A NEW TRIANGLE OF NUMBERS: PROPERTIES AND OBSERVATIONS

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Abstract

We introduce a newly discovered path on a grid and explore the sequence of numbers associated with it. These numbers reveal interesting properties and are deeply linked to binomial coefficients, Fine numbers, and generalized Catalan numbers.

1. Introduction

Grids serve as the basis for many well-known sequences in combinatorics. The path we study in this work follows a distinctive movement across the grid, producing a new triangle of numbers that exhibits surprising properties. Unlike a traditional Delannoy path, which is a lattice path in which only north, east, and northeast steps are allowed (*i.e.*, $\uparrow\longrightarrow\nearrow$), our path obeys a specific set of rules that lead to a novel numerical pattern. Before defining our paths, we would like to recall two exemplary numbers in combinatorics.

The classical Catalan numbers (this is sequence A001008 in the OEIS [4]) are given by

$$C(n) := \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 1.$$

Their ordinary generating function is

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

To avoid confusion with the famous Fibonacci sequence, we will denote the n^{th} Fine number by f_n. They are related to Catalan numbers by

$$C_n = 2f_n + f_{n-1}, n \geq 1.$$

The *Fine numbers* (this is sequence A000957 in the OEIS [4]), as defined in [1], are

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given by their generating function

$$F = F(x) = \sum_{n=0}^{\infty} f_n x^n = \frac{1}{x}\frac{1-\sqrt{1-4x}}{3-\sqrt{1-4x}} = 1 + x^2 + 2x^3 + 6x^4 + 18x^5 + ...$$

One combinatorial interpretation of the Fine numbers is *Dyck* paths. These are paths starting and ending on the horizontal axis using $(1,1)$ and $(1,-1)$ steps and not going below the ground.

2. New Paths and Their Triangle of Numbers

In odd columns, the following rules apply.

- Up steps are allowed in two forms: straight (S) and twisted (T).
- If you start straight, you remain straight along that column.
- If you start twisted, you can:
  - stay twisted,
  - make a right turn,
  - be straightened up; once straight, you cannot return to twisted.

Example 1. The following sequences illustrate possible behaviors of a path in an odd column. Each sequence begins with twisted up steps and shows the possible transitions allowed by the rules above.

$$T \ldots TT$$

$$T \ldots TR1$$

$$T \ldots TSS$$

$$T \ldots TSR1$$

- From the starting point, one step to the right is not allowed; two steps to the right are.

In even columns, the following rules apply.

- Only twisted up steps are allowed.
- Right steps are always allowed.
- If you go to the right, the next step cannot be twisted.

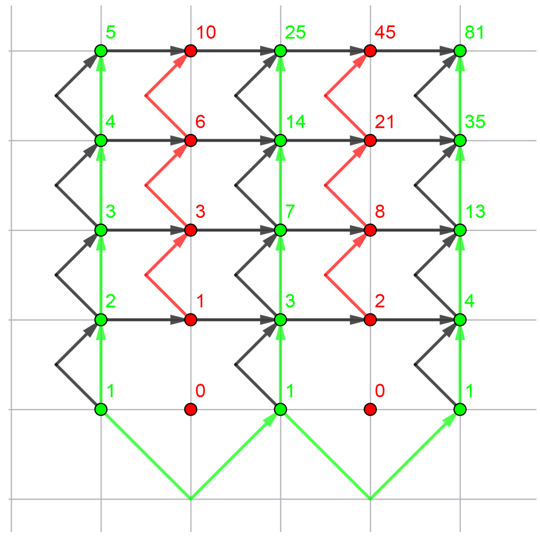

Figure 1: All possible paths on a 5×5 lattice grid.

The allowed paths are illustrated in Figure 1 and the triangle of numbers associated with these paths is shown in Table 1.

The number of possible paths is given by

$$s\binom{n}{i} := \sum_{j=i+1}^{n} (-1)^{n-j}\binom{j}{i}.$$

The triangle $\left(s\binom{n}{i}\right)_{0\le i\le n-1}$ is sequence A341091 in the OEIS [4]. The following identities follow directly from the definition of $s\binom{n}{i}$ and the Pascal identity:

$$\begin{aligned} s\binom{n}{i} &= s\binom{n-1}{i-1} + s\binom{n-1}{i} + (-1)^{n-i+1}, \\ \binom{n}{i} &= s\binom{n}{i} + s\binom{n-1}{i}, \\ s\binom{n}{i} &= s\binom{n-2}{i} + \binom{n-1}{i-1}. \end{aligned}$$

The first two identities combined can be written as

$$2s\binom{n}{i} = \binom{n}{i} + s\binom{n-1}{i-1} + (-1)^{n-i+1}.$$

$n\backslash i$	0	1	2	3	4	5	6	7	8	9
1	1									
2	0	2								
3	1	1	3							
4	0	3	3	4						
5	1	2	7	6	5					
6	0	4	8	14	10	6				
7	1	3	13	21	25	15	7			
8	0	5	15	35	45	41	21	8		
9	1	4	21	49	81	85	63	28	9	
10	0	6	24	71	129	167	147	92	36	10

Table 1: Triangle of numbers associated with the allowed paths.

Before stating the main theorem of this section, we remind the reader of the *Gauss hypergeometric function*, defined by

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k}\,\frac{z^k}{k!}, $$

where $(a)_k$ denotes the *Pochhammer symbol*, also known as the *rising factorial*. For $k \geq 1$, it is defined by

$$ (a)_k = a(a+1)\cdots(a+k-1), $$

with the convention that $(a)_0 = 1$.

The next result relates our coefficients to hypergeometric functions. It is the best "closed form" that we have found so far. Out of scientific integrity, we should mention that a similar result was stated a year ago, without proof (as the sequence A341091 in the OEIS [4]). We have refined this result and provided a proof.

Theorem 1. *For* $0 \leq i < n$,

$$ \begin{aligned} s\binom{n}{i} &= \binom{n+1}{i} \cdot {}_2F_1\left(1, n+2, n+2-i, -1\right) + (-1)^{n-i+1}\left(1 - \frac{1}{2^{i+1}}\right) \\ &= \binom{n}{i}\left[1 - {}_2F_1\left(1, n+1, n+1-i, -1\right)\right] + (-1)^{n-i+1}\left(1 - \frac{1}{2^{i+1}}\right). \end{aligned} $$

Proof. We proceed by induction on n. We now need to show that

$$ s\binom{i+1}{i} = \binom{i+2}{i} \cdot {}_2F_1\left(1, i+3, 3, -1\right) + 1 - \frac{1}{2^{i+1}}, $$

but we know that $s\binom{i+1}{i} = i+1$, so what we need to prove is

$$ {}_2F_1\left(1, i+3, 3, -1\right) = \frac{i + 2^{-i-1}}{\binom{i+2}{i}}. $$

From the third Euler transformation formula, we get

$$ {}_2F_1\left(1, i+3, 3, -1\right) = 2^{-i-1} \cdot {}_2F_1\left(2, -i, 3, -1\right). $$

By definition, we have

$$ {}_2F_1\left(2, -i; 3; -1\right) = \sum_{k=0}^{\infty} \frac{(2)_k\,(-i)_k}{(3)_k} \frac{(-1)^k}{k!}, $$

where

$$ (-i)_k = \begin{cases} (-1)^k \binom{i}{k} k!, & \text{if } k \leq i, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } (2)_k = (k+1)! \text{ and } (3)_k = \frac{(k+2)!}{2}. $$

We get

$$ {}_2F_1\big(2, -i; 3; -1\big) = 2 \sum_{k=0}^{i} \frac{\binom{i}{k}}{k+2}. $$

We have

$$ \sum_{k=0}^{i} \frac{\binom{i}{k}}{k+2} = \sum_{k=0}^{i} \binom{i}{k} \int_0^1 x^{k+1}\, dx = \int_0^1 x(1+x)^i\, dx. $$

The substitution $t = 1 + x$ gives the following integral:

$$ \begin{aligned} \int_1^2 (t-1)\, t^i dt &= \left[\frac{t^{i+2}}{i+2}\right]_1^2 - \left[\frac{t^{i+1}}{i+1}\right]_1^2 = \frac{2^{i+2}-1}{i+2} - \frac{2^{i+1}-1}{i+1} \\ &= \frac{i2^{i+1}+1}{(i+1)\,(i+2)}. \end{aligned} $$

We obtain

$$ {}_2F_1\left(1, i+3, 3, -1\right) = 2^{-i-1}.2.\frac{i2^{i+1}+1}{(i+1)\,(i+2)} = \frac{i + 2^{-i-1}}{\binom{i+2}{i}}. $$

Thus, the base case for the induction is proved; we now need to show that

$$ s\binom{n+1}{i} = \binom{n+2}{i} \cdot {}_2F_1\left(1, n+3, n+3-i, -1\right) + (-1)^{n-i}\left(1 - \frac{1}{2^{i+1}}\right). $$

Using the identity
$$\binom{n+1}{i} = s\binom{n+1}{i} + s\binom{n}{i},$$
we get
$$\begin{aligned} s\binom{n+1}{i} &= \binom{n+1}{i} - s\binom{n}{i} = \binom{n+1}{i} \\ &\quad -\binom{n+1}{i} \cdot {}_2F_1\left(1, n+2, n+2-i, -1\right) + (-1)^{n-i}\left(1 - \frac{1}{2^{i+1}}\right) \\ &= \binom{n+1}{i}\left[1 - {}_2F_1\left(1, n+2, n+2-i, -1\right)\right] + (-1)^{n-i}\left(1 - \frac{1}{2^{i+1}}\right). \end{aligned}$$

Now, from Gauss' contiguous relations we learn that
$${}_2F_1\left(a, b+1, c+1, z\right) = z\frac{c}{b}\left({}_2F_1\left(a, b, c, z\right) - {}_2F_1\left(a-1, b, c, z\right)\right).$$

We set
$$a = 1,\ b = n+2,\ c = n+2-i,\ z = -1, \text{ and } {}_2F_1(a-1, b, c, z) = 1.$$

Thus,
$${}_2F_1\left(1, n+3, n+3-i, -1\right) = \frac{n+2-i}{n+2}\left[1 - {}_2F_1\left(1, n+2, n+2-i, -1\right)\right].$$

Therefore,
$$s\binom{n+1}{i} = \binom{n+2}{i} \cdot {}_2F_1\left(1, n+3, n+3-i, -1\right) + (-1)^{n-i}\left(1 - \frac{1}{2^{i+1}}\right).$$

Hence, we obtain the desired result. □

As the exploration of our coefficients continued, we derived certain summation formulae relating them to harmonic numbers and to sums of inverses of binomial coefficients. Furthermore, a direct connection with Fine numbers and also the so-called Generalized Catalan numbers was discovered. Hence, a new formula is found that links Fine and generalized Catalan.

3. Weighted Sums of Sabar's Numbers

Let H_n denote the n^{th} harmonic number and define
$$C_n := \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\binom{n-1}{i}}.$$

These two classic sums are extensively studied in the literature. However, little is known about their connection. We now claim that H_n and C_n are related by means of a simple weighted sum of our coefficients.

Theorem 2. *For* $n \geq 2$,

$$H_n = C_n + \sum_{i=0}^{n-2} (-1)^{n-i} \frac{s\binom{n-1}{i}}{n-i}.$$

Proof. We prove the statement by induction on n. Assume that the formula is true for n, and prove it for $n+1$:

$$H_{n+1} = C_{n+1} + \sum_{i=0}^{n-1} (-1)^{n-i+1} \frac{s\binom{n}{i}}{n+1-i}.$$

We have

$$\begin{aligned}
\sum_{i=0}^{n-1} (-1)^{n-(i-1)} \frac{s\binom{n}{i}}{n-(i-1)} &= (-1)^{n+1} \frac{s\binom{n}{0}}{n+1} + \sum_{i=1}^{n-1} (-1)^{n-(i-1)} \frac{s\binom{n}{i}}{n-(i-1)} \\
&= (-1)^{n+1} \frac{s\binom{n}{0}}{n+1} + \sum_{i=0}^{n-2} (-1)^{n-i} \frac{s\binom{n}{i+1}}{n-i} \\
&= (-1)^{n+1} \frac{s\binom{n}{0}}{n+1} + \frac{1}{2} \sum_{i=0}^{n-2} (-1)^{n-i} \frac{\binom{n}{i+1}}{n-i} \\
&\quad + \frac{1}{2} \sum_{i=0}^{n-2} (-1)^{n-i} \frac{s\binom{n-1}{i}}{n-i} + \frac{1}{2} \sum_{i=0}^{n-2} \frac{1}{n-i} \\
&= (-1)^{n+1} \frac{s\binom{n}{0}}{n+1} + \frac{1}{2} \sum_{i=0}^{n-2} (-1)^{n-i} \frac{\binom{n}{i+1}}{n-i} \\
&\quad + \frac{1}{2}(H_n - C_n) + \frac{1}{2} H_n - \frac{1}{2}.
\end{aligned}$$

On the other hand, using the result from [5], we obtain

$$\sum_{i=0}^{n-2}(-1)^{n-i}\frac{\binom{n}{i+1}}{n+1-(i+1)} = -\sum_{i=1}^{n-1}(-1)^{n-i}\frac{\binom{n}{i}}{n+1-i}$$
$$= -\sum_{i=0}^{n}(-1)^{n-i}\frac{\binom{n}{i}}{n+1-i} + 1 - (-1)^{n+1}\frac{1}{n+1}$$
$$= -\frac{1}{n+1} + 1 - (-1)^{n+1}\frac{1}{n+1}.$$

Therefore,

$$\sum_{i=0}^{n-1}(-1)^{n-(i-1)}\frac{s\binom{n}{i}}{n-(i-1)} = (-1)^{n+1}\frac{s\binom{n}{0}}{n+1} - \frac{1}{2}\left(\frac{1}{n+1} - 1 + (-1)^{n+1}\frac{1}{n+1}\right)$$
$$+ \frac{1}{2}(H_n - C_n) + \frac{1}{2}H_n - \frac{1}{2}$$
$$= H_n - \frac{1}{2}C_n$$
$$= H_{n+1} - C_{n+1}.$$

The last equalities follow from the following identities:

$$s\binom{n}{0} = \begin{cases} 0 & \text{for even } n, \\ 1 & \text{for odd } n, \end{cases} \quad \text{and} \quad C_{n+1} = \frac{1}{2}C_n + \frac{1}{n+1}.$$

This completes the induction. □

Corollary 1. *For $n \geq 2$,*

$$C_{n+1} = 1 + \sum_{i=0}^{n-2}(-1)^{n-i}\frac{s\binom{n-1}{i}}{(n-i)(n+1-i)}.$$

Proof. For $n \geq 2$,

$$
\begin{aligned}
H_{n+1} - H_n &= C_{n+1} - C_n + \sum_{i=0}^{n-1} (-1)^{n-i-1} \frac{s\binom{n}{i}}{n+1-i} - \sum_{i=0}^{n-2} (-1)^{n-i} \frac{s\binom{n-1}{i}}{n-i} \\
&= C_{n+1} - C_n + \frac{n}{2} + \sum_{i=0}^{n-2} (-1)^{n-i-1} \Big[-\frac{s\binom{n-1}{i}}{n+1-i} + \frac{\binom{n}{i}}{n+1-i} + \frac{s\binom{n-1}{i}}{n-i} \Big] \\
&= C_{n+1} - C_n + \sum_{i=0}^{n-2} (-1)^{n-i-1} \frac{s\binom{n-1}{i}}{(n-i)(n+1-i)} + \Big[-\sum_{i=0}^{n} (-1)^{n-i} \frac{\binom{n}{i}}{n+1-i} + 1 - \frac{n}{2} \Big] \\
&= C_{n+1} - C_n + \sum_{i=0}^{n-2} (-1)^{n-i-1} \frac{s\binom{n-1}{i}}{(n-i)(n+1-i)} - \frac{1}{n+1} + 1 \\
&= -C_{n+1} + \frac{1}{n+1} + 1 + \sum_{i=0}^{n-2} (-1)^{n-i-1} \frac{s\binom{n-1}{i}}{(n-i)(n+1-i)}.
\end{aligned}
$$

Here we used the last result from [5], together with

$$C_{n+1} = \frac{1}{2} C_n + \frac{1}{n+1}.$$

□

Now we write the harmonic number as an alternating weighted sum of our coefficients.

Theorem 3. *For $n \geq 2$,*

$$\sum_{i=0}^{2n-2} (-1)^i \frac{s\binom{2n-1}{i}}{i+1} = H_n.$$

Proof. We have

$$\begin{aligned}
\sum_{i=0}^{2n+2}(-1)^i\frac{s\binom{2n+1}{i}}{i+1} &= \sum_{i=1}^{2n-2}(-1)^i\frac{s\binom{2n+1}{i}}{i+1} + s\binom{2n+1}{0} \\
&\quad - \frac{s\binom{2n+1}{2n-1}}{2n} + \frac{s\binom{2n+1}{2n}}{2n+1} \\
&= \sum_{i=1}^{2n-2}(-1)^i\frac{s\binom{2n-1}{i}}{i+1} + \sum_{i=1}^{2n-2}(-1)^i\frac{\binom{2n}{i-1}}{i+1} + s\binom{2n+1}{0} \\
&\quad - \frac{s\binom{2n+1}{2n-1}}{2n} + \frac{s\binom{2n+1}{2n}}{2n+1} \\
&= H_n - s\binom{2n-1}{0} + \sum_{i=1}^{2n-2}(-1)^i\frac{\binom{2n}{i-1}}{i+1} + s\binom{2n+1}{0} \\
&\quad - \frac{\binom{2n}{2}}{2n} + 1 \\
&= H_n + \sum_{i=1}^{2n-1}(-1)^i\frac{\binom{2n}{i-1}}{i+1} + 1 \\
&= H_n - \sum_{i=0}^{2n-2}(-1)^i\frac{\binom{2n}{i}}{i+2} + 1.
\end{aligned}$$

Here we used the basic fact that

$$s\binom{n}{i} = s\binom{n-2}{i} + \binom{n-1}{i-1}.$$

Finally, applying formula (1.47) from [2] with $j = 0$ and $x = 2$, we obtain

$$\sum_{i=0}^{2n}(-1)^i\frac{\binom{2n}{i}}{i+2} = \frac{1}{2(n+1)(2n+1)}.$$

This completes the proof. □

4. Central Coefficients and Fine Numbers

In this section, we present the relationship between Fine numbers and our central coefficients.

Theorem 4. *For $n \geq 2$,*

$$f_n = s\binom{2n}{n+1} - s\binom{2n}{n-1}.$$

Proof. We have

$$s\binom{2n+2}{n+2} - s\binom{2n+2}{n} + f_n = a - b,$$

with

$$a = s\binom{2n+2}{n+2} + s\binom{2n}{n+1} \text{ and } b = s\binom{2n+2}{n} + s\binom{2n}{n-1}.$$

From the basic facts, we learn that

$$\begin{aligned} s\binom{2n+2}{n+2} &= \frac{1}{2}\binom{2n+2}{n+2} + \frac{1}{2}s\binom{2n+1}{n+1} + \frac{1}{2}(-1)^n \text{ and} \\ s\binom{2n+1}{n+1} &= \binom{2n+1}{n+1} - s\binom{2n}{n+1}. \end{aligned}$$

Thus,

$$\begin{aligned} a &= \frac{1}{2}\binom{2n+2}{n+2} + \frac{1}{2}\binom{2n+1}{n+1} + \frac{1}{2}s\binom{2n}{n+1} + \frac{1}{2}(-1)^n \text{ and} \\ b &= \frac{1}{2}\binom{2n+2}{n} + \frac{1}{2}\binom{2n+1}{n-1} + \frac{1}{2}s\binom{2n}{n-1} + \frac{1}{2}(-1)^n. \end{aligned}$$

Therefore,

$$a - b = \frac{1}{2}\left[\binom{2n+2}{n+2} + \binom{2n+1}{n+1} - \binom{2n+2}{n} - \binom{2n+1}{n-1}\right] + \frac{1}{2}f_n.$$

And since

$$\begin{aligned} \binom{2n+2}{n+2} &= \frac{n+1}{n+2}\binom{2n+2}{n+1}, \; \binom{2n+1}{n+1} = \frac{1}{2}\binom{2n+2}{n+1} \text{ and} \\ \binom{2n+2}{n} &= \frac{n+1}{n+2}\binom{2n+2}{n+1}, \; \binom{2n+1}{n-1} = \frac{n}{2.(n+2)}\binom{2n+2}{n+1}, \end{aligned}$$

we get

$$\begin{aligned} a - b &= s\binom{2n+2}{n+2} - s\binom{2n+2}{n} + f_n \\ &= \frac{1}{2}\frac{1}{n+2}\binom{2n+2}{n+1} + \frac{1}{2}f_n \\ &= \frac{1}{2}C(n+1) + \frac{1}{2}f_n. \end{aligned}$$

The formula
$$\frac{1}{2}C(n+1) = f_{n+1} + \frac{1}{2}f_n,$$
completes the proof. □

As a direct result, we get the following nice new formula.

Corollary 2. *For $n \geq 2$,*
$$\sum_{k=1}^{n} (-1)^{k+1} \left(1 - 2\frac{k}{n+1} + \frac{n}{k+1}\right) \binom{n+k}{k} = n.$$

Proof. In [1], f_n is defined by
$$f_n = \sum_{k=0}^{n} (-1)^{n-k} \left(1 - \frac{k}{n+1}\right) \binom{n+k}{k}.$$

Moreover, we have
$$s\binom{2n}{n+1} = \sum_{k=2}^{n} (-1)^{n-k} \binom{n+k}{n+1} \quad \text{and} \quad s\binom{2n}{n-1} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n-1}.$$

Thus,
$$\begin{aligned}(-1)^n + \sum_{k=1}^{n} (-1)^{n-k} \left(1 - \frac{k}{n+1}\right) \binom{n+k}{k} &= (-1)^n + \sum_{k=1}^{n} (-1)^{n-k} \binom{n+k}{n+1} \\ &\quad - \sum_{k=1}^{n} (-1)^{n-k} \binom{n+k}{n-1} - (-1)^n n.\end{aligned}$$

This is the desired formula. □

Another immediate consequence is an alternative expression of f_n.

Corollary 3. *For $n \geq 2$,*
$$f_n = \frac{1}{2}(-1)^n + \frac{1}{2}\sum_{k=0}^{n} (-1)^{n-k} \left(1 - \frac{n}{k+1}\right) \binom{n+k}{k}.$$

Now, we give an expression of our central coefficients in terms of Fine numbers.

Theorem 5. *For $n \geq 1$,*
$$\begin{aligned} s\binom{2n}{n} &= \frac{1}{6}\binom{2n+2}{n+1} + \frac{2}{3}f_n + (-1)^{n+1} \\ &= \frac{1}{3}(2n+1)\, C(n) + \frac{2}{3}f_n + (-1)^{n+1} \\ &= \frac{2}{3}\binom{2n}{n} - \frac{1}{3}f_{n-1} + (-1)^{n+1}. \end{aligned}$$

Consequently,

$$s\binom{2n-1}{n} = \frac{2}{3}\binom{2n-1}{n} + \frac{1}{3}f_{n-1} + (-1)^n.$$

Proof. We have

$$\begin{aligned}
s\binom{2n+2}{n+1} &= \frac{1}{2}\binom{2n+2}{n+1} + \frac{1}{2}s\binom{2n+1}{n} + \frac{1}{2}(-1)^n \\
&= \frac{1}{2}\binom{2n+2}{n+1} + \frac{1}{2}\left(\binom{2n+1}{n} - s\binom{2n}{n}\right) + \frac{1}{2}(-1)^n \\
&= \frac{1}{2}\binom{2n+2}{n+1} + \frac{1}{2}\binom{2n+1}{n} - \frac{1}{12}\binom{2n+2}{n+1} - \frac{1}{3}f_n + (-1)^n \\
&= \frac{1}{2}\binom{2n+2}{n+1} + \frac{1}{2}\binom{2n+1}{n} - \frac{1}{12}\binom{2n+2}{n+1} \\
&\quad -\frac{1}{3}\frac{1}{n+2}\binom{2n+2}{n+1} + \frac{2}{3}f_{n+1} + (-1)^n \\
&= \frac{1}{6}\frac{2(2n+3)}{n+2}\binom{2n+2}{n+1} + \frac{2}{3}f_{n+1} + (-1)^n \\
&= \frac{1}{6}\binom{2n+4}{n+2} + \frac{2}{3}f_{n+1} + (-1)^n.
\end{aligned}$$

Hence, the induction is complete. □

As a result, the generating function of our central coefficients is given by

$$\begin{aligned}
S(x) &= \sum_{n=0}^{\infty} s\binom{2n}{n}x^n = \frac{1}{6x}\sum_{n=0}^{\infty}\binom{2(n+1)}{n+1}x^{n+1} + \frac{2}{3}F + \frac{x}{x+1} \\
&= \frac{1}{6x}(C-1) + \frac{2}{3}F + \frac{x}{x+1} \\
&= \frac{4x^2 + x + 2 + 3x\sqrt{1-4x}}{(x+1)\left(3-\sqrt{1-4x}\right)\sqrt{1-4x}}.
\end{aligned}$$

Before leaving this section, we would like to mention that the number

$$\breve{O}(n) = s\binom{2n-1}{n} + (-1)^{n+1} = \frac{2}{3}\binom{2n-1}{n} + \frac{1}{3}f_{n-1}$$

is also studied in the context of [1].

5. Connection with the Generalized Catalan Numbers

The *generalized Catalan numbers* (this is sequence A064062 in the OEIS [4]), as defined in [6] , are given explicitly by

$$C(\alpha, n+1) := \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \binom{n+k}{k} \alpha^k.$$

In the same paper, the authors proposed the following sequence:

$$V_n(\alpha) := \sum_{k=0}^{n} \binom{n+k}{k} \alpha^k,$$

with α as a positive integer in both definitions. However, we are only interested in the case $\alpha = 2$ in this article.

It is also mentioned in [6] that Equation (5.137) of [3] gives the following recurrence relation:

$$V_n(\alpha) + (\alpha - 1) V_{n+1}(\alpha) = (2\alpha - 1) \alpha^{n+1} V_n(1).$$

We noticed that $V_{n+1}(1)$ satisfies

$$\begin{aligned} V_{n+1}(1) &= \sum_{k=0}^{n+1} \binom{n+1+k}{k} = \sum_{k=0}^{n+1} \left(1 + \frac{k}{n+1}\right) \binom{n+k}{k} \\ &= \sum_{k=0}^{n+1} \binom{n+k}{k} + \sum_{k=1}^{n+1} \binom{n+k}{k-1} \\ &= \binom{2n+2}{n+1} + \sum_{k=0}^{n} \binom{n+k}{k} + \sum_{k=0}^{n} \binom{n+1+k}{k} \\ &= \binom{2n+1}{n+1} + V_n(1) + V_{n+1}(1) - \binom{2n+2}{n+1}, \end{aligned}$$

which implies

$$V_n(1) = \binom{2n+2}{n+1} - \binom{2n+1}{n+1} = \frac{1}{2} \binom{2n+2}{n+1}.$$

This, in turn, leads to the following interesting identity.

Lemma 1. *For $n \geq 1$,*

$$V_n(2) + V_{n+1}(2) = 3.2^n \binom{2n+2}{n+1}.$$

We are now in a position to present the following new observation on the generalized Catalan numbers.

Theorem 6. *For $n \geq 1$,*

$$C\,(2,n) + C\,(2,n+1) = 2^{n+1} C\,(n)\,.$$

Proof. From the definition of $C\,(2,n)$ and the previous lemma, we have the following chain of equalities:

$$\begin{aligned}
C(2,n)+C(2,n+1) &= \sum_{k=0}^{n-1}\Big(1-\frac{k}{n}\Big)\binom{n-1+k}{k}2^k + \sum_{k=0}^{n}\Big(1-\frac{k}{n+1}\Big)\binom{n+k}{k}2^k\\
&= V_{n-1}(2)+V_n(2) - \sum_{k=1}^{n-1}\binom{n-1+k}{k-1}2^k - \sum_{k=1}^{n}\binom{n+k}{k-1}2^k\\
&= 3\cdot 2^{n-1}\binom{2n}{n} - 2\sum_{k=0}^{n-2}\binom{n+k}{k}2^k - 2\sum_{k=0}^{n-1}\binom{n+1+k}{k}2^k\\
&= 3\cdot 2^{n-1}\binom{2n}{n} - 2\Big[3\cdot 2^n\binom{2n+2}{n+1} - 2^{n-1}\binom{2n-1}{n-1}\\
&\qquad - 2^n\binom{2n}{n} - 2^n\binom{2n+1}{n} - 2^{n+1}\binom{2n+2}{n+1}\Big]\\
&= 7\cdot 2^{n-1}\binom{2n}{n} - 2^{n+1}\binom{2n+2}{n+1} + 2^n\binom{2n-1}{n-1}\\
&\quad + 2^{n+1}\binom{2n+1}{n}\\
&= \binom{2n}{n}\left[7\cdot 2^{n-1} - \frac{2n+1}{n+1}2^{n+2} + 2^{n-1} + \frac{2n+1}{n+1}2^{n+1}\right]\\
&= \binom{2n}{n}\Big[2^{n+2} - \frac{2n+1}{n+1}2^{n+1}\Big]\\
&= 2^{n+1}\binom{2n}{n}\Big[2 - \frac{2n+1}{n+1}\Big].
\end{aligned}$$

This is the desired result. □

Next, we answer the main question of this section: how are our central coefficients and the generalized Catalan numbers related?

Theorem 7. *For $n \geq 1$,*

$$s\binom{2n}{n} = \frac{2}{3}\binom{2n}{n} - \frac{1}{3}\frac{C\,(2,n)}{2^{n+1}} + (-1)^{n+1} + \frac{(-1)^n}{2^{n+1}}.$$

Proof. Again, we proceed with induction. We have

$$\begin{aligned}
s\binom{2n+2}{n+1} &= \frac{1}{2}\binom{2n+2}{n+1}+\frac{1}{2}\left(\binom{2n+1}{n}-s\binom{2n}{n}\right)+\frac{1}{2}(-1)^n\\
&= \frac{1}{2}\binom{2n+2}{n+1}+\frac{1}{2}\binom{2n+1}{n}-\frac{1}{3}\binom{2n}{n}\\
&\quad +\frac{1}{6}\frac{C(2,n)}{2^{n+1}}-\frac{1}{2}(-1)^{n+1}-\frac{1}{2}\frac{(-1)^n}{2^{n+1}}+\frac{1}{2}(-1)^n\\
&= \frac{1}{2}\binom{2n+2}{n+1}+\frac{1}{2}\binom{2n+1}{n}-\frac{1}{3}\binom{2n}{n}\\
&\quad +\frac{1}{3}\frac{C(2,n)}{2^{n+2}}+(-1)^n+\frac{(-1)^{n+1}}{2^{n+2}}.
\end{aligned}$$

Now it is left to show that

$$\frac{1}{2}\binom{2n+2}{n+1}+\frac{1}{2}\binom{2n+1}{n}-\frac{1}{3}\binom{2n}{n}+\frac{1}{3}\frac{C(2,n)}{2^{n+2}}=\frac{2}{3}\binom{2n+2}{n+1}-\frac{1}{3}\frac{C(2,n+1)}{2^{n+2}}.$$

Notice that the following list of equations are equivalent:

$$\begin{aligned}
\begin{pmatrix}\frac{4}{6}\binom{2n+2}{n+1}-\frac{3}{6}\binom{2n+2}{n+1}\\ -\frac{1}{2}\binom{2n+1}{n}+\frac{1}{3}\binom{2n}{n}\end{pmatrix} &= \frac{1}{3}\frac{C(2,n)}{2^{n+2}}+\frac{1}{3}\frac{C(2,n+1)}{2^{n+2}}\\
\frac{1}{6}\binom{2n+2}{n+1}-\frac{1}{2}\binom{2n+1}{n}+\frac{1}{3}\binom{2n}{n} &= \frac{1}{3}\frac{C(2,n)+C(2,n+1)}{2^{n+2}}\\
\frac{1}{6}\binom{2n+2}{n+1}-\frac{1}{4}\binom{2n+2}{n+1}+\frac{1}{3}\binom{2n}{n} &= \frac{1}{3}\frac{C(2,n)+C(2,n+1)}{2^{n+2}}\\
\frac{1}{3}\binom{2n}{n}-\frac{1}{12}\binom{2n+2}{n+1} &= \frac{1}{3}\frac{C(2,n)+C(2,n+1)}{2^{n+2}}\\
\binom{2n}{n}-\frac{1}{4}\binom{2n+2}{n+1} &= \frac{C(2,n)+C(2,n+1)}{2^{n+2}}\\
\frac{1}{(n+1)}\binom{2n}{n} &= \frac{C(2,n)+C(2,n+1)}{2^{n+1}}\\
2^{n+1}C(n) &= C(2,n)+C(2,n+1)
\end{aligned}$$

The last equation was shown right before the statement of this result. □

As an immediate consequence, we obtain a link between Fine numbers and generalized Catalan numbers.

Corollary 4. *For* $n \geq 1$,

$$2^{n+2}f_n = C(2,n+1)+3(-1)^n.$$

Proof. Combine the two main results of the last two sections. □

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References

[1] E. Deutsch and L. Shapiro , A survey of the Fine numbers, *Discrete Math.* **241** (2001), 241-265.

[2] H.W. Gould, *Combinatorial Identities: A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, Morgantown Printing and Binding Co, Morgantown, WV, 1972.

[3] R.L. Graham, D.E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, MA, 1994.

[4] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, https://oeis.org.

[5] T. Sabar, A note on the difference of powers and falling powers, *Int. J. Math. Math. Sci.* **5** (2021), 1-4.

[6] W. Whitt, Brownian motion and the generalized Catalan numbers, *J. Integer Seq.* **14** (2011), 11-26.