IRRATIONALITY OF THE RECIPROCAL SUM OF DOUBLY EXPONENTIAL SEQUENCES

Junnosuke Koizumi
RIKEN iTHEMS, Wako, Saitama, Japan
`junnosuke.koizumi@riken.jp`

Abstract

We show that sequences of positive integers whose ratios a_n^2/a_{n+1} lie within a specific range are almost uniquely determined by their reciprocal sums. For instance, the Sylvester sequence is uniquely characterized as the only sequence with $a_n^2/a_{n+1} \in [2/3, 4/3]$ whose reciprocal sum is equal to 1. This result has applications to irrationality problems. We prove that for almost every real number $\alpha > 1$, sequences asymptotic to α^{2^n} have irrational reciprocal sums. Furthermore, our observations provide heuristic insight into an open problem by Erdős and Graham.

Introduction

The asymptotic behavior of a sequence of positive integers is related to the irrationality of its reciprocal sum. It is a folklore result that if a sequence satisfies $\lim_{n\to\infty} a_n^{2^{-n}} = \infty$, then its reciprocal sum is irrational. Therefore, the largest possible asymptotic growth of a sequence for which the rationality of the reciprocal sum can be expected is C^{2^n} for some constant $C > 1$. A well-known example of such sequence is the (shifted) *Sylvester sequence* [1, A129871], which is defined by

$$s_1 = 2, \quad s_{n+1} = s_n^2 - s_n + 1.$$

It is straightforward to see that the sum of their reciprocals is 1:

$$1 = \sum_{n=1}^{\infty} \frac{1}{s_n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \cdots.$$

Also, it is known that there is a constant $c = 1.2640847\cdots$ such that $s_n \approx c^{2^n}$ (see [2, p. 109]). On the other hand, the *Millin series* [3] provides an example of a doubly exponential sequence with an irrational reciprocal sum:

$$\frac{5 - \sqrt{5}}{2} = \sum_{n=1}^{\infty} \frac{1}{F_{2^n}} = \frac{1}{1} + \frac{1}{3} + \frac{1}{21} + \frac{1}{987} + \frac{1}{2178309} + \cdots.$$

The purpose of this paper is to point out that these doubly exponential sequences are often *almost uniquely* determined by their reciprocal sum. Our main result is the following.

Theorem 1. *Let $\beta \geq 0$ be a real number, and $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers satisfying*

$$\left| \frac{a_n^2}{a_{n+1}} - \beta \right| \leq \frac{1}{3}, \quad \sum_{n=1}^{\infty} \frac{1}{a_n} = r < \infty.$$

Then, for every n satisfying $a_n \geq 8(\beta + (1/3))^2$, we have

$$a_n = \left\lfloor \left(r - \sum_{k=1}^{n-1} \frac{1}{a_k} \right)^{-1} + \beta \right\rceil,$$

where $\lfloor x \rceil = \lfloor x + (1/2) \rfloor$ is the integer closest to x. Moreover, if $\lim_{n \to \infty} a_n^2/a_{n+1} = \beta$, then we have

$$\lim_{n \to \infty} \left| \left(r - \sum_{k=1}^{n-1} \frac{1}{a_k} \right)^{-1} + \beta - a_n \right| = 0.$$

This yields the following characterizations of the Sylvester sequence and the Millin series.

Corollary 1. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers satisfying*

$$\frac{2}{3} \leq \frac{a_n^2}{a_{n+1}} \leq \frac{4}{3}, \quad \sum_{n=1}^{\infty} \frac{1}{a_n} = 1.$$

Then, we have $a_n = s_n$, where $(s_n)_{n=1}^{\infty}$ is the Sylvester sequence.

Corollary 2. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers satisfying*

$$\frac{a_n^2}{a_{n+1}} \leq \frac{2}{3}, \quad \sum_{n=1}^{\infty} \frac{1}{a_n} = \frac{5 - \sqrt{5}}{2}.$$

Then, we have $a_n = F_{2^n}$, where $(F_n)_{n=1}^{\infty}$ is the Fibonacci sequence.

Note that Corollary 1 resembles Badea's characterization of the Sylvester-like sequences [4]: if $(a_n)_{n=1}^{\infty}$ is a sequence of positive integers satisfying $a_{n+1} \geq a_n^2 - a_n + 1$ for $n \gg 1$ and $\sum_{n=1}^{\infty}(1/a_n) \in \mathbb{Q}$, then the equality $a_{n+1} = a_n^2 - a_n + 1$ holds for $n \gg 1$.

Theorem 1 has an immediate application to the study of *irrationality sequences* defined by Erdős and Straus [5] and by Erdős and Graham [6]. Following the

terminology introduced by Kovač and Tao [7], we say that an increasing sequence[1] of positive integers

$$a_1 \leq a_2 \leq a_3 \leq \cdots$$

is a *Type 2 irrationality sequence* if for every sequence of positive integers $(b_n)_{n=1}^{\infty}$ such that $a_n \approx b_n$, we have $\sum_{n=1}^{\infty}(1/b_n) \notin \mathbb{Q}$. For example, any sequence with $a_n^{2^{-n}} \to \infty$ is a Type 2 irrationality sequence by the folklore result mentioned above. On the other hand, Kovač and Tao [7] proved that any sequence satisfying $a_n^2/a_{n+1} \to \infty$ (e.g. $\lfloor 2^{(2-\varepsilon)^n} \rfloor$ for $0 < \varepsilon < 1$) cannot be a Type 2 irrationality sequence. An unsolved problem of Erdős and Graham [6, p. 63] asks whether 2^{2^n} is a Type 2 irrationality sequence; it is also listed in the website *Erdős Problems* [8, Problem #263]. As a consequence of Theorem 1, we get the following result.

Theorem 2. *Let $\mathcal{I}$ denote the following subset of $(1, \infty)$:*

$$\mathcal{I} := \{\alpha \in (1, \infty) \mid \lfloor \alpha^{2^n} \rfloor \text{ is a Type 2 irrationality sequence}\}.$$

Then, its complement $(1, \infty) \setminus \mathcal{I}$ is countable.

In the latter half of the paper, we study the following unsolved problem of Erdős and Graham.

Question 1 ([6]). Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers satisfying

$$\lim_{n \to \infty} \frac{a_n^2}{a_{n+1}} = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{a_n} \in \mathbb{Q}.$$

Is it true that $a_{n+1} = a_n^2 - a_n + 1$ holds for $n \gg 1$?

In order to state our result, we introduce the concept of *pseudo-greedy expansion*, which is a variant of the greedy expansion of a positive real number into unit fractions. For a positive real number r, its pseudo-greedy expansion is the sequence of positive integers $(a_n)_{n=1}^{\infty}$ defined by

$$a_n = \left\lfloor \left(r - \sum_{k=1}^{n-1} \frac{1}{a_k} \right)^{-1} + 1 \right\rceil,$$

where $\lfloor x \rceil = \lfloor x + (1/2) \rfloor$ is the integer closest to x. We can show that $r = \sum_{n=1}^{\infty}(1/a_n)$, so it indeed gives an expansion of r into unit fractions. We define the *gap sequence* of this expansion by

$$\varepsilon_n = \left(r - \sum_{k=1}^{n-1} \frac{1}{a_k} \right)^{-1} + 1 - a_n.$$

Then, we prove that Question 1 is equivalent to the following conjecture.

---

[1] In the original definition [6], the sequence is required to be strictly increasing, i.e., $a_1 < a_2 < a_3 < \cdots$. In this paper, we relax this condition by allowing equality, enabling us to consider sequences such as $\lfloor \alpha^{2^n} \rfloor$.

Conjecture 1. Let r be a positive rational number and $(\varepsilon_n)_{n=1}^\infty$ be the gap sequence of the pseudo-greedy expansion of r. If $\lim_{n\to\infty} \varepsilon_n = 0$, then $\varepsilon_n = 0$ holds for $n \gg 1$.

We expect that Conjecture 1 is true even without assuming $\lim_{n\to\infty} \varepsilon_n = 0$. We confirmed by a computer that Conjecture 1 holds for $r = p/q$ with $0 < p \le q \le 10^5$. In Remark 1, we provide a heuristic argument showing that Conjecture 1 is likely to be correct.

In this paper, we use the following asymptotic notations. For two sequences of real numbers $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$ with $y_n > 0$, we write:

- $x_n = O(y_n)$ if $\limsup_{n\to\infty}(|x_n|/y_n) < \infty$;

- $x_n = o(y_n)$ if $\lim_{n\to\infty}(x_n/y_n) = 0$;

- $x_n \approx y_n$ if $\lim_{n\to\infty}(x_n/y_n) = 1$.

We say that a proposition $P(n)$ is *true for $n \gg 1$* if there exists a positive integer n_0 such that $P(n)$ is true for all $n \ge n_0$.

1. Proof of the Main Results

The idea of the proof of Theorem 1 is very simple. Let $(a_n)_{n=1}^\infty$ be a sequence of positive integers such that a_n^2/a_{n+1} is sufficiently close to $\beta \ge 0$ and a_n is sufficiently large. Then, we have

$$r - \sum_{k=1}^{n-1} \frac{1}{a_k} = \frac{1}{a_n} + \frac{1}{a_{n+1}} + \cdots = \frac{1}{a_n}\left(1 + \sum_{k=1}^\infty \frac{a_n}{a_{n+k}}\right). \tag{1}$$

Using that the sum in the right-hand side is very small, we get

$$\left(r - \sum_{k=1}^{n-1} \frac{1}{a_k}\right)^{-1} \approx a_n\left(1 - \sum_{k=1}^\infty \frac{a_n}{a_{n+k}}\right) = a_n - \frac{a_n^2}{a_{n+1}} - \sum_{k=2}^\infty \frac{a_n^2}{a_{n+k}}.$$

Since the last sum is also very small and a_n^2/a_{n+1} is close to β, it follows that the left-hand side is close to $a_n - \beta$, which is what we want. We will make this precise:

Proof of Theorem 1. Let $\beta \ge 0$ be a real number, and $(a_n)_{n=1}^\infty$ be a sequence of positive integers such that

$$\left|\frac{a_n^2}{a_{n+1}} - \beta\right| \le \frac{1}{3}, \quad \sum_{n=1}^\infty \frac{1}{a_n} = r < \infty.$$

We write $\beta_+ = \beta + (1/3)$. Fix a positive integer n satisfying $a_n \geq 8\beta_+^2$. Combining with $a_n \geq 1$, we obtain $a_n^2 \geq 8\beta_+^2$ and hence $a_n \geq 2\sqrt{2}\beta_+$. In other words, we have

$$\frac{\beta_+^2}{a_n} \leq \frac{1}{8}, \quad \frac{\beta_+}{a_n} \leq \frac{1}{2\sqrt{2}}.$$

Our assumption shows that $a_{k+1} \geq a_k^2/\beta_+$ holds for all $k > 0$. Using this inequality iteratively, we obtain

$$a_{n+k} \geq \frac{a_{n+k-1}^2}{\beta_+} \geq \frac{a_{n+k-2}^4}{\beta_+^3} \geq \cdots \geq \frac{a_n^{2^k}}{\beta_+^{2^k-1}} = \beta_+ \left(\frac{a_n}{\beta_+}\right)^{2^k}$$

for all $k \geq 0$. Therefore, we have

$$A := \sum_{k=1}^{\infty} \frac{a_n}{a_{n+k}} \leq \frac{a_n}{\beta_+} \sum_{k=1}^{\infty} \left(\frac{\beta_+}{a_n}\right)^{2^k} \leq \frac{a_n}{\beta_+} \sum_{k=1}^{\infty} \left(\frac{\beta_+}{a_n}\right)^{2k} \tag{2}$$

$$= \frac{\beta_+}{a_n} \cdot \frac{1}{1 - (\beta_+/a_n)^2} \leq \frac{\beta_+}{a_n} \cdot \frac{1}{1 - (1/8)} = \frac{8}{7} \cdot \frac{\beta_+}{a_n}.$$

Similarly, we have

$$B := \sum_{k=2}^{\infty} \frac{a_n^2}{a_{n+k}} \leq \frac{a_n^2}{\beta_+} \sum_{k=2}^{\infty} \left(\frac{\beta_+}{a_n}\right)^{2^k} \leq \frac{a_n^2}{\beta_+} \sum_{k=1}^{\infty} \left(\frac{\beta_+}{a_n}\right)^{4k} \tag{3}$$

$$= \frac{\beta_+^2}{a_n} \cdot \frac{\beta_+}{a_n} \cdot \frac{1}{1 - (\beta_+/a_n)^4} \leq \frac{2\sqrt{2}}{63}.$$

Now we use Equation (1). Taking the reciprocal, we get

$$\left(r - \sum_{k=1}^{n-1} \frac{1}{a_k}\right)^{-1} = \frac{a_n}{1 + A} = a_n - a_n A + \frac{a_n A^2}{1 + A} \tag{4}$$

$$= a_n - \frac{a_n^2}{a_{n+1}} - B + \frac{a_n A^2}{1 + A}.$$

By Equation (3), we have $B < (1/6)$. On the other hand, by Equation (2), we have

$$\frac{a_n A^2}{1 + A} \leq a_n A^2 \leq \frac{64}{49} \cdot \frac{\beta_+^2}{a_n} \leq \frac{8}{49} < \frac{1}{6}.$$

Therefore, we have

$$\left| \left(r - \sum_{k=1}^{n-1} \frac{1}{a_k}\right)^{-1} + \frac{a_n^2}{a_{n+1}} - a_n \right| = \left| -B + \frac{a_n A^2}{1 + A} \right| < \frac{1}{6}.$$

Finally, by our assumption that $|(a_n^2/a_{n+1}) - \beta| \le (1/3)$, we obtain

$$\left| \left(r - \sum_{k=1}^{n-1} \frac{1}{a_k} \right)^{-1} + \beta - a_n \right| < \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$

This shows that a_n is the integer closest to $(r - \sum_{k=1}^{n-1}(1/a_k))^{-1} + \beta$.

Suppose, moreover, that $\lim_{n\to\infty}(a_n^2/a_{n+1}) = \beta$. Equation (2) and Equation (3) shows that $a_n A^2$ and B converge to 0 as $n \to \infty$. Therefore, by Equation (4), we have

$$\left(r - \sum_{k=1}^{n-1} \frac{1}{a_k} \right)^{-1} = a_n - \frac{a_n^2}{a_{n+1}} + o(1) = a_n - \beta + o(1).$$

In other words, the difference $|(r - \sum_{k=1}^{n-1}(1/a_k))^{-1} + \beta - a_n|$ converges to 0 as $n \to \infty$. $\qquad\square$

Proof of Corollary 1. First, we note that the Sylvester sequence satisfies

$$\sum_{k=1}^{n-1} \frac{1}{s_k} = 1 - \frac{1}{s_n - 1},$$

by the recurrence relation $s_{n+1} = s_n^2 - s_n + 1$. Therefore, we also have

$$s_n = \left(1 - \sum_{k=1}^{n-1} \frac{1}{s_k} \right)^{-1} + 1. \qquad (5)$$

Let $(a_n)_{n=1}^{\infty}$ be another sequence of positive integers satisfying

$$\frac{2}{3} \le \frac{a_n^2}{a_{n+1}} \le \frac{4}{3}, \quad \sum_{n=1}^{\infty} \frac{1}{a_n} = 1.$$

Applying Theorem 1 with $\beta = 1$, we obtain

$$a_n \ge 15 \quad \text{implies} \quad a_n = \left\lfloor \left(1 - \sum_{k=1}^{n-1} \frac{1}{a_k} \right)^{-1} + 1 \right\rceil. \qquad (6)$$

Since $\sum_{n=1}^{\infty}(1/a_n) = 1$, we have $a_1 \ge 2$. Using the inequality $a_{n+1} \ge (3/4)a_n^2$ inductively, we obtain

$$a_1 \ge 2, \quad a_2 \ge 3, \quad a_3 \ge 7, \quad a_4 \ge 37.$$

In particular, Equation (6) implies

$$37 \le a_4 = \left\lfloor \left(1 - \sum_{k=1}^{3} \frac{1}{a_k} \right)^{-1} + 1 \right\rceil,$$

and hence

$$\sum_{k=1}^{3} \frac{1}{a_k} \geq \frac{69}{71} > \frac{1}{2} + \frac{1}{3} + \frac{1}{8}.$$

Therefore, we must have $a_1 = 2$, $a_2 = 3$, $a_3 = 7$, and the rest of the sequence is determined by Equation (6). Comparing this with Equation (5), we conclude that $(a_n)_{n=1}^{\infty}$ coincides with the Sylvester sequence. $\square$

Proof of Corollary 2. Let $\phi = (1+\sqrt{5})/2$ and $\overline{\phi} = (1-\sqrt{5})/2$. Then, the Fibonacci sequence can be written as $F_n = (\phi^n - \overline{\phi}^n)/\sqrt{5}$, and hence we have

$$\frac{F_{2^n}^2}{F_{2^{n+1}}} = \frac{1}{\sqrt{5}} \cdot \frac{\phi^{2^{n+1}} - 2 + \overline{\phi}^{2^{n+1}}}{\phi^{2^{n+1}} - \overline{\phi}^{2^{n+1}}} \leq \frac{1}{\sqrt{5}} \leq \frac{2}{3}.$$

Applying Theorem 1 with $\beta = 1/3$, we obtain for $n \geq 3$ that

$$F_{2^n} = \left\lfloor \left(\frac{5-\sqrt{5}}{2} - \sum_{k=1}^{n-1} \frac{1}{F_{2^k}} \right)^{-1} + \frac{1}{3} \right\rfloor. \tag{7}$$

Let $(a_n)_{n=1}^{\infty}$ be another sequence of positive integers satisfying

$$\frac{a_n^2}{a_{n+1}} \leq \frac{2}{3}, \quad \sum_{n=1}^{\infty} \frac{1}{a_n} = \frac{5-\sqrt{5}}{2}.$$

Applying Theorem 1 with $\beta = (1/3)$, we obtain

$$a_n \geq 4 \quad \text{implies} \quad a_n = \left\lfloor \left(\frac{5-\sqrt{5}}{2} - \sum_{k=1}^{n-1} \frac{1}{a_k} \right)^{-1} + \frac{1}{3} \right\rfloor. \tag{8}$$

Using the inequality $a_{n+1} \geq (3/2)a_n^2$ inductively, we obtain

$$a_1 \geq 1, \quad a_2 \geq 2, \quad a_3 \geq 6, \quad a_4 \geq 54.$$

In particular, Equation (8) implies

$$54 \leq a_4 = \left\lfloor \left(\frac{5-\sqrt{5}}{2} - \sum_{k=1}^{3} \frac{1}{a_k} \right)^{-1} + \frac{1}{3} \right\rfloor,$$

and hence

$$\sum_{k=1}^{3} \frac{1}{a_k} \geq \frac{1583 - 319\sqrt{5}}{638} = 1.3631572\cdots. \tag{9}$$

This easily implies that $a_1 = 1$ and $a_2 = 3$. The rest of the sequence is determined by Equation (8). Comparing this with Equation (7), we conclude that $(a_n)_{n=1}^{\infty}$ coincides with $(F_{2^n})_{n=1}^{\infty}$. $\square$

Proof of Theorem 2. By definition, $(1, \infty) \setminus \mathcal{I}$ consists of real numbers $\alpha > 1$ such that there exists a sequence of positive integers $(a_n)_{n=1}^\infty$ satisfying

$$a_n \approx \alpha^{2^n} \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{a_n} \in \mathbb{Q}.$$

Let $\mathcal{S}$ denote the set of all such sequences, i.e.,

$$\mathcal{S} := \bigcup_{\alpha > 1} \left\{ (a_n)_{n=1}^\infty \,\middle|\, a_n \approx \alpha^{2^n}, \ \sum_{n=1}^\infty \frac{1}{a_n} \in \mathbb{Q} \right\}.$$

Then, there is a surjective map

$$\mathcal{S} \to (1, \infty) \setminus \mathcal{I}; \quad (a_n)_{n=1}^\infty \mapsto \lim_{n \to \infty} a_n^{2^{-n}}.$$

It suffices to show that $\mathcal{S}$ is countable. Let $(a_n)_{n=1}^\infty$ be an element of $\mathcal{S}$. Then, we have $\lim_{n \to \infty} (a_n^2/a_{n+1}) = 1$ and $\lim_{n \to \infty} a_n = \infty$. By Theorem 1, there exists an integer $n_0 > 0$ such that

$$n \geq n_0 \quad \text{implies} \quad a_n = \left\lfloor \left(r - \sum_{k=1}^{n-1} \frac{1}{a_k} \right)^{-1} + 1 \right\rfloor,$$

where $r = \sum_{n=1}^\infty (1/a_n)$. In particular, the sequence $(a_n)_{n=1}^\infty$ is uniquely determined by the tuple $(a_1, \ldots, a_{n_0-1}, r)$. Since the set of all such tuples is countable, it follows that $\mathcal{S}$ is countable. $\square$

2. Pseudo-Greedy Expansion

Given a positive real number r, there are many possible ways to express r as a (possibly infinite) sum of unit fractions. The simplest choice is the *greedy expansion*, which is given by

$$r = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots, \quad a_n = \left\lceil \left(r - \sum_{k=1}^{n-1} \frac{1}{a_k} \right)^{-1} \right\rceil.$$

It can be easily shown that the greedy expansion of any positive rational number terminates in finite steps. Another interesting choice is the *odd greedy expansion*, where a_n is defined to be the smallest odd number greater than or equal to $(r - \sum_{k=1}^{n-1} (1/a_k))^{-1}$. It is an open problem whether the odd greedy expansion of a positive rational number with odd denominator terminates in finite steps (see [9, p. 88]).

Motivated by Theorem 1, we study the following variant of the greedy expansion.

Definition 1. Let r be a positive real number. Define a sequence of positive integers $(a_n)_{n=1}^\infty$ by

$$a_n = \left\lfloor \left(r - \sum_{k=1}^{n-1} \frac{1}{a_k} \right)^{-1} + 1 \right\rfloor.$$

We call $(a_n)_{n=1}^\infty$ the *pseudo-greedy expansion* of r.

Lemma 1. *Let r be a positive real number and $(a_n)_{n=1}^\infty$ be its pseudo-greedy expansion. Then, we have $r = \sum_{n=1}^\infty (1/a_n)$.*

Proof. We define a sequence of positive real numbers $(x_n)_{n=1}^\infty$ by $x_n = r - \sum_{k=1}^{n-1}(1/a_k)$. Since $(x_n)_{n=1}^\infty$ is decreasing, we have $\lim_{n\to\infty} x_n = \gamma$ for some real number $\gamma \geq 0$. By definition, we have $a_n = \lfloor x_n^{-1} + 1 \rfloor \leq x_n^{-1} + 2$ and hence

$$x_{n+1} = x_n - \frac{1}{a_n} \leq x_n - \frac{1}{x_n^{-1} + 2} = x_n \left(1 - \frac{1}{1 + 2x_n} \right).$$

Taking the limit as $n \to \infty$, we obtain

$$\gamma \leq \gamma \left(1 - \frac{1}{1 + 2\gamma} \right)$$

and hence $\gamma = 0$. Therefore, we have $r = \sum_{n=1}^\infty (1/a_n)$. $\qquad\square$

Definition 2. Let r be a positive real number and $(a_n)_{n=1}^\infty$ be its pseudo-greedy expansion. We define a sequence of positive real numbers $(x_n)_{n=1}^\infty$ by

$$x_n = r - \sum_{k=1}^{n-1} \frac{1}{a_k}.$$

We call $(x_n)_{n=1}^\infty$ the *remainder sequence* of the pseudo-greedy expansion of r. By definition, we have $a_n = \lfloor x_n^{-1} + 1 \rfloor$. We also define a sequence of real numbers $(\varepsilon_n)_{n=1}^\infty$ by

$$\varepsilon_n = x_n^{-1} + 1 - a_n.$$

We call $(\varepsilon_n)_{n=1}^\infty$ the *gap sequence* of the pseudo-greedy expansion of r.

As an immediate consequence of Theorem 1, we obtain the following result.

Corollary 3. *Let $(a_n)_{n=1}^\infty$ be a sequence of positive integers satisfying*

$$\lim_{n\to\infty} \frac{a_n^2}{a_{n+1}} = 1 \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{a_n} < \infty.$$

Then, there is some integer $N \geq 0$ such that $(a_{N+n})_{n=1}^\infty$ is the pseudo-greedy expansion of $\sum_{n=1}^\infty (1/a_{N+n})$. Moreover, the gap sequence of this expansion satisfies $\lim_{n\to\infty} \varepsilon_n = 0$.

Example 1. The pseudo-greedy expansion of 1 is given by the Sylvester sequence:

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \cdots .$$

More generally, for any integer $m > 0$, the pseudo-greedy expansion of $1/m$ is given by $a_n = s_n(m)$, where

$$s_1(m) = m + 1, \quad s_{n+1}(m) = s_n(m)^2 - s_n(m) + 1.$$

The remainder/gap sequences are given by $x_n = 1/(s_n(m) - 1)$, $\varepsilon_n = 0$. It is known that there is a constant $c(m) > 1$ such that $s_n(m) \approx c(m)^{2^n}$ (see [2, p. 109]). Wagner and Ziegler [10] showed that $c(m)$ is irrational, and Dubickas [11] showed that $c(m)$ is transcendental.

The recurrence relation $a_{n+1} = a_n^2 - a_n + 1$ appearing in Example 1 is a special case of the following lemma.

Lemma 2. *Let r be a positive real number, and $(a_n)_{n=1}^\infty$, $(\varepsilon_n)_{n=1}^\infty$ be its pseudo-greedy expansion and the gap sequence. Then, we have*

$$a_{n+1} = \frac{1}{1 - \varepsilon_n} a_n^2 - a_n + (1 - \varepsilon_{n+1}).$$

Proof. By definition, we have $x_n^{-1} = a_n - (1 - \varepsilon_n)$ and hence

$$x_{n+1} = x_n - \frac{1}{a_n} = \frac{1}{a_n - (1 - \varepsilon_n)} - \frac{1}{a_n} = \frac{1 - \varepsilon_n}{a_n^2 - (1 - \varepsilon_n)a_n}.$$

Substituting this into $a_{n+1} = x_{n+1}^{-1} + 1 - \varepsilon_{n+1}$, we obtain the desired formula. $\square$

Lemma 3. *Let r be a positive real number, and $(\varepsilon_n)_{n=1}^\infty$ be the gap sequence of the pseudo-greedy expansion of r. If $\varepsilon_n = 0$ for some $n > 0$, then $\varepsilon_{n+1} = 0$.*

Proof. Let $(a_n)_{n=1}^\infty$, $(x_n)_{n=1}^\infty$ be the pseudo-greedy expansion of r and its remainder sequence. If $\varepsilon_n = 0$, then x_n^{-1} is an integer, so we can write $x_n = 1/m$ for some positive integer m. Then, we have $a_n = m + 1$ and hence

$$x_{n+1} = x_n - \frac{1}{a_n} = \frac{1}{m(m+1)}.$$

This shows that x_{n+1}^{-1} is an integer and thus $\varepsilon_{n+1} = 0$. $\square$

Example 2. The pseudo-greedy expansion of $11/29$ is given in Table 1. We see that $(a_5, a_6, \ldots)$ is the pseudo-greedy expansion of $1/10684296$, which is given as in Example 1. In other words, we have $\varepsilon_5 = \varepsilon_6 = \cdots = 0$.

n	1	2	3	4	5	6	$\cdots$
x_n	$\dfrac{11}{29}$	$\dfrac{15}{116}$	$\dfrac{19}{1044}$	$\dfrac{5}{14616}$	$\dfrac{1}{10684296}$	$\dfrac{1}{114154191699912}$	$\cdots$
a_n	4	9	56	2924	10684297	114154191699913	$\cdots$
ε_n	$-\dfrac{4}{11}$	$-\dfrac{4}{15}$	$-\dfrac{1}{19}$	$\dfrac{1}{5}$	0	0	$\cdots$

Table 1: The pseudo-greedy expansion of $11/29$

We conjecture that for any positive rational number r, the gap sequence of the pseudo-greedy expansion satisfies $\varepsilon_n = 0$ for $n \gg 1$, at least when $\varepsilon_n \to 0$ (Conjecture 1). We confirmed by a computer that this conjecture is true for $r = p/q$ with $0 < p \le q \le 10^5$. Note that this conjecture resembles the termination problem of the odd greedy expansion.

Let us take a closer look at the pseudo-greedy expansion of positive rational numbers.

Lemma 4. *Let* $r = p/q$ *be a positive rational number, and* $(a_n)_{n=1}^{\infty}$, $(x_n)_{n=1}^{\infty}$, $(\varepsilon_n)_{n=1}^{\infty}$ *be its pseudo-greedy expansion, the remainder sequence, and the gap sequence, respectively.*

1. *Let* $d_n = q a_1 a_2 \cdots a_{n-1}$. *Then we can write*

$$x_n = \frac{c_n}{d_n}, \quad \varepsilon_n = \frac{e_n}{c_n},$$

where c_n *is a positive integer and* e_n *is an integer.*

2. *The pair* (c_n, d_n) *determines* (e_n, a_n) *by*

$$\begin{cases} e_n \equiv d_n \pmod{c_n}, \quad -\dfrac{c_n}{2} \le e_n < \dfrac{c_n}{2}, \\ a_n = \dfrac{d_n - e_n}{c_n} + 1. \end{cases}$$

The tuple (c_n, d_n, e_n, a_n) *determines* (c_{n+1}, d_{n+1}) *by*

$$\begin{cases} c_{n+1} = c_n - e_n, \\ d_{n+1} = d_n a_n. \end{cases}$$

3. *We have the following asymptotic estimates:*

$$c_n = O\left(\frac{a_1 a_2 \cdots a_{n-1}}{a_n}\right), \quad c_n = O(1.5^n).$$

Proof. 1. By definition, we have

$$x_n = \frac{p}{q} - \sum_{k=1}^{n-1} \frac{1}{a_k} \in \frac{1}{qa_1 \cdots a_{n-1}} \mathbb{Z}.$$

Therefore, we can write $x_n = c_n/d_n$ for some positive integer c_n. The formula $\varepsilon_n = x_n^{-1} + 1 - a_n$ implies that $\varepsilon_n \in (1/c_n)\mathbb{Z}$, so we can write $\varepsilon_n = e_n/c_n$ for some integer e_n.

2. The definition of ε_n can be reformulated as $\varepsilon_n = x_n^{-1} - \lfloor x_n^{-1} \rceil$. Therefore, ε_n is characterized by

$$\varepsilon_n \equiv x_n^{-1} \pmod{1}, \quad -\frac{1}{2} \le \varepsilon_n < \frac{1}{2}.$$

Since $\varepsilon_n = e_n/c_n$, this yields the characterization of e_n. The formula for a_n follows from $a_n = x_n^{-1} - \varepsilon_n + 1$. The equality $d_{n+1} = d_n a_n$ is immediate from the definition. Finally, the formula for c_{n+1} follows from

$$x_{n+1} = x_n - \frac{1}{a_n} = \frac{c_n a_n - d_n}{d_n a_n} = \frac{c_n - e_n}{d_{n+1}}.$$

3. The first estimate follows from $d_n = qa_1 a_2 \cdots a_{n-1}$ and

$$a_n = \lfloor x_n^{-1} + 1 \rceil \approx x_n^{-1} = \frac{d_n}{c_n}.$$

The second one follows from $c_{n+1} = c_n - e_n \le 1.5 c_n$. $\qquad\square$

Now we prove that Question 1 by Erdős and Graham is equivalent to our Conjecture 1.

Theorem 3. *Conjecture 1 is true if and only if Question 1 has an affirmative answer.*

Proof. First, suppose that Conjecture 1 is true. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers satisfying

$$\lim_{n \to \infty} \frac{a_n^2}{a_{n+1}} = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{a_n} \in \mathbb{Q}.$$

By Corollary 3, we may assume that $(a_n)_{n=1}^{\infty}$ is the pseudo-greedy expansion of some positive rational number r, and that its gap sequence satisfies $\lim_{n\to\infty} \varepsilon_n = 0$. By our assumption that Conjecture 1 is true, we have $\varepsilon_n = 0$ for $n \gg 1$. The formula

$$a_{n+1} = \frac{1}{1 - \varepsilon_n} a_n^2 - a_n + (1 - \varepsilon_{n+1}) \tag{10}$$

given in Lemma 2 shows that $a_{n+1} = a_n^2 - a_n + 1$ for $n \gg 1$.

Conversely, suppose that Question 1 has an affirmative answer. Let r be a positive rational number and $(a_n)_{n=1}^\infty$ be its pseudo-greedy expansion. We define the quantities c_n and e_n as in Lemma 4. Assume that the gap sequence satisfies $\lim_{n\to\infty} \varepsilon_n = 0$. Then, Equation (10) shows that

$$\lim_{n\to\infty} \frac{a_n^2}{a_{n+1}} = 1.$$

Therefore, by our assumption that Question 1 has an affirmative answer, we have $a_{n+1} = a_n^2 - a_n + 1$ for $n \gg 1$. Comparing this with Equation (10), we obtain

$$\frac{\varepsilon_n a_n^2}{1 - \varepsilon_n} = \varepsilon_{n+1} = o(1).$$

In particular, we have $\varepsilon_n = o(a_n^{-2})$. Combining this with the estimate $c_n = O(1.5^n)$ given in Lemma 4 (3), we obtain $e_n = c_n \varepsilon_n = o(1)$. Since e_n is an integer, we see that $e_n = 0$ holds for $n \gg 1$. This shows that Conjecture 1 is true. $\qquad\square$

Remark 1. Using Lemma 4, we can provide a heuristic argument showing that Conjecture 1 is likely to be correct even without assuming $\lim_{n\to\infty} \varepsilon_n = 0$. Let r be a positive rational number, and define c_n, d_n, and e_n as in Lemma 4. By Lemma 4 (2), we have $c_{n+1} = c_n - e_n$ and hence

$$\frac{c_{n+1}}{c_n} = 1 - \frac{e_n}{c_n}, \quad -\frac{1}{2} \leq \frac{e_n}{c_n} < \frac{1}{2}.$$

Thus the behavior of $(c_n)_{n=1}^\infty$ can be modeled by the multiplicative random walk $c_{n+1} = t_n c_n$, where t_n is chosen uniformly randomly from $[1/2, 3/2)$. Since we have

$$\mathbb{E}[\log t_n] = \int_{1/2}^{3/2} \log t \; dt = \frac{3}{2} \log 3 - \log 2 - 1 = -0.0452287\cdots < 0,$$

c_n tends to shrink exponentially on average, so it is natural to expect that $\varepsilon_n = 0$ holds for some n.

Remark 2. When actually computing the values of ε_n for a given rational number r, directly performing the calculation will cause the values of a_n and d_n to grow explosively large. We can use modular arithmetic to avoid this problem. When the values of $c_1, \ldots, c_n$ and $e_1, \ldots, e_{n-1}$ are known, one can use the recurrence relation from Lemma 4 (2) to inductively compute, for $k = 1, 2, \ldots, n-1$, the residue classes

$$a_k \pmod{c_{k+1} c_{k+2} \cdots c_n} \quad \text{and} \quad d_{k+1} \pmod{c_{k+1} c_{k+2} \cdots c_n}.$$

In particular, we can compute $d_n \pmod{c_n}$, and this value can then be used to determine e_n and c_{n+1}. We provide below a pseudocode for computing the gap sequence $(\varepsilon_n)_{n=1}^\infty$ of the pseudo-greedy expansion of a positive rational number $r = p/q$.

Algorithm 1 Pseudocode for computing $(\varepsilon_n)_{n=1}^{\infty}$ for $r = p/q$

1: $c_1 \leftarrow p$
2: **for all** $n \leftarrow 1$ to $n_{\max}$ **do**
3:      $d_1 \leftarrow q$
4:      **for all** $k \leftarrow 1$ to $n-1$ **do**
5:          $a_k \leftarrow ((d_k - e_k)/c_k) + 1 \pmod{c_{k+1} \cdots c_n}$
6:          $d_{k+1} \leftarrow d_k a_k \pmod{c_{k+1} \cdots c_n}$
7:      **end for**
8:      $e_n \leftarrow d_n - c_n \lfloor d_n/c_n \rceil$
9:      $c_{n+1} \leftarrow c_n - e_n$
10:      $\varepsilon_n \leftarrow e_n/c_n$
11:      Print ε_n
12: **end for**

Finally, we reinterpret the partial results on Question 1 due to Erdős and Straus [12] and Badea [4] within our framework. In terms of the pseudo-greedy expansion, their results can be regarded as the following special cases of Conjecture 1.

Proposition 1. *Let r be a positive rational number, and $(\varepsilon_n)_{n=1}^{\infty}$ be the gap sequence of the pseudo-greedy expansion of r. Suppose that one of the following conditions is satisfied:*

1. $\liminf_{n \to \infty} \varepsilon_n \prod_{k=1}^{n-1}(1 - \varepsilon_k) \geq 0$.

2. $\varepsilon_n \geq 0$ holds for $n \gg 1$.

Then, we have $\varepsilon_n = 0$ for $n \gg 1$.

Proof. We define the quantities c_n and e_n as in Lemma 4.

1. By Lemma 4 (2), we have $c_{n+1} = c_n - e_n$ and hence

$$
e_n = \varepsilon_n c_n = \varepsilon_n c_1 \prod_{k=1}^{n-1} \left(1 - \frac{e_k}{c_k} \right) = c_1 \cdot \varepsilon_n \prod_{k=1}^{n-1}(1 - \varepsilon_k).
$$

Our assumption shows that $\liminf_{n \to \infty} e_n \geq 0$. Since e_n is an integer, we conclude that $e_n \geq 0$ holds for $n \gg 1$. Therefore, this case is reduced to (2).

2. By Lemma 4 (2), we have $c_{n+1} = c_n - e_n$ and hence $(c_n)_{n=1}^{\infty}$ is eventually non-increasing. Since c_n is a positive integer, $(c_n)_{n=1}^{\infty}$ is eventually constant and thus $e_n = 0$ holds for $n \gg 1$. $\square$

Corollary 4 ([4, 12]). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers satisfying*

$$
\lim_{n \to \infty} \frac{a_n^2}{a_{n+1}} = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{a_n} \in \mathbb{Q}.
$$

Suppose that one of the following conditions is satisfied:

1. $\liminf_{n\to\infty} \dfrac{a_1 a_2 \cdots a_{n-1}}{a_n}\left(1 - \dfrac{a_n^2}{a_{n+1}}\right) \geq 0.$

2. $a_{n+1} \geq a_n^2 - a_n + 1$ *holds for* $n \gg 1.$

Then, $a_{n+1} = a_n^2 - a_n + 1$ *holds for* $n \gg 1.$

Proof. By Corollary 3, we may assume that $(a_n)_{n=1}^\infty$ is the pseudo-greedy expansion of some positive rational number r, and that its gap sequence satisfies $\lim_{n\to\infty} \varepsilon_n = 0$. By Lemma 2, it suffices to show that $\varepsilon_n = 0$ holds for $n \gg 1$.

1. By Lemma 2, we have $a_{n+1} = (a_n^2/(1 - \varepsilon_n))(1 + O(a_n^{-1}))$, and hence we can write

$$\frac{a_n^2}{a_{n+1}} = (1 - \varepsilon_n)(1 + \beta_n), \tag{11}$$

where $\beta_n = O(a_n^{-1})$. We can write $a_1 a_2 \cdots a_{n-1}/a_n$ as

$$\frac{a_1 a_2 \cdots a_{n-1}}{a_n} = \frac{1}{a_1}\prod_{k=1}^{n-1}\frac{a_k^2}{a_{k+1}} = \frac{1}{a_1}\prod_{k=1}^{n-1}(1 - \varepsilon_k)\prod_{k=1}^{n-1}(1 + \beta_k). \tag{12}$$

The infinite product $\prod_{k=1}^\infty(1+\beta_k)$ converges since $\beta_k = O(a_k^{-1})$. In particular, the quantity

$$B_n := \frac{1}{a_1}\prod_{k=1}^{n}(1 + \beta_k) > 0$$

is bounded. Combining Equation (11) with Equation (12), we obtain

$$\frac{a_1 a_2 \cdots a_{n-1}}{a_n}\left(1 - \frac{a_n^2}{a_{n+1}}\right) = B_{n-1}\varepsilon_n\prod_{k=1}^{n-1}(1 - \varepsilon_k) - B_{n-1}\beta_n\prod_{k=1}^{n}(1 - \varepsilon_k).$$

The second term of the right-hand side is $o(1)$ because $\beta_n = O(a_n^{-1})$ and $|\varepsilon_k| \leq (1/2)$. Thus, our assumption is equivalent to $\liminf_{n\to\infty} \varepsilon_n \prod_{k=1}^{n-1}(1-\varepsilon_k) \geq 0$. By Proposition 1 (1), we conclude that $\varepsilon_n = 0$ for $n \gg 1$.

2. By Lemma 2, our assumption can be reformulated as

$$\frac{\varepsilon_n a_n^2}{1 - \varepsilon_n} \geq \varepsilon_{n+1} \quad (n \gg 1).$$

In particular, for sufficiently large n, we have

$$\varepsilon_n < 0 \quad \text{implies} \quad \varepsilon_n > \varepsilon_{n+1}.$$

Suppose that we have $\varepsilon_n < 0$ for infinitely many n. Then, the above implication shows that ε_n is decreasing for $n \gg 1$. This contradicts the fact that ε_n converges to 0. Therefore, we have $\varepsilon_n \geq 0$ for $n \gg 1$. By Proposition 1 (2), we conclude that $\varepsilon_n = 0$ for $n \gg 1$. $\square$

Remark 3. The aforementioned result of Erdős and Straus solves Question 1 under the condition

$$\frac{a_n^2}{a_{n+1}} = 1 + o(n^{-1}).$$

Indeed, if we write $(a_n^2/a_{n+1}) = 1 - \gamma_n$ with $\gamma_n = o(n^{-1})$, then we have

$$\frac{a_1 a_2 \cdots a_{n-1}}{a_n}\left(1 - \frac{a_n^2}{a_{n+1}}\right) = \frac{1}{a_1}\left(1 - \frac{a_n^2}{a_{n+1}}\right)\prod_{k=1}^{n-1}\frac{a_k^2}{a_{k+1}} = \frac{\gamma_n}{a_1}\prod_{k=1}^{n-1}(1 - \gamma_k) = o(1),$$

so we can apply Corollary 4 (1).

Remark 4. Suppose that Question 1 has an affirmative answer, or equivalently, that Conjecture 1 is true. Recall the set $\mathcal{I}$ from Theorem 2, and let $\alpha \in (1, \infty) \setminus \mathcal{I}$. By definition, there is a sequence of positive integers $(a_n)_{n=1}^\infty$ satisfying $a_n \approx \alpha^{2^n}$ whose reciprocal sum is rational. By our assumption, $(a_n)_{n=1}^\infty$ eventually follows the recurrence relation

$$a_{n+1} = a_n^2 - a_n + 1.$$

In other words, there is some integer $N \geq 0$ and a positive integer m such that $a_{N+n} = s_n(m)$, where $s_n(m)$ is the sequence defined in Example 1. In particular, α is given by

$$\alpha = \lim_{n\to\infty} a_{N+n}^{2^{-(N+n)}} = \lim_{n\to\infty} s_n(m)^{2^{-(N+n)}} = c(m)^{2^{-N}},$$

where $c(m)$ is the constant defined in Example 1. Therefore, assuming that Question 1 has an affirmative answer, we can conclude that

$$(1, \infty) \setminus \mathcal{I} = \{c(m)^{2^{-N}} \mid N \geq 0,\ m > 0\}.$$

Since $c(m)$ is transcendental by the result of Dubickas [11], this implies that $\overline{\mathbb{Q}} \cap (1, \infty) \subset \mathcal{I}$. In particular, an affirmative answer to Question 1 will imply that 2^{2^n} is a Type 2 irrationality sequence.

References

[1] OEIS Foundation Inc, The on-line encyclopedia of integer sequences, `http://oeis.org/`.

[2] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley Publishing Company, 1989.

[3] D. A. Miller, Problem h-237, *Fibonacci Quart.* **12** (3) (1974), 309.

[4] C. Badea, A theorem on irrationality of infinite series and applications, *Acta Arithmetica* **63** (4) (1993), 313–323.

[5] P. Erdős, Some problems and results on the irrationality of the sum of infinite series, *J. Math. Sci.* **10** (1975), 1–7.

[6] P. Erdős, R. L. Graham, *Old and new problems and results in combinatorial number theory*, Université de Genève, L'Enseignement Mathématique, Geneva, 1980.

[7] V. Kovač and T. Tao, On several irrationality problems for ahmes series, *Acta Math. Hungar.* **175** (2025), 572–608.

[8] T. Bloom, Erdős problems, `https://www.erdosproblems.com/`.

[9] R. K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, 1981.

[10] S. Wagner and V. Ziegler, Irrationality of growth constants associated with polynomial recursions, *J. Integer Seq.* **24** (1) (2021), Article 21.1.6.

[11] A. Dubickas, Transcendency of some constants related to integer sequences of polynomial iterations, *Ramanujan J.* **57** (2) (2022), 569 – 581.

[12] P. Erdős and E. G. Straus, On the irrationality of certain ahmes series, *J. Indian Math. Soc.* **27** (1963), 129–133.