerden theorem, there exists a positive integer w such that every n^{m+2}-coloring of $1, 2, \ldots, w$ has a monochromatic arithmetic progression of length $m+1$. We consider two cases.

`Case 1:` every block of length w multiplies to 0. Because the van der Waerden theorem guarantees the existence of m consecutive zero-sum blocks $B_1, \ldots, B_m$ each of length $l \geq w$, it must be that $0 = \prod B_j = \sum B_j$ for all $j = 1, \ldots, m$.

`Case 2:` there exists a block $B = (b_1, \ldots b_w) \in \{1, \ldots, n-1\}^w$ such that $\prod B \neq 0$. Define $b'_k = \begin{cases} b_k & \text{if } b_k \text{ is odd} \\ 1 & \text{otherwise} \end{cases}$ for all $k = 1, \ldots, w$ and consider a n^{m+2}-coloring

$$\chi(k) = \left(\sum_{i=1}^{k} b_i, \prod_{i=1}^{k} b_i, \prod_{i=1}^{k} b'_i, \begin{cases} b_{k-1} & \text{if } k > 1 \\ 0 & \text{otherwise} \end{cases}, \ldots, \begin{cases} b_{k-m+1} & \text{if } k > m-1 \\ 0 & \text{otherwise} \end{cases} \right)$$

for all $k = 1, \ldots, w$, induced by B. Let $\chi(s) = \chi(s+l) = \ldots = \chi(s+ml)$ describe its monochromatic arithmetic progression of length $m+1$, so in particular the blocks $B_j = (b_{s+(j-1)l+1}, \ldots, b_{s+jl})$ satisfy $\sum B_j = 0$ for all $j = 1, \ldots, m$. Because $\prod_{i=1}^{s} b_i = \prod_{i=1}^{s+ml} b_i$ and the order of 2 in the prime factorization of $\prod_{i=1}^{k} b_i$ is non-decreasing in k, it must be that b_k is odd for $s < k \leq s+ml$, which implies $\prod B_j = \prod_{k=s+(j-1)l+1}^{s+jl} b'_k = 1$ for all $j = 1, \ldots, m$. Moreover, because shifting the j-th block left by $j-1$ for $j = 1, \ldots, m$ does not change its elements due to the identity

$$b_{s+jl-t} = b_{s+(j-1)l-t} \text{ for all } t = 1, \ldots, j-1,$$

the blocks $\overline{B}_j = (b_{s+(j-1)l-j+2}, \ldots, b_{s+jl-j+1})$ each of length l must also each sum to 0 and multiply to 1. Note that this entails $l \geq 3$, and observe that every odd residue in $\mathbb{Z}_n$ is its own multiplicative inverse. It follows that the consecutive blocks $\overline{B}_j^- = (b_{s+(j-1)l-j+2}, \ldots, b_{s+jl-j})$ each of length $l-1$, obtained by removing the right-most element from $\overline{B}_j$, satisfy $\sum \overline{B}_j^- = -\prod \overline{B}_j^-$ for all $j = 1, \ldots, m$. □

Lemma 3. *If $n = 6$, then $\mathcal{F}_1$ is 1-vanishing.*

Proof. Assume that some A does not contain a block whose sum and product add to 0, which implies that it contains neither $(1,5)$ nor $(5,1)$. By the van der Waerden theorem, there exists a positive integer w such that every 6^5-coloring of $1,\dots,w$ has a monochromatic arithmetic progression of length 2 whose difference is at least 3. By the assumption, a zero-sum block of length at least $w+3$, whose existence is also guaranteed by the van der Waerden theorem, cannot multiply to 0. Let $B = (b_1,\dots,b_{w+3})$ satisfy $\prod B \neq 0$. Then one of the following two scenarios must hold.

`Case 1:` $B \in \{1,3,5\}^{w+3}$. Define $b'_k = \begin{cases} b_k & \text{if } b_k \neq 3 \\ 1 & \text{otherwise} \end{cases}$ for $k = 2,\dots,w+1$ and consider a 6^4-coloring

$$\chi(k) = \left(\sum_{i=2}^{k} b_i, \prod_{i=2}^{k} b'_i, b_k, b_{k+1}\right) \text{ for all } k = 2,\dots,w+1,$$

induced by B. Let $\chi(s) = \chi(s+l)$ describe its monochromatic arithmetic progression of length 2 for some $l \geq 3$, so in particular $\sum_{k=s+1}^{s+l} b_k = 0$ and $\prod_{k=s+1}^{s+l} b'_k = 1$. This implies $3 \in \{b_{s+1},\dots,b_{s+l}\}$ or else $\prod_{k=s+1}^{s+l} b_k = 1$ which violates the assumption by implying $\prod_{k=s+2}^{s+l} b_k = b_{s+1}$, due to the identity $\sum_{k=s+2}^{s+l} b_k = -b_{s+1}$. Therefore, $\prod_{k=s+1}^{s+l} b_k = \prod_{k=s}^{s+l} b_k = \prod_{k=s+1}^{s+l+1} b_k = 3$, which entails $3 \neq b_s = b_{s+l}$ and $3 \neq b_{s+1} = b_{s+l+1}$ or else $(b_s,\dots,b_{s+l})$ or $(b_{s+1},\dots,b_{s+l+1})$ violates the assumption. Because 1 and 5 cannot be neighbors in A, it must be that $b_s = b_{s+1} = b_{s+l} = b_{s+l+1}$. But b_{s+l+2} can be neither b_{s+l+1} (or else $(b_s,\dots,b_{s+l+2})$ violates the assumption) nor 3 (or else $(b_{s+2},\dots,b_{s+l+2})$ does), and we arrive at a contradiction.

`Case 2:` $B \in \{1,2,4,5\}^{w+3}$. Define $b'_k = \begin{cases} b_k & \text{if } b_k \notin \{2,4\} \\ 1 & \text{otherwise} \end{cases}$, $b''_k = \begin{cases} 3 & \text{if } b_k = 2 \\ 0 & \text{otherwise} \end{cases}$ for $k = 2,\dots,w+1$ and consider a 6^5-coloring

$$\chi(k) = \left(\sum_{i=2}^{k} b_i, \prod_{i=2}^{k} b'_i, \sum_{i=2}^{k} b''_i, b_k, b_{k+1}\right) \text{ for all } k = 2,\dots,w+1,$$

induced by B. Let $\chi(s) = \chi(s+l)$ describe its monochromatic arithmetic progression of length 2 for some $l \geq 3$, so in particular $\sum_{k=s+1}^{s+l} b_k = 0$ and $\prod_{k=s+1}^{s+l} b'_k = 1$. Analogously to Case 1, block $(b_{s+1},\dots,b_{s+l})$ contains an even residue, and because its count of 2's is even due to the identity $\sum_{k=s+1}^{s+l} b''_k = 0$, it must be that $\prod_{k=s+1}^{s+l} b_k = 4$. It follows that neither b_{s+1} nor b_{s+l} is an even residue, or else removing it would yield a block violating the assumption. Therefore, $b_s = b_{s+1} = b_{s+l} = b_{s+l+1} \notin \{2,4\}$. But this value can be neither 1 (or else $(b_s,\dots,b_{s+l+1})$

violates the assumption) nor 5 (or else $(b_{s+2}, \dots, b_{s+l-1})$ does), and we arrive at a contradiction. $\square$

Lemma 4. *If $n = 6$ and $m > 1$, then neither $\mathcal{F}_{-1}$ nor $\mathcal{F}_1$ is m-vanishing.*

Proof. Consider $A = 1, 3, 5, 3, 1, 3, 5, 3, \dots$, and assume that it contains consecutive blocks B_1, B_2 each of length l such that $\sum B_j - \prod B_j = 0$ or $\sum B_j + \prod B_j = 0$ for $j = 1, 2$. Because every block of A contains a 3 it must be that $\prod B_j = 3$ and therefore $\sum B_j = 3$ for $j = 1, 2$. It follows that B_1 and B_2 each contain an odd number of 3's and an even number of 1's and 5's combined, so in particular $l = 4k + r$ for $r \in \{1, 3\}$. Then $4 \nmid 2l$ and therefore $2l$ consecutive elements from A must contain an odd number of 3's, which yields a contradiction. $\square$

Lemma 5 (Theorem 3.6 in [4])**.** *If n is a prime satisfying $n \equiv 1 \bmod 4$, then $\mathcal{F}_1$ is not m-vanishing for any $m > 0$.*

Lemma 6. *If $n > 3$ is a prime satisfying $n \equiv 3 \bmod 4$, then there exist $x, y \in \mathbb{Z}_n \setminus \{0\}$ and $r \in \{2, 3\}$ such that $x + ry = 0$ and $xy^r = 1$.*

Proof. One of the following scenarios must hold.

`Case 1:` 4 is a cubic residue. Let $x \in \mathbb{Z}_n$ satisfy $x^3 = 4$, then $y = -\frac{x}{2}$ satisfy $x + 2y = 0$ and $xy^2 = \frac{x^3}{4} = 1$.

`Case 2:` 4 is not a cubic residue. Because every $a \in \mathbb{Z}_n$ is a cubic residue when $n \equiv 2 \bmod 3$, it must be that $n \equiv 1 \bmod 3$, and therefore every $a \in \mathbb{Z}_n$ has either 0 or 3 distinct cubic roots in $\mathbb{Z}_n$. In particular, $0 = a^3 - 1 = (a-1)(a^2+a+1)$ has two solutions besides the unity, at least one of them being a root of $a^2 + a + 1 = 0$. The discriminant of this quadratic polynomial is -3, and therefore there exists $z \in \mathbb{Z}_n$ satisfying $z^2 = -3$. It follows that $x = (3z)^{\frac{n+1}{4}}, y = -\frac{x}{3}$ satisfy $x + 3y = 0$ and $xy^3 = \frac{-(3z)^{n+1}}{27} = 1$. $\square$

Lemma 7. *If $n > 3$ is a prime satisfying $n \equiv 3 \bmod 4$, then $\mathcal{F}_1$ is not m-vanishing for any $m > 0$.*

Proof. One can verify that the sequences $A = 2, 3, 3, 3, 3, 2, 3, 3, 3, 3, \dots$ and $A = 5, 3, 3, 5, 3, 3, \dots$ satisfy the statement of the theorem for, respectively, $n = 7$ and $n = 11$. Therefore, it only remains to consider the case of $n > 11$.

Let $x, y \in \mathbb{Z}_n \setminus \{0\}$ and $r \in \{2, 3\}$ satisfy $x = -ry$ and $ry^{r+1} = -1$ as in Proposition 6, and consider the sequence $A = x, \underbrace{y, \dots, y}_{r \text{ times}}, x, \underbrace{y, \dots, y}_{r \text{ times}}, \dots$. Any block B must then satisfy $\sum B = sx + ty$ and $\prod B = x^s y^t$ for some $s \in \{0, 1\}, t \in \{0, \dots, r\}$ such that $s + t < 1 + r$. Assume some B satisfies $\sum B = -\prod B$, which is equivalent to $(rs - t)y = (-r)^s y^{s+t}$. It immediately follows that $(s, t) \notin \{(0, 0), (0, 1), (1, 0)\}$, and the remaining cases are examined below.

`Case 1:` $s = 0, t = 2$. Then $-2y = y^2$, which implies $y = 2$, and therefore $-1 = ry^{r+1} \in \{32, 48\}$. It follows that $n \mid 33$ or $n \mid 49$.

`Case 2:` $s = 0, t = r = 3$. Then $-3y = y^3$, which implies $y^2 = -3$, and therefore $-1 = 3y^4 = 27$. It follows that $n \mid 28$.

`Case 3:` $s = 1, t = 1$. Then $(r-1)y = -ry^2$, which implies $y = (1-r)r^{-1}$ and therefore $-1 = (1-r)^{r+1}r^{-r} \in \{-2^{-2}, 16 \cdot 3^{-3}\}$. It follows that $n \mid 5$ or $n \mid 25$.

`Case 4:` $s = 1, t = 2, r = 3$. Then $y = -3y^3$, which implies $y^2 = -3^{-1}$ and therefore $-1 = 3y^4 = 3^{-1}$. It follows that $n \mid 4$.

Because the prime divisors of $33, 49, 28, 5, 25, 4$ are at most 11, we arrive at a contradiction. □

Lemma 8. *If $n = 8u$ for $u > 1$, then $\mathcal{F}_1$ is not m-vanishing for any $m > 0$.*

Proof. Consider the sequence $A = 3, -3, 3, -3, \ldots$ and assume that some block B of length l satisfies $\sum B + \prod B = 0$. We consider three cases.

`Case 1:` l is even. Then $\sum B = 0$ but $0 \neq \prod B \in \{\pm 3^k : k = 1, 2, \ldots\}$, which yields a contradiction.

`Case 2:` $l = 4r + 3$ for some integer $r \geq 0$. Then either $\sum B = 3, \prod B = -3^{4r+3}$ or $\sum B = -3, \prod B = 3^{4r+3}$, so $0 = \sum B + \prod B$ entails

$$0 = 3(3^{4r+2} - 1) = 3^{4r+2} - 1 = 2(3^{4r+1} + 3^{4r} + \ldots + 1)$$

and therefore $4u \mid 3^{4r+1} + 3^{4r} + \ldots + 1$. Notice that $u > 1$ entails $r \neq 0$ due to the fact that $4u \nmid 4$. Because $3^{2h} \equiv 1 \bmod 4$ and $3^{2h+1} \equiv 3 \bmod 4$ for any integer $h \geq 0$, it must be that $3^{4r+1} + 3^{4r} + \ldots + 1 \equiv 3^{4r+1} \equiv 3 \bmod 4$, which yields a contradiction.

`Case 3:` $l = 4r + 1$ for some integer $r > 0$. Then either $\sum B = 3, \prod B = 3^{4r+1}$ or $\sum B = -3, \prod B = -3^{4r+1}$, so $0 = \sum B + \prod B$ entails $0 = 3(3^{4r} + 1) = 3^{4r} + 1$ which yields a contradiction because $3^{4r} \equiv 1 \bmod 4$. □

Lemma 9. *If $n = 9u$ for $u > 0$, then $\mathcal{F}_1$ is not m-vanishing for any $m > 0$.*

Proof. The proof will assume the context of $\mathbb{Z}_9$ arithmetic unless stated otherwise. Consider the sequence $A = 7, 4, 4, 7, 4, 4, \ldots$. Any block B in this sequence is comprised of $r + t$ 7's and $2r + s$ 4's for some $r = 0, 1, \ldots$, $t \in \{0, 1\}$, and $s \in \{0, 1, 2\}$ such that $t + s < 3$, and therefore $\sum B = 6r + 4s + 7t \equiv s + t \bmod 3$ and $\prod B = 4^{r+s}7^t \equiv 1 \bmod 3$. Assume that some block B satisfies $\sum B + \prod B = 0$, so in particular $\sum B + \prod B \equiv 0 \bmod 3$ which implies $s + t \equiv 2 \bmod 3$. We consider two cases.

`Case 1:` $s = t = 1$. Then $\sum B + \prod B = 6r + 2 + 4^{r+1}7 = 3$, which yields a contradiction.

`Case 2:` $s = 2, t = 0$. Then $\sum B + \prod B = 6r + 8 + 4^{r+2} = 6$, which yields a contradiction. Because A does not contain any block B satisfying $\sum B + \prod B = 0$, it also cannot contain a block B such that $\sum B + \prod B \equiv 0 \bmod n$. □

4. The Case of $c = -1$

In this section we provide a classification of the m-vanishing property of $\mathcal{F}_{-1}$ for all n, excluding only the case where n is square-free.

Theorem 4. *If $n = p^k$ for a prime p and $k > 0$, then $\mathcal{F}_{-1}$ is vanishing.*

Proof. By the van der Waerden theorem, there exists a positive integer w such that every n^{m+2}-coloring of $1, 2, \dots, w$ has a monochromatic arithmetic progression of length $m + 1$. We consider two cases.

`Case 1:` every block of length w in the sequence multiplies to 0. Because the van der Waerden theorem guarantees the existence of m consecutive zero-sum blocks $B_1, \dots, B_m$ each of length $l \geq w$, it must be that $0 = \prod B_j = \sum B_j$ for all $j = 1, \dots, m$.

`Case 2:` there exists a block $B = (b_1, \dots b_w) \in \{1, \dots, n-1\}^w$ such that $\prod B \neq 0$. Define $b'_k = \begin{cases} b_k & \text{if } p \nmid b_k \\ 1 & \text{otherwise} \end{cases}$ for $k = 1, \dots, w$ and consider a n^{m+2}-coloring

$$\chi(k) = \left(\sum_{i=1}^{k} b_i, \prod_{i=1}^{k} b_i, \prod_{i=1}^{k} b'_i, b_{k+1}, \dots, b_{k+m-1} \right) \text{ for all } k = 1, \dots, w,$$

induced by B. Let $\chi(s) = \chi(s + l) = \dots = \chi(s + ml)$ describe its monochromatic arithmetic progression of length $m + 1$, so in particular the blocks $B_j = (b_{s+(j-1)l+1}, \dots, b_{s+jl})$ satisfy $\sum B_j = 0$ for all $j = 1, \dots, m$. Because $\prod_{i=1}^{s} b_i = \prod_{i=1}^{s+ml} b_i$ and the order of p in the prime factorization of $\prod_{i=1}^{k} b_i$ is non-decreasing in k, it must be that $p \nmid b_k$ for $s < k \leq s + md$ and therefore

$$\prod B_j = \prod_{i=s+(j-1)l+1}^{s+jl} b'_i = 1 \text{ for all } j = 1, \dots, m.$$

Moreover, because shifting the j-th block right by $j - 1$ for $j = 1, \dots, m$ does not change its elements due to the identity

$$b_{s+(j-1)l+t} = b_{s+jd+t} \text{ for all } t = 1, \dots, j-1,$$

the blocks $\overline{B}_j = (b_{s+(j-1)l+j}, \dots, b_{s+jl+j-1})$ must also each sum to 0 and multiply to 1. It follows that the consecutive blocks $\overline{B}_j^+ = (b_{s+(j-1)l+j}, \dots, b_{s+jl+j})$ each of length $l+1$, obtained via extending $\overline{B}_j$ by one element on the right, satisfy $\sum \overline{B}_j^+ = \prod \overline{B}_j^+$ for all $j = 1, \ldots, m$. □

Theorem 5. *If n satisfies $pq^2 \mid n$ for distinct primes p, q, then neither $\mathcal{F}_1$ nor $\mathcal{F}_{-1}$ is m-vanishing for any $m > 0$.*

Proof. We repurpose the proof of Theorem 3.5 (b) in [4] as follows. Consider the sequence $A = q, -q, q, -q, \ldots$ and an arbitrary block B within it. Trivially, $\sum B \in \{0, q, -q\}$ and $\prod B \in \{\pm q^k : k = 2, 3, \ldots\}$. Assume $\sum B + c \prod B = 0$ for some $c \in \{1, -1\}$, which is equivalent to $n \mid \sum B + c \prod B$. It follows that $\sum B \neq 0$, or otherwise $\sum B + c \prod B \in \{q^k, -q^k\}$ for some integer $k > 1$, which yields a contradiction because $p \nmid q^k$ implies that $n \nmid q^k$. But $\sum B \in \{q, -q\}$ would entail $\sum B + c \prod B \in \{q+q^k, -q+q^k, q-q^k, -q-q^k\}$ for some integer $k > 1$, and therefore $n \mid q(q^{k-1} - 1)$ or $n \mid q(q^{k-1} + 1)$. However, neither $q \nmid q^{k-1} - 1$ nor $q \nmid q^{k-1} + 1$, which contradiction concludes the proof. □

5. Multivariate Generalization of the van der Waerden Theorem

The family of sums $\mathcal{F} = \{(b_1, \ldots, b_l) \mapsto \sum_{i=1}^{l} b_i : l = 2, 3, \ldots\}$ can be viewed as an instance of families of *transformation sums*,

$$\mathcal{F}_g = \{(b_1, \ldots, b_l) \mapsto \sum_{i=1}^{l} g(b_i) : l = 2, 3, \ldots\},$$

in which the *transformation* $g : \mathbb{Z}_n \to \mathbb{Z}_n$ is taken as the identity. Another common example of a transformation is $x \mapsto x^r$ for some integer $r > 1$, which yields the family of power sums of degree r.

Note that the m-vanishing property is naturally extended to families of vector-valued functions $f^{(l)} : \mathbb{Z}_n \to \mathbb{Z}_n^d$ for $l = 2, 3, \ldots$ by replacing the scalar zero in $f^{(l)}(B_1) = \ldots = f^{(l)}(B_m) = 0$ with its vector counterpart $\mathbf{0} \in \mathbb{Z}^d$, i.e., by requiring all d components of $f^{(l)}$ to simultaneously attain 0 on each of the m consecutive blocks. This allows a further generalization of the families of transformation sums by considering multiple transformations at once:

$$\mathcal{F}_{(g_1, \ldots, g_d)} = \left\{ (b_1, \ldots, b_l) \mapsto \left(\sum_{i=1}^{l} g_1(b_i), \ldots, \sum_{i=1}^{l} g_d(b_i) \right) : l = 2, 3, \ldots \right\}.$$

In the following simple result, we show that any function whose finite-dimensional value is comprised of the transformation sums yields a vanishing family:

Theorem 6. *Let n and d be positive integers and $g_i : \mathbb{Z}_n \to \mathbb{Z}_n$ for $i = 1, \ldots, d$. Then the family $\mathcal{F}_{(g_1, \ldots, g_d)}$ is vanishing.*

Proof. Consider an n^d-coloring

$$\chi(k) = \left(\sum_{i=1}^{l} g_1(b_i), \ldots, \sum_{i=1}^{l} g_d(b_i)\right) \text{ for all } k = 1, 2 \ldots,$$

induced by A. Let $\chi(s) = \chi(s+l) = \ldots = \chi(s+ml)$ describe its monochromatic arithmetic progression of length $m+1$ for some $l \geq 2$. Because the blocks $B_j = (b_{s+(j-1)l+1}, \ldots, b_{s+jl})$ satisfy $\sum_{k=1}^{l} g_i(b_{s+(j-1)l+k}) = 0$ for all $j = 1, \ldots, m$ and $i = 1, \ldots, d$, the result immediately follows. □

In particular, choosing any integer $r > 0$ and applying Theorem 6 for $d = r$ and $g_i = x \mapsto x^i$ for $i = 1, \ldots, d$ implies that the family of the combined first r power sums

$$\mathcal{F}_{\text{powers},r} = \left\{(b_1, \ldots, b_l) \mapsto \left(\sum_{i=1}^{l} b_i, \sum_{i=1}^{l} b_i^2, \ldots, \sum_{i=1}^{l} b_i^r\right) : l = 2, 3, \ldots\right\}$$

is vanishing. Because the elementary symmetric polynomial of degree r can be expressed as a polynomial in the first r power sums with no constant term via the Newton identities, it follows that the family of elementary symmetric polynomials of degree r

$$\mathcal{F}_{\text{elem},r} = \left\{(b_1, \ldots, b_l) \mapsto \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq l} b_{i_1} b_{i_2} \ldots b_{i_r} : l = 2, 3, \ldots\right\}$$

is also vanishing. This presents a simpler proof of Theorem 3.4 from [4].

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