SOLVABILITY OF $\binom{2K}{K} = \binom{2A}{A}\binom{X+2B}{B}$

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Abstract

Suppose k, x, and b are positive integers, and a is a nonnegative integer such that $k = a + b$. In this paper, we will prove $\binom{2k}{k} = \binom{2a}{a}\binom{x+2b}{b}$ if and only if $x = a = 1$. We do this by looking at different cases depending on the values of x and k. We use various techniques to prove the cases, such as direct proof, verification through Maple software, and a proof technique found in Moser's paper. Previous results from Hanson, Stănică, Shanta, Nair, and Shorey are also used.

1. Introduction

It was discovered by Moser in [4] that the equation

$$\binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$$

has no solutions. This result was further extended by P. Erdős in [1], where he proved that

$$\binom{2m}{m} \nmid \binom{2n}{n}$$

for $2m > n$. Following the line investigated by Moser in [4] and Erdős in [1], the purpose of this paper is to prove the following result.

Theorem 1. *Let k, x, b be positive integers and a be a nonnegative integer such that $k = a + b$. Then*

$$\binom{2k}{k} = \binom{2a}{a}\binom{x+2b}{b} \quad \textit{if and only if} \quad x = a = 1.$$

To prove this result, we need to overcome the difficulty that integers a and b are no longer symmetric in the equation $\binom{2k}{k} = \binom{2a}{a}\binom{x+2b}{b}$, unlike the case discussed in

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[1] and [4]. The key tool used in our proof is an analysis of the existence of prime numbers in the product of consecutive integers, which was extensively investigated in [3] and [5].

2. Proof of Main Result

To prove our main result, we will break the proof up into different cases. We will state these cases as propositions, and prove them throughout the paper by proving smaller lemmas. First, we will prove the result when $x \geq a$.

Proposition 1. *Let k, x, b be positive integers and a be a nonnegative integer such that $k = a + b$. Assume that $x \geq a$. Then*

$$\binom{2k}{k} \leq \binom{2a}{a}\binom{x+2b}{b}.$$

Moreover, the equality holds if and only if $x = a = 1$.

Proof. Note that

$$\begin{aligned}
\frac{\binom{2a}{a}\binom{a+2b}{b}}{\binom{2k}{k}} &= \frac{(2a)!}{a! \cdot a!} \cdot \frac{(a+2b)!}{b!(a+b)!} \cdot \frac{k! \cdot k!}{(2k)!} \\
&= \frac{(2a)(2a-1)\cdots(a+1)}{a(a-1)\cdots 1} \cdot \frac{k(k-1)\cdots(b+1)}{(2k)(2k-1)\cdots(a+2b+1)} \\
&= 2 \cdot \frac{2a-1}{a-1}\frac{2a-2}{a-2}\cdots\frac{a+1}{1} \cdot \frac{1}{2} \cdot \frac{k-1}{2k-1}\frac{k-2}{2k-2}\cdots\frac{k-a+1}{2k-a+1} \\
&= \left(\frac{2a-1}{a-1}\frac{k-1}{2k-1}\right)\left(\frac{2a-2}{a-2}\frac{k-2}{2k-2}\right)\cdots\left(\frac{a+1}{1}\frac{k-a+1}{2k-a+1}\right) \\
&\begin{cases} > 1 & \text{for } 1 < a < k \\ = 1 & \text{for } a = 0, 1 \end{cases}
\end{aligned}$$

Now, if $x > a$, then

$$\binom{2a}{a}\binom{x+2b}{b} > \binom{2a}{a}\binom{a+2b}{b} \geq \binom{2k}{k}.$$

Furthermore, if $x = a > 1$, then

$$\binom{2a}{a}\binom{x+2b}{b} = \binom{2a}{a}\binom{a+2b}{b} > \binom{2k}{k}.$$

When $x = a = 1$, we have

$$\binom{2a}{a}\binom{x+2b}{b} = \binom{2a}{a}\binom{a+2b}{b} = \binom{2}{1}\binom{1+2b}{b} = \binom{2k}{k}.$$

□

Next, we will prove the result for $1 \le k \le 10$.

Lemma 1. *Let k, x, a, b be positive integers such that $k = a + b$. Assume that $1 \le k \le 10$. Then*

$$\binom{2k}{k} = \binom{2a}{a}\binom{x+2b}{b}$$

if and only if $x = a = 1$.

Proof. By Proposition 1, we may assume that $x \le a$. All cases for $1 \le k \le 10$ and $x \le a$ are verified by direct computation with Maple software. $\square$

In the following three lemmas, we will prove the result for $a - x = 1, 2, 3$ when $4 < 2a \le k$. We will fully prove Lemma 2, and omit the proofs of Lemma 3 and Lemma 4 since they are similar.

Lemma 2. *Let k, x, a, b be positive integers such that $k = a + b$. Then*

$$\binom{2k}{k} \ne \binom{2a}{a}\binom{a+2b-1}{b}, \qquad \text{for } \ 4 < 2a \le k.$$

Proof. Let k, x, a, b be positive integers such that $k = a + b$. Assume that

$$\binom{2k}{k} = \binom{2a}{a}\binom{a+2b-1}{b} \quad \text{for } 4 < 2a \le k. \tag{1}$$

So,

$$\frac{(2k)!}{(k!)^2} = \frac{(2a)!}{(a!)^2} \cdot \frac{(2k-a-1)!}{(k-a)!(k-1)!}.$$

Thus,

$$\frac{2k(2k-1)(2k-2)\cdots(2k-a)}{k \cdot k \cdot (k-1)\cdots(k-a+1)} = \frac{(2a)(2a-1)\cdots(a+2)(a+1)}{a(a-1)\cdots 2\cdot 1}.$$

We have, for $k > a$,

$$\frac{2k-1}{k} < 2, \ \frac{2k-2}{k-1} < \frac{2a-1}{a-1}, \ \frac{2k-3}{k-2} < \frac{2a-2}{a-2}, \cdots, \frac{2k-a+1}{k-a+2} < \frac{a+2}{2}$$

and

$$2 \cdot \frac{2k-a}{k-a+1} = 2\Big(2 + \frac{a-2}{k-a+1}\Big) < 6 \le a+1 \qquad \text{for } a \ge 5.$$

Thus, if $a \ge 5$, then

$$\binom{2k}{k} < \binom{2a}{a}\binom{a+2b-1}{b} \qquad \text{for } k = a + b.$$

When $a = 4$, the assumption (1) becomes

$$\binom{2k}{k} = \binom{8}{4}\binom{2k-5}{k-4}, \qquad \text{for } k \geq 8,$$

or

$$\frac{(2k)(2k-1)(2k-2)(2k-3)(2k-4)}{k \cdot k(k-1)(k-2)(k-3)} = \binom{8}{4}.$$

A simple computation shows that

$$8(2k-1)(2k-3) = 70k(k-3),$$

which has a unique integer solution $k = 4$. Thus

$$\binom{2k}{k} \neq \binom{8}{4}\binom{2k-5}{k-4}, \qquad \text{for } k \geq 8.$$

When $a = 3$, the assumption (1) becomes

$$\binom{2k}{k} = \binom{6}{3}\binom{2k-4}{k-3}, \qquad \text{for } k \geq 6.$$

Equivalently, we have,

$$\frac{(2k)(2k-1)(2k-2)(2k-3)}{k \cdot k(k-1)(k-2)} = \binom{6}{3}.$$

It implies that

$$4(2k-1)(2k-3) = 20k(k-2),$$

where $k = -1, 3$ are integer solutions. Thus,

$$\binom{2k}{k} \neq \binom{6}{3}\binom{2k-4}{k-3}, \qquad \text{for } k \geq 6.$$

The proof is complete. □

Lemma 3. *Let k, x, a, b be positive integers such that $k = a + b$. Then*

$$\binom{2k}{k} \neq \binom{2a}{a}\binom{a+2b-2}{b}, \qquad \textit{for } 4 < 2a \leq k.$$

Lemma 4. *Let k, x, a, b be positive integers such that $k = a + b$. Then*

$$\binom{2k}{k} \neq \binom{2a}{a}\binom{a+2b-3}{b}, \qquad \textit{for } \ 4 < 2a \leq k.$$

Next, we will prove the main result in the case that $a \geq \frac{1}{2}k$ and $b \geq 3$. To do so, we will need the following results from Hanson [2] and Stănică [6].

Lemma 5 ([2]). *The product of m consecutive integers $n(n+1)\cdots(n+m-1)$ greater than m contains a prime divisor greater than $\frac{3}{2}m$ with the exceptions of $3\cdot 4$, $8\cdot 9$, and $6\cdot 7\cdot 8\cdot 9\cdot 10$.*

Lemma 6 ([6]). *Let m, n, r be positive integers, with $m > r \geq 1$ and $n \geq 1$. Then*

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{8n}}n^{-\frac{1}{2}}\frac{m^{mn+\frac{1}{2}}}{(m-r)^{(m-r)n+\frac{1}{2}}r^{rn+\frac{1}{2}}} < \binom{mn}{rn} < \frac{1}{\sqrt{2\pi}}n^{-\frac{1}{2}}\frac{m^{mn+\frac{1}{2}}}{(m-r)^{(m-r)n+\frac{1}{2}}r^{rn+\frac{1}{2}}}$$

Using the previous two results, we can prove the following Lemma.

Lemma 7. *Let k, x, a, b be positive integers such that $k = a+b$. If $a \geq \frac{1}{2}k$, $b \geq 3$, then*

$$\binom{2k}{k} \neq \binom{2a}{a}\binom{x+2b}{b}.$$

Proof. We follow the strategy used in [4]. Suppose that $a \geq \frac{1}{2}k$. Since $b \geq 3$, we know that $k \geq 6$. Assume $\binom{2k}{k} = \binom{2a}{a}\binom{x+2b}{b}$. Then

$$\frac{\binom{2k}{k}}{\binom{2a}{a}} = \frac{(2k)(2k-1)\cdots(2a+1)}{(k(k-1)\cdots(a+1))^2} = \binom{x+2b}{b}$$

is an integer. By Lemma 5, there exists a prime divisor p of the product $(2k)(2k-1)\cdots(2a+1)$ such that $p > \frac{3}{2}(2k-2a) = 3b > 2$. We claim that p is not a divisor of $k(k-1)\cdots(a+1)$. In fact assume that p divides $k-i$ for some $0 \leq i \leq b-1$ and α is the largest positive integer such that p^α divides $k-i$. Then α is also the largest positive integer such that p^α divides $2(k-i)$, and p is not a divisor of other terms $2k-j$, with $j \neq 2i$, in the numerator. In other words, α is the largest positive integer such that p^α divides $(2k)(2k-1)\cdots(2a+1)$. However, $p^{2\alpha}$ divides $(k(k-1)\cdots(a+1))^2$. This contradicts the assumption that $\frac{\binom{2k}{k}}{\binom{2a}{a}}$ is an integer. Hence p does not divide the denominator $k(k-1)\cdots(a+1)$.

So, we have found that p divides $\binom{x+2b}{b}$. Since p is prime, we have $x+2b \geq p > 3b$. By Lemma 6, we obtain, for $b \geq 3$,

$$\binom{x+2b}{b} > \binom{3b}{b} \geq \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{8b}}b^{-\frac{1}{2}}\frac{3^{3b+\frac{1}{2}}}{2^{2b+\frac{1}{2}}} = \left(\sqrt{\frac{3}{4\pi}}\cdot\frac{1}{\sqrt{b}e^{\frac{1}{8b}}}\cdot\left(\frac{27}{16}\right)^b\right)\cdot 4^b > 4^b.$$

However,

$$\frac{\binom{2k}{k}}{\binom{2a}{a}} = \frac{(2k)(2k-1)\cdots(2a+1)}{(k(k-1)\cdots(a+1))^2} = 2^b\cdot\frac{(2k-1)(2k-3)\cdots(2a+1)}{k(k-1)\cdots(a+1)} < 4^b.$$

This is a contradiction. Thus, for $b \geq 3$,

$$\binom{2k}{k} \neq \binom{2a}{a}\binom{x+2b}{b}.$$

□

Using the previous lemma, we can now prove our main result when $a \geq \frac{1}{2}k$ and $x \geq 2$.

Lemma 8. *Let k, x, a, b be positive integers such that $k = a + b$. If $a \geq \frac{1}{2}k$ and $x \geq 2$, then*

$$\binom{2k}{k} \neq \binom{2a}{a}\binom{x+2b}{b}.$$

Proof. By Lemma 7, we may assume that $1 \leq b < 3$. When $b = 1$, for $x \geq 2$,

$$\begin{aligned}\binom{2(k-1)}{k-1}\binom{x+2}{1} &= \frac{(2(k-1))!}{((k-1)!)^2}(x+2) \\ &= \binom{2k}{k}(x+2) \cdot \frac{k^2}{2k(2k-1)} \\ &> \binom{2k}{k}.\end{aligned}$$

When $b = 2$, we have

$$\frac{\binom{2(k-2)}{k-2}\binom{x+4}{2}}{\binom{2k}{k}} = \frac{(x+4)(x+3)}{2} \cdot \frac{k^2(k-1)^2}{(2k)(2k-1)(2k-2)(2k-3)}.$$

If $x \geq 3$,

$$\begin{aligned}\frac{(x+4)(x+3)}{2} \cdot \frac{k^2(k-1)^2}{2k(2k-1)(2k-2)(2k-3)} &= \frac{(x+4)(x+3)}{8} \cdot \frac{k(k-1)}{(2k-1)(2k-3)} \\ &> \frac{(x+4)(x+3)}{32} \\ &> 1.\end{aligned}$$

If $x = 2$,

$$\frac{(2+4)(2+3)}{2} \cdot \frac{k^2(k-1)^2}{2k(2k-1)(2k-2)(2k-3)} = 1$$

has no integer solution. Thus, when $b = 2$, for all $x \geq 2$,

$$\binom{2(k-2)}{k-2}\binom{x+4}{2} \neq \binom{2k}{k}.$$

□

We will now prove our main result when $x = 1$.

Proposition 2. *Let k, a, b be positive integers such that $k = a + b$. Then*

$$\binom{2k}{k} = \binom{2a}{a}\binom{1+2b}{b} \quad \textit{if and only if } a = 1.$$

Proof. First, assume $a = 1$. Then,

$$\begin{aligned}
\frac{\binom{2k}{k}}{\binom{1+2b}{b}} &= \frac{(2k)! \cdot b! \cdot (b+1)!}{(1+2b)! \cdot k! \cdot k!} \\
&= \frac{(2+2b)(1+2b)\dots(b+2)}{(1+2b)(2b)\dots(b+2)} \cdot \frac{b(b-1)\dots 1}{(b+1)b(b-1)\dots 1} \\
&= \frac{2(1+b)}{1+b} \\
&= 2 \\
&= \binom{2a}{a}.
\end{aligned}$$

Next, assume $\binom{2k}{k} = \binom{2a}{a}\binom{1+2b}{b}$. By Lemma 1, we may assume that $k > 10$. Lemma 7 implies that, if $a \geq \frac{1}{2}k$ and $b \geq 3$, then $\binom{2k}{k} \neq \binom{2a}{a}\binom{1+2b}{b}$. Moreover, when $b = 1, 2$, the equation

$$\binom{2k}{k} = \binom{2a}{a}\binom{1+2b}{b}$$

has no integer solution for $k > 10$. Therefore, we may assume that $a < \frac{1}{2}k$, so $b > \frac{1}{2}k$. For the purpose of the contradiction, we further assume that $a > 1$.

Then

$$\begin{aligned}
\frac{\binom{2k}{k}}{\binom{1+2b}{b}} &= \frac{(2k)(2k-1)\cdots(2b+2)}{(k(k-1)\cdots(b+1))(k(k-1)\cdots(b+2))} \\
&= \frac{(2k)(2k-1)\cdots(2b+3)\cdot 2}{(k(k-1)\cdots(b+2))^2} \\
&= \binom{2a}{a}
\end{aligned}$$

is an integer. By Lemma 5, there exists a prime divisor p of the product $(2k)(2k-1)\cdots(2b+3)$ such that $p > \frac{3}{2}(2k-2b-2) = 3a-3 > 2$. We claim that p is not a divisor of $k(k-1)\cdots(b+2)$. In fact, assume that p divides $k-i$ for some $0 \leq i \leq a-1$ and α is the largest positive integer such that p^α divides $k-i$. Then α is also the largest positive integer such that p^α divides $2(k-i)$, and p is not a divisor of other terms $2k-j$, with $j \neq 2i$, in the numerator. In other words, α is the largest positive integer such that p^α divides $(2k)(2k-1)\cdots(2b+3)\cdot 2$. However, $p^{2\alpha}$ divides $(k(k-1)\cdots(b+2))^2$. This contradicts the assumption that $\frac{\binom{2k}{k}}{\binom{1+2b}{b}}$ is an integer. Hence p does not divide the denominator $k(k-1)\cdots(b+2)$.

So, we must have p divides $\binom{2a}{a}$, whence $2a \geq p > 3a-3$. So $a < 3$. From the assumption that $a > 1$, we have $a = 2$. Now the equation $\binom{2k}{k} = \binom{2a}{a}\binom{1+2b}{b}$ becomes

$$\binom{2k}{k} = \binom{4}{2}\binom{2k-3}{k-2}, \qquad \text{when } a = 2.$$

Equivalently, we have,

$$\frac{(2k)(2k-1)(2k-2)}{k \cdot k(k-1)} = 6,$$

which has no integer solution when $k > 10$, a contradiction.

Therefore, $a \le 1$. Since a is a positive integer, it follows that $a = 1$. This completes the proof of the result. □

For the next Lemma we will prove, we need the following results from [3] and [5].

Lemma 9 ([3])**.** *The product of m consecutive integers $n(n+1)\cdots(n+m-1)$ contains a prime divisor greater than $1.8m$ if $n > m > 2$ and $n + m \ge 150$.*

Lemma 10 ([5])**.** *The product of m consecutive integers $n(n+1)\cdots(n+m-1)$ contains a prime divisor greater than $4.42m$ if $n > 4m$, $m > 3$ and $n + m \ge 150$.*

Lemma 11. *Let k, x, a, b be positive integers such that $k = a + b \ge 75$ and $x \ge 2$. If $a \le 0.9k$ and $1 \le b \le 0.8k$, then*

$$\binom{2k}{k} \ne \binom{2a}{a}\binom{x+2b}{b}.$$

Proof. Assume that $a \le 0.9k$ and $1 \le b \le 0.8k$. Assume, by means of contradiction, that $\binom{2k}{k} = \binom{2a}{a}\binom{x+2b}{b}$. By Lemma 9, there exists a prime divisor of the product $(2k)(2k-1)\cdots(k+1)$ such that $p > 1.8k$. Obviously, p is also a divisor of $\binom{2k}{k}$, thus a divisor of $\binom{2a}{a}\binom{x+2b}{b}$. It follows that $2a > 1.8k$ or $x + 2b > 1.8k$. By Proposition 1, we may assume that $x < a$. Thus, if $x + 2b > 1.8k$, then $b = a + 2b - (a + b) > x + 2b - k > 0.8k$. Therefore, we conclude that either $a > 0.9k$ or $b > 0.8k$, a contradiction to $a \le 0.9k$ and $1 \le b \le 0.8k$. Thus,

$$\binom{2k}{k} \ne \binom{2a}{a}\binom{x+2b}{b}.$$

□

Lemma 12. *Let k, x, a, b be positive integers such that $x \ge 2$ and $k = a + b \ge 150$. If $a \ge \frac{171}{121}x$, then*

$$\binom{2k}{k} \ne \binom{2a}{a}\binom{x+2b}{b}.$$

Proof. Assume that $x \ge 2$, $a \ge \frac{171}{121}x$ and $k \ge 150$. Assume, by means of contradiction, that $\binom{2k}{k} = \binom{2a}{a}\binom{x+2b}{b}$ for some $b \ge 1$. By Lemma 8, we can assume that $a < \frac{1}{2}k$. By Lemma 11, we can assume that $b > 0.8k$, thus $a = k - b < 0.2k$. In particular, $b > 4a$. By Lemma 2, Lemma 3, and Lemma 4 we can assume that $a - x > 3$. So,

$$\binom{2a}{a} = \frac{\binom{2k}{k}}{\binom{x+2b}{b}} = \frac{(2k)(2k-1)\cdots(x+2b+1)}{(k\cdot(k-1)\cdots(b+1))(k\cdot(k-1)\cdots(x+b+1))}.$$

Then

$$Q = \frac{(2k)(2k-1)\cdots(x+2b+1)}{(k\cdot(k-1)\cdots(x+b+1))^2} = \binom{2a}{a}\cdot(b+1)\cdots(b+x)$$

is an integer. Let $n = b + x + 1$ and $m = a - x$. Since $k - (x + b) = m > 3$, we have, $n > b > 4a > 4m$, and $n + m = k + 1 \geq 150$. So, by Lemma 10, there exists a prime divisor p of $k \cdot (k-1)\cdots(x+b+1)$ such that $p > 4.42(a - x)$. Thus, for some $0 \leq i < a - x$, p divides $k - i$ in the denominator $k \cdot (k-1)\cdots(x+b+1)$ of Q. Let α be the largest positive integer such that p^α divides $k - i$. Then α is also the largest positive integer such that p^α divides $2(k - i)$, and no other term $(2k - t)$, for $t \neq 2i$, in the numerator $(2k)(2k-1)\cdots(x+2b+1)$ of Q is divisible by p as $a \geq \frac{171}{121}x$ and $p > 4.42(a - x) \geq (2a - x)$. However, $p^{2\alpha}$ is a divisor of the denominator $(k\cdot(k-1)\cdots(x+b+1))^2$ of Q. That contradicts the fact that Q is an integer. Therefore, if $x \geq 2$, $a \geq \frac{171}{121}x$, and $k \geq 150$, then

$$\binom{2k}{k} \neq \binom{2a}{a}\binom{x+2b}{b}.$$

□

Using the previous lemmas, we can prove our main result when $x \geq 2$ and $k \geq 150$.

Proposition 3. *Let k, x, a, b be positive integers such that $x \geq 2$ and $k = a + b \geq 150$. Then*

$$\binom{2k}{k} \neq \binom{2a}{a}\binom{x+2b}{b}.$$

Proof. Assume, by means of contradiction, that $\binom{2k}{k} = \binom{2a}{a}\binom{x+2b}{b}$ for some $b \geq 1$. By Proposition 1 and Lemma 12, we may assume that $x < a < \frac{171}{121}x < \frac{3}{2}x$. By Lemma 11, we assume that $b > 0.8k$, which gives us $a < 0.2k$. Note that

$$\binom{x+2b}{b} = \frac{(x+2b)!}{b!\cdot(x+b)!} = \frac{(2b)!}{b!\cdot b!}\cdot\frac{(2b+1)(2b+2)\cdots(2b+x)}{(b+1)(b+2)\cdots(b+x)}.$$

By Lemma 6 and the assumption that $a < \frac{3}{2}x$,

$$\begin{aligned}\frac{\binom{2k}{k}}{\binom{2a}{a}\binom{2b}{b}} &\leq \frac{\frac{1}{\sqrt{2\pi}}k^{-\frac{1}{2}}2^{2k+\frac{1}{2}}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{8a}}a^{-\frac{1}{2}}2^{2a+\frac{1}{2}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{8b}}b^{-\frac{1}{2}}2^{2b+\frac{1}{2}}} \\ &= e^{\frac{1}{8a}+\frac{1}{8b}}\sqrt{\frac{\pi ab}{k}} \\ &< e^{1/4}\sqrt{\frac{3\pi x}{2}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{(2b+1)(2b+2)\cdots(2b+x)}{(b+1)(b+2)\cdots(b+x)} &= \left(2-\frac{1}{b+1}\right)\left(2-\frac{2}{b+2}\right)\cdots\left(2-\frac{x}{b+x}\right)\\ &> \left(2-\frac{x}{b+x}\right)^x > \left(2-\frac{a}{b+a}\right)^x \geq \left(2-\frac{1}{5}\right)^x.\end{aligned}$$

Note that $\left(\frac{9}{5}\right)^x - e^{1/4}\sqrt{3\pi x/2}$ is an increasing function for $x \geq 2$ and $\left(\frac{9}{5}\right)^3 > e^{1/4}\sqrt{9\pi/2}$. So, for $x \geq 3$ and $k \geq 150$,

$$\frac{(2b+1)(2b+2)\cdots(2b+x)}{(b+1)(b+2)\cdots(b+x)} > \left(\frac{9}{5}\right)^x > e^{1/4}\sqrt{\frac{3\pi x}{2}} > \frac{\binom{2k}{k}}{\binom{2a}{a}\binom{2b}{b}},$$

which gives us

$$\binom{x+2b}{b} = \binom{2b}{b}\cdot\frac{(2b+1)(2b+2)\cdots(2b+x)}{(b+1)(b+2)\cdots(b+x)} > \frac{\binom{2k}{k}}{\binom{2a}{a}}.$$

Therefore,

$$\binom{2a}{a}\binom{x+2b}{b} > \binom{2k}{k}$$

which is a contradiction to our assumption. Also, we know that $x < a < \frac{3}{2}x$, so when $x = 2$, we must have $2 < a < 3$, a contradiction to a an integer. Therefore, for $x \geq 2$ and $k \geq 150$,

$$\binom{2k}{k} \neq \binom{2a}{a}\binom{x+2b}{b}.$$

□

Finally, we will prove our main result when $x \geq 2$ and $x < k < 150$.

Proposition 4. *Let k, x, a, b be positive integers such that $x \geq 2$, $k = a + b$, and $x < k < 150$. Then*

$$\binom{2k}{k} \neq \binom{2a}{a}\binom{x+2b}{b}.$$

Proof. Using Maple software, we verified that, for $x \geq 2$ and $x < k < 150$,

$$\binom{2k}{k} \neq \binom{2a}{a}\binom{x+2b}{b}.$$

□

Now, we can prove the following, main theorem of the paper.

Theorem 1. *Let k, x, b be positive integers and a be a nonnegative integer such that $k = a + b$. Then*

$$\binom{2k}{k} = \binom{2a}{a}\binom{x+2b}{b} \quad \textit{if and only if} \quad x = a = 1.$$

Proof. When $x = 1$, we obtain the result from Proposition 2. When $x \geq 2$, and $k \geq 150$, we obtain the result from Proposition 3. When $x \geq 2$, and $x < k \leq 150$, we obtain the result from Proposition 4. When $x \geq 2$, and $x \geq k$, we obtain the result from Proposition 1. □

References

[1] P. Erdős, On some divisibility properties of $\binom{2n}{n}$, *Canad. Math. Bull.* **7** (1964), 513-518.

[2] D. Hanson, On a theorem of Sylvester and Schur, *Canad. Math. Bull.* **16** (1973), 195-199.

[3] S. Laishram and T.N. Shorey, The greatest prime divisor of a product of consecutive integers, *Acta Arith.* **120** (3) (2005), 299–306.

[4] L. Moser, Insolvability of $\binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$, *Canad. Math. Bull.* **6** (2) (1963), 167-169.

[5] S. Nair and T.N. Shorey, Lower bounds for the greatest prime factor of product of consecutive positive integers, *J. Number Theory* **159** (2016), 307–328.

[6] P. Stănică, Good lower and upper bounds on binominal coefficients, *J. Inequal. Pure Appl. Math.* **2** (3) (2001), Article 30, 5 pp.