A NOTE ON THE EVALUATION OF TWO ARITHMETIC FUNCTIONS OF KRONECKER

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Abstract

We give explicit evaluations of two arithmetic functions of Kronecker, $F_o(n)$ and $F_e(n)$, for all $n \in \mathbb{N}$ in terms of the class number $h(d)$ for a suitable discriminant d depending upon n.

1. Introduction and the Main Result

As usual, we let $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ denote the sets of positive integers, nonnegative integers, and all integers, respectively.

Let $ax^2+bxy+cy^2$ be an integral binary quadratic form which is positive-definite, so that a is positive and its discriminant $d = b^2 - 4ac$ is negative. Clearly, $d \equiv 0, 1 \pmod 4$. The quadratic form $ax^2 + bxy + cy^2$ is said to be *reduced* if a, b, and c satisfy

$$-a < b \leq a < c \text{ or } 0 \leq b \leq a = c.$$

The set of reduced positive-definite binary quadratic forms of discriminant d is denoted by $R(d)$. The set $R(d)$ is nonempty as it contains the form $x^2 - (d/4)y^2$ if $d \equiv 0 \pmod 4$ and the form $x^2 + xy + ((1-d)/4)y^2$ if $d \equiv 1 \pmod 4$. For a fixed discriminant d, the set $R(d)$ is finite as every form in $R(d)$ satisfies

$$1 \leq a \leq \sqrt{|d|/3},\ |b| \leq \sqrt{|d|/3},\ c = (b^2-d)/(4a).$$

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The *Hurwitz class number* $H(d)$ is given by

$$H(d) := \sum_{ax^2+bxy+cy^2 \in R(d)} 1 \tag{1}$$

and the *Gauss class number* $h(d)$ by

$$h(d) := \sum_{\substack{ax^2+bxy+cy^2 \in R(d) \\ \gcd(a,b,c)=1}} 1.$$

We remark that there are differences in the definition of the Hurwitz class number in the literature. Collecting together the terms in (1) having a common value k for $\gcd(a, b, c)$, we deduce that

$$H(d) = \sum_{k^2 \mid d} h(d/k^2). \tag{2}$$

The *weight* of $ax^2 + bxy + cy^2 \in R(d)$ is denoted by $wt(ax^2 + bxy + cy^2)$ and is defined by

$$wt(ax^2 + bxy + cy^2) = \begin{cases} 1/2 & \text{if } a = c,\ b = 0, \\ 1/3 & \text{if } a = b = c, \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

For $n \in \mathbb{N}$, $d = -4n$ is a discriminant and we define

$$F_o(n) := \sum_{\substack{ax^2+bxy+cy^2 \in R(-4n) \\ a \text{ or } c \equiv 1 \pmod 2}} wt(ax^2 + bxy + cy^2), \tag{4}$$

and

$$F_e(n) := \sum_{\substack{ax^2+bxy+cy^2 \in R(-4n) \\ a \equiv c \equiv 0 \pmod 2}} wt(ax^2 + bxy + cy^2). \tag{5}$$

In this notation, the subscript 'o' indicates that at least one of a and c is odd, and the subscript 'e' that a and c are both even. These sums occur as $F(n)(= F_o(n))$ and $G(n)(= F_o(n) + F_e(n))$ in the seminal work of Kronecker [5, p. 187] on class numbers of binary quadratic forms, and as $F(n)(= F_o(n))$ and $F_1(n)(= F_e(n))$ in the papers of Bell [2, 3]. The following properties of $F_o(n)$ and $F_e(n)$ are well-known and can be found in Kronecker [5, p. 189] and Bell [2, p. 127], [3, p. 494]:

$$F_e(n) = 0 \text{ if } n \equiv 1 \pmod 4, \tag{6}$$

$$F_e(n) = 0 \text{ if } n \equiv 2 \pmod 4, \tag{7}$$

$$F_o(n) = 3F_e(n) \text{ if } n \equiv 3 \pmod 8, \tag{8}$$

$$F_o(4n) = 2F_o(n), \tag{9}$$

$$F_e(4n) = F_o(n) + F_e(n). \tag{10}$$

An important relation between $F_o(n)$, $F_e(n)$, and the number $r_3(n)$ of representations of n as a sum of three squares was found by Gauss [4, Arts. 292, 293], namely

$$F_o(n) - F_e(n) = \frac{1}{12} r_3(n). \tag{11}$$

By repeated application of (9) and (10), and the use of (7), we obtain for $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, and $\ell \equiv 1 \pmod 2$ the following equalities:

$$F_o(2^k \ell) = 2^{k/2} F_o(\ell) \text{ if } k \equiv 0 \,(\mathrm{mod}\ 2), \tag{12}$$

$$F_o(2^k \ell) = 2^{(k-1)/2} F_o(2\ell) \text{ if } k \equiv 1 \,(\mathrm{mod}\ 2), \tag{13}$$

$$F_e(2^k \ell) = (2^{k/2} - 1) F_o(\ell) + F_e(\ell) \text{ if } k \equiv 0 \,(\mathrm{mod}\ 2),$$

$$F_e(2^k \ell) = (2^{(k-1)/2} - 1) F_o(2\ell) \text{ if } k \equiv 1 \,(\mathrm{mod}\ 2).$$

By (6) and (11) we have

$$F_o(\ell) = \frac{1}{12} r_3(\ell), \;\; F_e(\ell) = 0 \text{ if } \ell \equiv 1 \,(\mathrm{mod}\ 4). \tag{14}$$

By (7) and (11) we have

$$F_o(\ell) = \frac{1}{12} r_3(\ell), \;\; F_e(\ell) = 0 \text{ if } \ell \equiv 2 \,(\mathrm{mod}\ 4). \tag{15}$$

By (8) and (11) we have

$$F_o(\ell) = \frac{1}{8} r_3(\ell), \;\; F_e(\ell) = \frac{1}{24} r_3(\ell) \text{ if } \ell \equiv 3 \,(\mathrm{mod}\ 8). \tag{16}$$

As far as the authors are aware, no explicit evaluation of $F_o(n)$ and $F_e(n)$ for all $n \in \mathbb{N}$ in terms of the class number $h(d)$ for a suitable discriminant d depending upon n has appeared in the literature. The purpose of this note is to give such an evaluation. It is clear from (14), (15), and (16) that a formula for $r_3(n)$ is going to be required. The authors have recently reformulated Gauss' formula for $r_3(n)$ into a simple, compact formula. This will be our starting point. We begin with some notation.

For $n \in \mathbb{N}$ we let 2^α ($\alpha \in \mathbb{N}_0$) be the largest power of 2 dividing n and we let h^2 ($h \in \mathbb{N}$) be the largest square dividing the odd positive integer $n/2^\alpha$. Thus we have

$$n = 2^\alpha g h^2, \tag{17}$$

where g and h are positive odd integers with g squarefree. The *discriminant associated with* n is

$$d(n) = \begin{cases} -4g & \text{if } \alpha \equiv 0 \,(\mathrm{mod}\ 2),\ g \equiv 1 \,(\mathrm{mod}\ 4), \\ -g & \text{if } \alpha \equiv 0 \,(\mathrm{mod}\ 2),\ g \equiv 3 \,(\mathrm{mod}\ 4), \\ -8g & \text{if } \alpha \equiv 1 \,(\mathrm{mod}\ 2). \end{cases}$$

We note that $d(n)$ is a fundamental discriminant as g is squarefree. We also define

$$\ell(n) := \prod_{p|h} \left(\frac{p^{\nu_p(h)+1} - 1}{p-1} - \left(\frac{d(n)}{p}\right) \frac{p^{\nu_p(h)} - 1}{p-1} \right),$$

where p runs through the (necessarily) odd primes dividing h, $\left(\frac{*}{p}\right)$ is the Legendre symbol, and $\nu_p(h)$ is the exponent of the largest power of p dividing h. The quantity $\ell(n)$ can also be expressed in the form

$$\ell(n) = \sum_{e|n} e \prod_{p|e} \left(1 - \left(\frac{d(n)}{p}\right)\frac{1}{p}\right).$$

Aygin and Williams [1, Theorem 3.2] proved that Gauss' formula for $r_3(n)$ can be reformulated in the following way.

Theorem 1. *Let $n \in \mathbb{N}$. Then*

$$r_3(n) = k(n)\ell(n)h(d(n)),$$

where the values of $k(n)$ are given in Table 1.

$k(n)$	conditions on α and g
6	$\alpha \equiv 0 \pmod 2$, $g = 1$
8	$\alpha \equiv 0 \pmod 2$, $g = 3$
12	$\alpha \equiv 0 \pmod 2$, $g \equiv 1 \pmod 4$, $g \neq 1$
24	$\alpha \equiv 0 \pmod 2$, $g \equiv 3 \pmod 8$, $g \neq 3$
0	$\alpha \equiv 0 \pmod 2$, $g \equiv 7 \pmod 8$
12	$\alpha \equiv 1 \pmod 2$

Table 1: Values of $k(n)$

Our main result is the following determination of $F_o(n)$ and $F_e(n)$ in terms of $h(d(n))$.

Theorem 2. *Let $n \in \mathbb{N}$. Express n in the form* (17)*. Then*

$$F_o(n) = k_o(n)\ell(n)h(d(n)),$$
$$F_e(n) = k_e(n)\ell(n)h(d(n)),$$

where the values of $k_o(n)$ and $k_e(n)$ are given in Table 2.

Case	$k_o(n)$	$k_e(n)$	conditions on α and g
Case 1	$2^{\alpha/2-1}$	$2^{\alpha/2-1} - 1/2$	$\alpha \equiv 0 \pmod 2$, $g = 1$
Case 2	$2^{\alpha/2}$	$2^{\alpha/2} - 2/3$	$\alpha \equiv 0 \pmod 2$, $g = 3$
Case 3	$2^{\alpha/2}$	$2^{\alpha/2} - 1$	$\alpha \equiv 0 \pmod 2$, $g \equiv 1 \pmod 4$, $g \neq 1$
Case 4	$3 \cdot 2^{\alpha/2}$	$3 \cdot 2^{\alpha/2} - 2$	$\alpha \equiv 0 \pmod 2$, $g \equiv 3 \pmod 8$, $g \neq 3$
Case 5	$2^{\alpha/2}$	$2^{\alpha/2}$	$\alpha \equiv 0 \pmod 2$, $g \equiv 7 \pmod 8$
Case 6	$2^{(\alpha-1)/2}$	$2^{(\alpha-1)/2} - 1$	$\alpha \equiv 1 \pmod 2$

Table 2: Values of $k_o(n)$ and $k_e(n)$

Proof. The proofs of Cases 1, 2, 3, 4, and 6 are similar. In these cases the evaluations of $F_o(n)$ follow from using the representation of n given by (17) and applying (12) or (13) (whichever is appropriate), (14), (15), or (16) (whichever is appropriate), and Theorem 1, in order. We only give the proof of Case 1 in detail.

If $\alpha \equiv 0 \pmod 2$ and $g = 1$, then by (17) we have

$$F_o(n) = F_o(2^\alpha h^2).$$

By using (12) we have

$$F_o(n) = 2^{\alpha/2} F_o(h^2),$$

and as $h^2 \equiv 1 \pmod 4$, by using (14) we obtain

$$F_o(n) = \frac{2^{\alpha/2}}{12} r_3(h^2).$$

Noting that $k(h^2) = 6$, by Theorem 1, we have

$$F_o(n) = 2^{\alpha/2-1} \ell(n) h(d(n)).$$

The evaluations of $F_e(n)$ for Cases 1, 2, 3, 4, and 6 follow from using (11), the evaluation of $F_o(n)$, and Theorem 1, in order. We only give the proof of Case 1 in detail. By (11), we have

$$F_e(n) = F_o(n) - \frac{1}{12} r_3(n).$$

By using the evaluation of $F_o(n)$ we have

$$F_e(n) = 2^{\alpha/2-1} \ell(n) h(d(n)) - \frac{1}{12} r_3(n).$$

By Theorem 1, in this case $r_3(n) = 6\ell(n)h(d(n))$, and therefore we have

$$F_e(n) = \left(2^{\alpha/2-1} - \frac{1}{2}\right) \ell(n) h(d(n)).$$

It remains to treat Case 5. In this case $n = 2^\alpha g h^2$, $\alpha \equiv 0 \pmod 2$, $g \equiv 7 \pmod 8$, so that $n = 4^{\alpha/2}(8m+7)$ for some $m \in \mathbb{N}_0$. Hence by Legendre's theorem, $r_3(n) = 0$, and by (11), we have

$$F_o(n) = F_e(n). \tag{18}$$

Now for any positive integer n we have by (1), (4), and (5)

$$\begin{aligned} H(-4n) - (F_o(n) + F_e(n)) &= \sum_{ax^2+bxy+cy^2 \in R(-4n)} (1 - wt(ax^2 + bxy + cy^2)) \\ &= S_1 + S_2, \end{aligned}$$

where

$$S_1 := \sum_{\substack{ax^2+bxy+cy^2 \in R(-4n) \\ wt(ax^2+bxy+cy^2)=1/2}} \frac{1}{2},$$

and

$$S_2 := \sum_{\substack{ax^2+bxy+cy^2 \in R(-4n) \\ wt(ax^2+bxy+cy^2)=1/3}} \frac{2}{3}.$$

By (3) we have

$$S_1 = \begin{cases} \frac{1}{2} & \text{if } a=c,\, b=0,\, b^2-4ac=-4n, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2} & \text{if } n=m^2, \\ 0 & \text{if } n \neq m^2, \end{cases}$$

and

$$S_2 = \begin{cases} \frac{2}{3} & \text{if } a=b=c,\, b^2-4ac=-4n, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{2}{3} & \text{if } n=3m^2, \\ 0 & \text{if } n \neq 3m^2. \end{cases}$$

Thus, for $n \in \mathbb{N}$ we have

$$F_o(n) + F_e(n) = \begin{cases} H(-4n) & \text{if } n \neq m^2, 3m^2, \\ H(-4n) - \frac{1}{2} & \text{if } n = m^2, \\ H(-4n) - \frac{2}{3} & \text{if } n = 3m^2. \end{cases} \tag{19}$$

Taking $n = 2^\alpha gh^2$, $\alpha \equiv 0 \pmod 2$, and $g \equiv 7 \pmod 8$, we deduce from (18) and (19) that

$$F_o(n) = F_e(n) = \frac{1}{2}H(-4n). \tag{20}$$

Taking $d = -4n$ in (2) we obtain

$$H(-4n) = \sum_{e^2 \mid -4n} h(-4n/e^2).$$

Now $n = 2^\alpha gh^2$, so $4n = 2^{\alpha+2}gh^2$ and $e^2 \mid 4n$ if and only if $e = 2^\beta f$, where $0 \leq \beta \leq \alpha/2 + 1$ and $f \mid h$. Thus

$$H(-4n) = \sum_{\beta=0}^{\alpha/2+1} \sum_{f \mid h} h(-2^{\alpha+2-2\beta} g(h/f)^2).$$

Mapping f to h/f, we deduce

$$H(-4n) = \sum_{\beta=0}^{\alpha/2+1} \sum_{f|h} h(-2^{\alpha+2-2\beta} g f^2).$$

Now, if Δ is a negative fundamental discriminant and $k \in \mathbb{N}$, then Δk^2 is a negative discriminant and

$$h(\Delta k^2) = h(\Delta) \frac{w(\Delta k^2)}{w(\Delta)} k \prod_{p|k} \left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right),$$

where $w(d)$ is the number of automorphs of a primitive reduced form of discriminant d; that is,

$$w(d) := \begin{cases} 6 & \text{if } d = -3, \\ 4 & \text{if } d = -4, \\ 2 & \text{if } d < -4; \end{cases}$$

see for example [6, p.774]. Taking $\Delta = -g \equiv 1 \pmod 8$ and $k = 2^{\alpha/2+1-\beta} f$ so that $\Delta k^2 = -2^{\alpha+2-2\beta} g f^2$, we have

$$H(-4n) = \sum_{\beta=0}^{\alpha/2+1} \sum_{f|h} h(-g) 2^{\alpha/2+1-\beta} f \prod_{p|2^{\alpha/2+1-\beta} f} \left(1 - \left(\frac{-g}{p}\right)\frac{1}{p}\right)$$

as $|\Delta| = g \geq 7$, so $w(\Delta) = w(\Delta k^2) = 2$. The terms with $\beta = \alpha/2 + 1$ comprise

$$h(-g) \sum_{f|h} f \prod_{p|f} \left(1 - \left(\frac{-g}{p}\right)\frac{1}{p}\right) = h(-g)\ell(n).$$

The terms with $\beta \neq \alpha/2 + 1$ comprise

$$\begin{aligned} & h(-g) \sum_{\beta=0}^{\alpha/2} 2^{\alpha/2+1-\beta} \sum_{f|h} f \left(1 - \left(\frac{-g}{2}\right)\frac{1}{2}\right) \prod_{p|f} \left(1 - \left(\frac{-g}{p}\right)\frac{1}{p}\right) \\ &= h(-g)(2^{\alpha/2+1} - 1) \sum_{f|h} f \prod_{p|f} \left(1 - \left(\frac{-g}{p}\right)\frac{1}{p}\right) \\ &= h(-g)(2^{\alpha/2+1} - 1)\ell(n) \end{aligned}$$

as $g \equiv 7 \pmod 8$ implies $\left(\frac{-g}{2}\right) = 1$, so

$$1 - \left(\frac{-g}{2}\right)\frac{1}{2} = \frac{1}{2}.$$

Thus we have

$$\begin{aligned} H(-4n) &= h(-g)\ell(n) + h(-g)(2^{\alpha/2+1} - 1)\ell(n) \\ &= 2^{\alpha/2+1} h(-g)\ell(n) \end{aligned}$$

so that by (20)

$$F_o(n) = F_e(n) = 2^{\alpha/2}\ell(n)h(-g).$$

This completes the proof of Theorem 2. □

As a byproduct of Theorem 2 and (19), we obtain the evaluation of $H(-4n)$ for all $n \in \mathbb{N}$.

Theorem 3. *Let $n \in \mathbb{N}$. Express n in the form (17).*

1. *If α is even and $g = 1$, then*

$$H(-4n) = \left(2^{\alpha/2} - \frac{1}{2}\right)\ell(n)h(d(n)) + \frac{1}{2}.$$

2. *If α is even and $g = 3$, then*

$$H(-4n) = \left(2^{\alpha/2+1} - \frac{2}{3}\right)\ell(n)h(d(n)) + \frac{2}{3}.$$

3. *If α is even, $g \neq 1$, and $g \equiv 1 \pmod 4$, then*

$$H(-4n) = \left(2^{\alpha/2+1} - 1\right)\ell(n)h(d(n)).$$

4. *If α is even, $g \neq 3$, and $g \equiv 3 \pmod 8$, then*

$$H(-4n) = \left(3 \cdot 2^{\alpha/2+1} - 2\right)\ell(n)h(d(n)).$$

5. *If α is even and $g \equiv 7 \pmod 8$, then*

$$H(-4n) = 2^{\alpha/2+1}\ell(n)h(d(n)).$$

6. *If α is odd, then*

$$H(-4n) = (2^{(\alpha+1)/2} - 1)\ell(n)h(d(n)).$$

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