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AN INTERPRETATION OF CONTINUED FRACTIONS AND FAREY GRAPHS IN THE FIELD OF RATIONAL FUNCTIONS

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Abstract

In this work, we construct Farey graphs in the field of rational functions analogous to the idea in the classical case. We explore their properties and establish some relationships between these graphs and their associated regular continued fraction expansions.

1. Introduction

In a 1991 article by Jones, Singerman, and Wicks [1], the graph $\mathcal{F}_{u,N}$, where u and N are natural numbers and $\gcd(u, N) = 1$, was introduced. The vertex set of the graph is

$$\chi_N = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q > 0, (p, q) = 1, \text{ and } N \mid q\right\} \cup \{\infty\}, \tag{1}$$

and any vertices $\frac{p}{q}$ and $\frac{r}{s}$ are joined by an edge if and only if $rq - sp = N$ with $p \equiv ur \pmod{N}$ or $rq - sp = -N$ with $p \equiv -ur \pmod{N}$. The graph $\mathcal{F}_{1,1}$ is known as the *Farey graph.* An interesting observation is that the Farey graph is intimately connected with the continued fractions arising from its subgraphs. Since then,

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numerous studies have investigated the relationships between $\mathcal{F}_{u,N}$ and continued fractions.

For example, in 2015, Sarma, Kushwaha, and Krishnan [4] introduced a specific kind of semi-regular continued fractions which is referred to as an $\mathcal{F}_{1,2}$-continued fraction, a finite continued fraction of the form

$$\frac{1}{0+}\frac{2}{b+}\frac{\epsilon_1}{a_1+}\frac{\epsilon_2}{a_2+}\cdots\frac{\epsilon_n}{a_n} \quad (n \geq 0)$$

or an infinite continued fraction of the form

$$\frac{1}{0+}\frac{2}{b+}\frac{\epsilon_1}{a_1+}\frac{\epsilon_2}{a_2+}\cdots\frac{\epsilon_n}{a_n+}\cdots,$$

where b is an odd integer, a_i is an even positive integer, and $\epsilon_i \in \{\pm 1\}$ for each i. They established that each finite $\mathcal{F}_{1,2}$-continued fraction corresponds to a path starting from ∞ and ending at its value. Also, in 2018, similar results for a graph $\mathcal{F}_{1,3}$ were established by Kushwaha and Sarma [2]. Recently, in 2022, Kushwaha and Sarma [3] relaxed the conditions of two adjacent vertices in the graph $\mathcal{F}_{u,N}$ and got a new family of graphs $\mathcal{F}_N$ defined as follows: the set of vertices is χ_N (as in Equation (1)) and two vertices $\frac{p}{q}$ and $\frac{r}{s}$ are connected by an edge if and only if $rq - sp = \pm N$. Similarly, they also constructed $\mathcal{F}_N$-continued fractions and established the parallel results of their earlier works.

Shaped by the article of Kushwaha and Sarma, we are very interested in the study of relationships between graphs and continued fractions, but in some other structure, namely, the field of rational functions. For the continued fraction part, we focus on the regular continued fractions that have already been constructed and well known; for example, see [5]. We next recall the definitions and basic results regarding regular continued fractions in the field of rational functions.

Throughout this paper, we let $\mathbb{F}$ be a field, $\mathbb{F}(x)$ the field of rational functions over $\mathbb{F}$, and $\mathbb{F}((x^{-1}))$ the field of formal series over $\mathbb{F}$ complete with respect to the degree valuation $|\cdot|$. Recall that for each nonzero element

$$\alpha = c_m x^m + \ldots + c_1 x + c_0 + \frac{c_{-1}}{x} + \frac{c_{-2}}{x^2} + \ldots \in \mathbb{F}((x^{-1})), \tag{2}$$

where $m \in \mathbb{Z}$ and $c_i \in \mathbb{F}$ $(i \leq m)$ with $c_m \neq 0$, the *degree valuation* is defined by $|\alpha| = e^m$ and $|0| = 0$. The *integral part* of α, denoted by $[\alpha]$, is defined to be $[\alpha] = c_m x^m + \ldots + c_1 x + c_0$. Note that when α, as expressed in (2), has no integral part, it becomes

$$\alpha = \frac{c_{-t}}{x^t} + \frac{c_{-(t+1)}}{x^{t+1}} + \frac{c_{-(t+2)}}{x^{t+2}} + \ldots,$$

where t is a positive integer.

Every element α in $\mathbb{F}((x^{-1}))$ can be uniquely represented as a finite or infinite expression of the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} = [a_0, a_1, a_2, a_3, \ldots],$$

where a_0 is in $\mathbb{F}[x]$ and the a_n are in $\mathbb{F}[x] \setminus \mathbb{F}$ $(n \geq 1)$. The polynomials a_n are called the *partial quotients* of the continued fraction expansion of α and each $\alpha_n = [a_n, a_{n+1}, \ldots]$ is called the nth *complete quotient* of α. The sequence of *convergents*, A_n/B_n, can be generated from two sequences $\{A_n\}$ and $\{B_n\}$ of polynomials in the following way:

$$\begin{array}{llll} A_{-1} = 1, & A_0 = a_0, & A_{n+1} = a_{n+1}A_n + A_{n-1} & (n \geq 0), \\ B_{-1} = 0, & B_0 = 1, & B_{n+1} = a_{n+1}B_n + B_{n-1} & (n \geq 0). \end{array}$$

We then have the following results.

Lemma 1 ([5])**.** *For any non-negative integer $n \geq 0$ and $\beta \in \mathbb{F}((x^{-1})) \setminus \{0\}$, we have*

1. $\dfrac{\beta A_n + A_{n-1}}{\beta B_n + B_{n-1}} = [a_0, a_1, a_2, \ldots, a_n, \beta]$,
2. $A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1}$,
3. $|B_n| > |B_{n-1}| > 0$, *and*
4. $\left|\alpha - \dfrac{A_n}{B_n}\right| = \dfrac{1}{|a_{n+1}||B_n|^2}$ $(n \geq 1)$.

From Lemma 1, every $\frac{A_n}{B_n} = [a_0, a_1, a_2, \ldots, a_n]$ is a reduced fraction and satisfies

$$\left|\alpha - \frac{A_n}{B_n}\right| \to 0 \qquad (n \to \infty).$$

Then we call $\frac{A_n}{B_n}$ the nth *convergent* of the regular continued fraction of α where A_n and B_n are called the nth *partial numerator* and nth *partial denominator*, respectively.

Theorem 1 ([5])**.** *Let $\alpha \in \mathbb{F}((x^{-1}))$. Then α is a rational function if and only if its continued fraction is finite.*

In this article, we construct Farey graphs, analogous to those defined in [1], over the field of rational functions. Also, we aim to explore their properties and establish some relationships between regular continued fractions and their associated Farey graphs.

2. Main Results

In this section, we first introduce the Farey graph over a field $\mathbb{F}$. It is the graph whose vertex set is

$$\chi' = \left\{ \frac{u}{v} : u, v \in \mathbb{F}[x] \text{ with } v \neq 0 \text{ and } (u, v) = 1 \right\}$$

and two vertices $\frac{u}{v}$ and $\frac{r}{s}$ are connected by an edge, denoted by $\frac{u}{v} \sim \frac{r}{s}$, if and only if

$$rv - su \in \mathbb{F} \setminus \{0\}.$$

So, we are able to consider the Farey graph over $\mathbb{F}$ as a simple and undirected graph.

It is clear to see that, for any c in $\mathbb{F}$ and for any nonzero polynomial v over $\mathbb{F}$, two vertices c and $c+\frac{1}{v}$ $\left(= \frac{cv+1}{v}\right)$ in χ' are adjacent. In addition, the path from c to $c+\frac{1}{v}$ defines the regular continued fraction of $c+\frac{1}{v}$ if v is a nonconstant polynomial. In the remainder, we focus on the path starting from a nonconstant polynomial over $\mathbb{F}$.

Remark 1. We denote the set of vertices by χ'_{p^r} when we consider the Farey graph over $\mathbb{F}_{p^r}$, the finite field of p^r elements (p is a prime and r is a positive integer). Also, for convenience, we usually use χ'_{p^r} to stand for the Farey graph over $\mathbb{F}_{p^r}$. We usually use long arrows to indicate and emphasize the direction of a path between two vertices, as it helps us visualize the relation between paths and their associated continued fractions.

Example 1. In the Farey graph χ'_3, here are some examples of paths starting from the vertex x:

$$x \longrightarrow \frac{x^2+1}{x} \longrightarrow \frac{x^3-x}{x^2+1} \longrightarrow \frac{x^4+1}{x^3-x} \longrightarrow \frac{x^5+x^3}{x^4+1},$$

$$x \longrightarrow \frac{x^2-1}{x} \longrightarrow \frac{x^3}{x^2+1} \longrightarrow \frac{x^4+x^2-1}{x^3-x} \longrightarrow \frac{-x^5+x}{-x^4-x^2+1},$$

$$x \longrightarrow \frac{x^2+1}{x} \longrightarrow \frac{x^3}{x^2-1} \longrightarrow \frac{x^4-x^2-1}{x^3+x} \longrightarrow \frac{-x^5-x^4+x^2+x+1}{-x^4-x^3+x^2-x+1},$$

and

$$x \longrightarrow \frac{x^2-1}{x} \longrightarrow \frac{x^3+x}{x^2-1} \longrightarrow \frac{-x^4+x^2+1}{-x^3} \longrightarrow \frac{x^6-x^4-x^3-x^2-x}{x^5-x^2+1}.$$

We now recall the concept of Farey series from [6] and establish its connections with our Farey graphs.

Definition 1 ([6])**.** Let p be a prime, and let n and r be positive integers. The *Farey series of order* n is defined as

$$\mathcal{F}_n = \left\{ \frac{u}{v} \mid u, v \in \mathbb{F}_{p^r}[x], \deg u < \deg v \leq n, (u, v) = 1, \text{ and } v \text{ is monic} \right\}.$$

Remark 2. From Definition 1, it follows immediately that $\mathcal{F}_n \subseteq \chi'_{p^r}$ with p, n, and r as above.

Definition 2 ([6]). For any distinct fractions $\frac{u}{v}$ and $\frac{h}{k}$ in a given $\mathcal{F}_n$ with v and k monic, we say that $\frac{h}{k}$ is a deg k-*neighbor* of $\frac{u}{v}$ if $\left|\frac{u}{v} - \frac{h}{k}\right| \leq \left|\frac{u}{v} - \frac{h'}{k'}\right|$ for every $\frac{h'}{k'} \in \mathcal{F}_n$ with $\deg k' = \deg k$.

Definition 3 ([6]). Any fractions $\frac{u}{v}$ and $\frac{h}{k}$ in a given $\mathcal{F}_n$ are called *neighbors* if $\frac{u}{v}$ is a $\deg v$-neighbor of $\frac{h}{k}$ and $\frac{h}{k}$ is a $\deg k$-neighbor of $\frac{u}{v}$.

Theorem 2 ([6]). *For any natural number n, two fractions $\frac{u}{v}$ and $\frac{h}{k}$ in $\mathcal{F}_n$ are neighbors if and only if* $\deg(uk - hv) = 0$.

Remark 3. From Theorem 2, we see that all elements that are neighbors in $\mathcal{F}_n$ are adjacent in the Farey graph χ'.

Theorem 3 ([6]). *Let p be a prime and let n be a natural number. If $\frac{u}{v}$ is in $\mathcal{F}_n$ and $q = \deg v$, then $\frac{u}{v}$ has exactly $(p-1)p^t$ neighbors of degree $q+t$ for any integer $t \geq 0$, and has only one neighbor of degree less than q.*

Example 2. Consider the vertex $\frac{x+1}{x^2}$ in $\mathcal{F}_2 \subseteq \chi'_3$. By Theorem 3, with $p = 3$ and $q = 2$, there is only one neighbor of degree 1, which must come from

$$\left\{\frac{0}{1}, \frac{1}{x}, \frac{2}{x}, \frac{1}{x+1}, \frac{2}{x+1}, \frac{1}{x+2}, \frac{2}{x+2}\right\}$$

and, through direct computation, it is found to be $\frac{1}{x+2}$. Note that a vertex of degree zero cannot be a neighbor, as it is not connected to any other vertex. Figure 1 shows the number of neighbors of $\frac{x+1}{x^2}$ for different degrees.

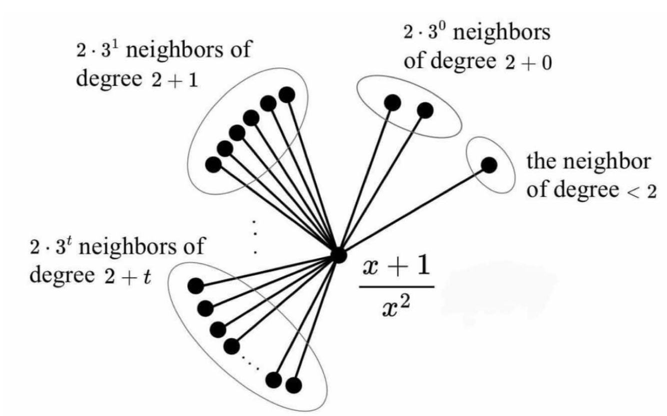

Figure 1: The number of neighbors of $\frac{x+1}{x^2}$ at each degree.

Next, we show how to construct a path in the Farey graph associated with a given continued fraction.

Theorem 4. *The value of every finite regular continued fraction belongs to χ' and every finite continued fraction defines a path from its integral part to its value with the convergents as the vertices.*

Proof. The first part is clear. The latter part comes automatically from Lemma 1 (2). ☐

Conversely, for a given path, we show how to define an associated continued fraction as follows.

Theorem 5. *Given the path $\frac{u_0}{v_0} \to \frac{u_1}{v_1} \to \frac{u_2}{v_2} \to \cdots \to \frac{u_n}{v_n}$, let $[a_0, a_1, a_2, ..., a_n]$ be the associated continued fraction of $\frac{u_n}{v_n}$ with all convergents as vertices of the path, and let $\frac{u_{n+1}}{v_{n+1}} \in \chi'$ be such that $\deg v_{n+1} > \deg v_n$. Then the path*

$$\frac{u_0}{v_0} \to \frac{u_1}{v_1} \to \cdots \to \frac{u_n}{v_n} \to \frac{u_{n+1}}{v_{n+1}}$$

defines the continued fraction $[a_0, a_1, a_2, ..., a_n, a_{n+1}]$ of $\frac{u_{n+1}}{v_{n+1}}$, where

$$a_{n+1} = \frac{v_{n+1}A_{n-1} - u_{n+1}B_{n-1}}{u_{n+1}B_n - v_{n+1}A_n} \in \mathbb{F}[x] \setminus \mathbb{F}.$$

Proof. Suppose that the path

$$\frac{u_0}{v_0} \to \frac{u_1}{v_1} \to \frac{u_2}{v_2} \to \cdots \to \frac{u_n}{v_n}$$

defines the continued fraction $[a_0, a_1, a_2, ..., a_n]$ with the convergents as vertices. Consider the path

$$\frac{u_0}{v_0} \to \frac{u_1}{v_1} \to \cdots \to \frac{u_n}{v_n} \to \frac{u_{n+1}}{v_{n+1}}.$$

Suppose that $u_{n+1}v_n - u_nv_{n+1} = c$ for some c in $\mathbb{F} \setminus \{0\}$ and $\frac{u_{n+1}}{v_{n+1}}$ is represented by the continued fraction of the form $[a_0, a_1, a_2, ..., a_n, a_{n+1}]$. We now solve for a_{n+1}. We have $\frac{u_{n+1}}{v_{n+1}} = \frac{a_{n+1}A_n + A_{n-1}}{a_{n+1}B_n + B_{n-1}}$, which implies that

$$a_{n+1} = \frac{v_{n+1}A_{n-1} - u_{n+1}B_{n-1}}{u_{n+1}B_n - v_{n+1}A_n}.$$

Next, we will show that $a_{n+1} \in \mathbb{F}[x] \setminus \mathbb{F}$. We first show that $a_{n+1} \in \mathbb{F}[x]$ by showing that $u_{n+1}B_n - v_{n+1}A_n$ is in $\mathbb{F} \setminus \{0\}$. Since $\frac{u_n}{v_n} = \frac{A_n}{B_n}$ and they are reduced fractions, there is a unit t in $\mathbb{F} \setminus \{0\}$ such that $A_n = tu_n$ and $B_n = tv_n$. Therefore,

$$u_{n+1}B_n - v_{n+1}A_n = u_{n+1}(tv_n) - v_{n+1}(tu_n) = tc \in \mathbb{F} \setminus \{0\}.$$

We now show that a_{n+1} is a nonconstant polynomial over $\mathbb{F}$. By Lemma 1 (2) and $|v_{n+1}| > |v_n| = |B_n| > |B_{n-1}|$, we have

$$\begin{aligned}\Big|\frac{v_{n+1}A_{n-1} - u_{n+1}B_{n-1}}{v_{n+1}B_{n-1}}\Big| &= \Big|\frac{u_{n+1}}{v_{n+1}} - \frac{A_{n-1}}{B_{n-1}}\Big| = \Big|\frac{tc}{v_{n+1}B_n} + \frac{(-1)^{n-1}}{B_nB_{n-1}}\Big| \\ &= \max\Big\{\frac{1}{|v_{n+1}B_n|}, \frac{1}{|B_nB_{n-1}|}\Big\} = \frac{1}{|B_nB_{n-1}|}.\end{aligned}$$

Then

$$|v_{n+1}A_{n-1} - u_{n+1}B_{n-1}| = \frac{|v_{n+1}B_{n-1}|}{|B_nB_{n-1}|} > 1.$$

Equivalently, we have $v_{n+1}A_{n-1} - u_{n+1}B_{n-1} \in \mathbb{F}[x] \setminus \mathbb{F}$. This implies that $a_{n+1} \in \mathbb{F}[x] \setminus \mathbb{F}$, as required. □

Theorem 6. *Let $u_{-1} = 1 = v_0$, $v_{-1} = 0$, $c_0 = 1$, and $u_0 \in \mathbb{F}[x]$. For each $i \in \{1, 2, ..., n\}$, let $u_i, v_i \in \mathbb{F}[x]$ with $(u_i, v_i) = 1, \deg(v_{i-1}) < \deg(v_i)$, and $u_iv_{i-1} - u_{i-1}v_i = c_i \in \mathbb{F} \setminus \{0\}$. Then the path*

$$a_0 := \frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1} \longrightarrow \frac{u_2}{v_2} \longrightarrow \frac{u_3}{v_3} \longrightarrow \cdots \longrightarrow \frac{u_n}{v_n}$$

from a_0 to $\frac{u_n}{v_n}$ defines the finite regular continued fraction of $\frac{u_n}{v_n}$, where each vertex $\frac{u_i}{v_i}$ defines its ith convergent. In particular, the partial quotients $a_1 = c_1^{-1}v_1$, $a_2 = -c_2^{-1}c_1(u_2v_0 - u_0v_2)$, and

$$a_i = (-1)^{i+1}c_i^{-1}c_{i-1}c_{i-2}^{-2}c_{i-3}^{2}\cdots c_1^{(-1)^i\cdot 2}(u_iv_{i-2} - u_{i-2}v_i) \quad (i \geq 3)$$

with the ith partial numerators and ith partial denominators $(i \geq 1)$

$$A_i = \begin{cases} c_i^{-1}c_{i-1}c_{i-2}^{-1}c_{i-3}\cdots c_1^{(-1)^i}u_i, & \text{if } i \equiv 0, 1 \pmod 4 \\ -c_i^{-1}c_{i-1}c_{i-2}^{-1}c_{i-3}\cdots c_1^{(-1)^i}u_i, & \text{if } i \equiv 2, 3 \pmod 4 \end{cases}$$

and

$$B_i = \begin{cases} c_i^{-1}c_{i-1}c_{i-2}^{-1}c_{i-3}\cdots c_1^{(-1)^i}v_i, & \text{if } i \equiv 0, 1 \pmod 4 \\ -c_i^{-1}c_{i-1}c_{i-2}^{-1}c_{i-3}\cdots c_1^{(-1)^i}v_i, & \text{if } i \equiv 2, 3 \pmod 4, \end{cases}$$

respectively.

Proof. Consider the path

$$\frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1}.$$

By the assumption, we have $u_1 = a_0v_1 + c_1$. Moreover, since $\deg(v_1) > \deg(v_0) = 0$, we have $v_1 \in \mathbb{F}[x] \setminus \mathbb{F}$. We then have the regular continued fraction of $\frac{u_1}{v_1}$ as

$$\frac{u_1}{v_1} = \frac{a_0v_1 + c_1}{v_1} = a_0 + \frac{1}{c_1^{-1}v_1} = [a_0, a_1]$$

with the partial quotient $a_1 = c_1^{-1}v_1$. Note that $A_0 = u_0, B_0 = 1 = v_0$, and the first partial denominator

$$B_1 = a_1B_0 + B_{-1} = a_1 = c_1^{-1}v_1 = c_1^{(-1)^1}v_1.$$

Since $\frac{u_1}{v_1} = \frac{A_1}{B_1}$ and they are reduced fractions, we have

$$A_1 = c_1^{(-1)^1}u_1.$$

Consider the path

$$\frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1} \longrightarrow \frac{u_2}{v_2}.$$

By Theorem 5, we have

$$a_2 = \frac{v_2A_0 - u_2B_0}{u_2B_1 - v_2A_1} = -c_2^{-1}c_1\big(u_2v_0 - u_0v_2\big) \in \mathbb{F}[x].$$

Since deg $v_2 >$ deg v_1, we have $a_2 \in \mathbb{F}[x] \setminus \mathbb{F}$. Therefore,

$$B_2 = a_2B_1 + B_0 = -c_2^{-1}c_1\big(u_2v_0 - u_0v_2\big)(c_1^{-1}v_1) + 1 = -c_2^{-1}c_1v_2 = -c_2^{-1}c_1^{(-1)^2}v_2$$

and it follows that

$$A_2 = -c_2^{-1}c_1u_2 - c_2^{-1}c_1^{(-1)^2}u_2.$$

Consider the path

$$\frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1} \longrightarrow \frac{u_2}{v_2} \longrightarrow \frac{u_3}{v_3}.$$

Again by Theorem 5, we have

$$a_3 = \frac{v_3A_1 - u_3B_1}{u_3B_2 - v_3A_2} = (-1)^4c_3^{-1}c_2c_1^{-2}\big(u_3v_1 - u_1v_3\big) \in \mathbb{F}[x] \setminus \mathbb{F}.$$

Similarly, by direct computation, we get

$$B_3 = a_3B_2 + B_1 = -c_3^{-1}c_2c_1^{-1}v_3 = -c_3^{-1}c_2c_1^{(-1)^3}v_3$$

and

$$A_3 = -c_3^{-1}c_2c_1^{(-1)^3}u_3.$$

Consider the path

$$\frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1} \longrightarrow \frac{u_2}{v_2} \longrightarrow \frac{u_3}{v_3} \longrightarrow \frac{u_4}{v_4}.$$

Similar to the previous cases, we obtain $a_4 = (-1)^5c_4^{-1}c_3c_2^{-2}c_1^2\big(u_4v_2 - u_2v_4\big) \in \mathbb{F}[x] \setminus \mathbb{F}$. By direct computation, we also have

$$B_4 = a_4B_3 + B_2 = c_4^{-1}c_3c_2^{-1}c_1v_4 = c_4^{-1}c_3c_2^{-1}c_1^{(-1)^4}v_4$$

and

$$A_4 = c_4^{-1} c_3 c_2^{-1} c_1^{(-1)^4} u_4.$$

For the inductive step, assume that the statement is true for $k \geq 4$, and that we are given a path

$$\frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1} \longrightarrow \frac{u_2}{v_2} \longrightarrow \frac{u_3}{v_3} \longrightarrow \cdots \longrightarrow \frac{u_{k+1}}{v_{k+1}}.$$

Then, by the induction hypothesis, the path

$$\frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1} \longrightarrow \frac{u_2}{v_2} \longrightarrow \frac{u_3}{v_3} \longrightarrow \cdots \longrightarrow \frac{u_k}{v_k}$$

from $\frac{u_0}{v_0}$ to $\frac{u_k}{v_k}$ defines the finite regular continued fraction of $\frac{u_k}{v_k}$, say

$$\frac{u_k}{v_k} = [a_0, a_1, \ldots, a_k],$$

where $a_1 = c_1^{-1} v_1, a_2 = -c_2^{-1} c_1(u_2, v_0 - u_0 v_2)$, and

$$a_i = (-1)^{i+1} c_i^{-1} c_{i-1} c_{i-2}^{-2} c_{i-3}^{2} \cdots c_1^{(-1)^i \cdot 2}(u_i v_{i-2} - u_{i-2} v_i) \in \mathbb{F}[x] \setminus \mathbb{F}$$

with $3 \leq i \leq k$. Moreover, for each $1 \leq i \leq k$, its ith partial numerator and denominator are

$$A_i = \begin{cases} c_i^{-1} c_{i-1} c_{i-2}^{-1} c_{i-3} \cdots c_1^{(-1)^i} u_i, & \text{if } i \equiv 0, 1 \pmod 4 \\ -c_i^{-1} c_{i-1} c_{i-2}^{-1} c_{i-3} \cdots c_1^{(-1)^i} u_i, & \text{if } i \equiv 2, 3 \pmod 4 \end{cases}$$

and

$$B_i = \begin{cases} c_i^{-1} c_{i-1} c_{i-2}^{-1} c_{i-3} \cdots c_1^{(-1)^i} v_i, & \text{if } i \equiv 0, 1 \pmod 4 \\ -c_i^{-1} c_{i-1} c_{i-2}^{-1} c_{i-3} \cdots c_1^{(-1)^i} v_i, & \text{if } i \equiv 2, 3 \pmod 4, \end{cases}$$

respectively. Note that the assumptions $u_{k+1} v_k - u_k v_{k+1} = c_{k+1}$ and $u_k v_{k-1} - u_{k-1} v_k = c_k$ imply that

$$\left(\frac{c_k v_{k+1} + c_{k+1} v_{k-1}}{v_k} \right) = u_{k+1} v_{k-1} - u_{k-1} v_{k+1}. \tag{3}$$

In order to show that

$$\frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1} \longrightarrow \frac{u_2}{v_2} \longrightarrow \frac{u_3}{v_3} \longrightarrow \cdots \longrightarrow \frac{u_{k+1}}{v_{k+1}}$$

satisfies the statement of the theorem, we divide the proof into four cases, as one might expect. However, due to the tedious nature of the argument, we present only the case when $k + 1 \equiv 1 \pmod 4$.

Assume that $k+1 \equiv 1 \pmod 4$. Then $k \equiv 0 \pmod 4$ and $k-1 \equiv 3 \pmod 4$. By the induction hypothesis, we have $A_k = c_k^{-1}c_{k-1}c_{k-2}^{-1}c_{k-3}\cdots c_2^{-1}c_1u_k$ and $B_k = c_k^{-1}c_{k-1}c_{k-2}^{-1}c_{k-3}\cdots c_2^{-1}c_1v_k$. Also, $A_{k-1} = -c_{k-1}^{-1}c_{k-2}c_{k-3}^{-1}\cdots c_2c_1^{-1}u_{k-1}$ and $B_{k-1} = -c_{k-1}^{-1}c_{k-2}c_{k-3}^{-1}\cdots c_2c_1^{-1}v_{k-1}$. Therefore, by Theorem 5,

$$\begin{aligned}
a_{k+1} &= \frac{v_{k+1}A_{k-1} - u_{k+1}B_{k-1}}{u_{k+1}B_k - v_{k+1}A_k} \\
&= \frac{v_{k+1}(-c_{k-1}^{-1}c_{k-2}c_{k-3}^{-1}\cdots c_1^{-1}u_{k-1}) - u_{k+1}(-c_{k-1}^{-1}c_{k-2}\cdots c_1^{-1}v_{k-1})}{u_{k+1}c_k^{-1}c_{k-1}c_{k-2}^{-1}c_{k-3}\cdots c_2^{-1}c_1v_k - v_{k+1}c_k^{-1}c_{k-1}c_{k-2}^{-1}\cdots c_2^{-1}c_1u_k} \\
&= \frac{-c_{k-1}^{-1}c_{k-2}\cdots c_1^{-1}(v_{k+1}u_{k-1} - u_{k+1}v_{k-1})}{c_k^{-1}c_{k-1}c_{k-2}^{-1}c_{k-3}\cdots c_1(u_{k+1}v_k - v_{k+1}u_k)} \\
&= (-1)^{k+2}c_{k+1}^{-1}c_kc_{k-1}^{-2}c_{k-2}^2\cdots c_2^2c_1^{-2}(u_{k+1}v_{k-1} - v_{k+1}u_{k-1}),
\end{aligned}$$

as desired.

Moreover, by Equation (3), we have

$$a_{k+1} = (-1)^{k+2}c_{k+1}^{-1}c_kc_{k-1}^{-2}c_{k-2}^2\cdots c_2^2c_1^{-2}\left(\frac{c_kv_{k+1} + c_{k+1}v_{k-1}}{v_k}\right).$$

Then one can see that

$$\begin{aligned}
B_{k+1} &= a_{k+1}B_k + B_{k-1} \\
&= a_{k+1}(c_k^{-1}c_{k-1}\cdots c_2^{-1}c_1v_k) + (-c_{k-1}^{-1}c_{k-2}\cdots c_2c_1^{-1}v_{k-1}) \\
&= c_{k+1}^{-1}c_k\cdots c_2c_1^{-1}v_{k+1}.
\end{aligned}$$

Similarly, replacing v_k by u_k and v_{k-1} by u_{k-1}, we obtain the analogous expression

$$A_{k+1} = c_{k+1}^{-1}c_k\cdots c_2c_1^{-1}u_{k+1},$$

as required. □

Next, adopting the same notation as in Theorem 6, we present special cases of Theorem 6 over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$. Note that in $\mathbb{F}_2(x)$, the results essentially coincide, since $1 \equiv -1 \pmod 2$. Over $\mathbb{F}_3$, however, the outcomes are more interesting and varied.

Corollary 1. *If $u_iv_{i-1} - u_{i-1}v_i = (-1)^{i-1}$ $(1 \le i \le n)$, then the path*

$$a_0 := \frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1} \longrightarrow \frac{u_2}{v_2} \longrightarrow \frac{u_3}{v_3} \longrightarrow \cdots \longrightarrow \frac{u_n}{v_n}$$

from a_0 to $\frac{u_n}{v_n}$ defines the finite regular continued fraction of $\frac{u_n}{v_n}$ where each vertex $\frac{u_i}{v_i}$ defines its ith convergent. In particular, the ith partial numerator and partial denominator are u_i and v_i, respectively, with the partial quotient

$$a_i = (-1)^i\big(u_iv_{i-2} - u_{i-2}v_i\big) \quad (1 \le i \le n).$$

Corollary 2. *If* $u_i v_{i-1} - u_{i-1} v_i = (-1)^i$ $(1 \le i \le n)$*, then the path*

$$a_0 := \frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1} \longrightarrow \frac{u_2}{v_2} \longrightarrow \frac{u_3}{v_3} \longrightarrow \cdots \longrightarrow \frac{u_n}{v_n}$$

from a_0 *to* $\frac{u_n}{v_n}$ *defines the finite regular continued fraction of* $\frac{u_n}{v_n}$ *where each vertex* $\frac{u_i}{v_i}$ *defines its* i*th convergent. In particular, the* i*th partial numerator and partial denominator are* $(-1)^i u_i$ *and* $(-1)^i v_i$*, respectively, with the partial quotient* $a_1 = -v_1$ *and*

$$a_i = (-1)^i \big(u_i v_{i-2} - u_{i-2} v_i\big) \quad (2 \le i \le n).$$

Corollary 3. *If* $u_i v_{i-1} - u_{i-1} v_i = 1$ $(1 \le i \le n)$*, then the path*

$$a_0 := \frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1} \longrightarrow \frac{u_2}{v_2} \longrightarrow \frac{u_3}{v_3} \longrightarrow \cdots \longrightarrow \frac{u_n}{v_n}$$

from a_0 *to* $\frac{u_n}{v_n}$ *defines the finite regular continued fraction of* $\frac{u_n}{v_n}$ *where each vertex* $\frac{u_i}{v_i}$ *defines its convergent. In particular, the* i*th partial numerator and partial denominator, respectively, are*

$$A_i = \begin{cases} u_i, & \text{if } i \equiv 0,1 \pmod 4 \\ -u_i, & \text{if } i \equiv 2,3 \pmod 4 \end{cases} \quad \text{and} \quad B_i = \begin{cases} v_i, & \text{if } i \equiv 0,1 \pmod 4 \\ -v_i, & \text{if } i \equiv 2,3 \pmod 4 \end{cases}$$

with the partial quotient $a_1 = v_1$ *and* $a_i = (-1)^{i+1}\big(u_i v_{i-2} - u_{i-2} v_i\big)$ $(2 \le i \le n)$*.*

Corollary 4. *If* $u_i v_{i-1} - u_{i-1} v_i = -1$ $(1 \le i \le n)$*, then the path*

$$a_0 := \frac{u_0}{v_0} \longrightarrow \frac{u_1}{v_1} \longrightarrow \frac{u_2}{v_2} \longrightarrow \frac{u_3}{v_3} \longrightarrow \cdots \longrightarrow \frac{u_n}{v_n}$$

from a_0 *to* $\frac{u_n}{v_n}$ *defines the finite regular continued fraction of* $\frac{u_n}{v_n}$ *where each vertex* $\frac{u_i}{v_i}$ *defines its convergent. In particular, the* i*th partial numerator and partial denominator, respectively, are*

$$A_i = \begin{cases} u_i, & \text{if } i \equiv 0,3 \pmod 4 \\ -u_i, & \text{if } i \equiv 1,2 \pmod 4 \end{cases} \quad \text{and} \quad B_i = \begin{cases} v_i, & \text{if } i \equiv 0,3 \pmod 4 \\ -v_i, & \text{if } i \equiv 1,2 \pmod 4 \end{cases}$$

with the partial quotient $a_i = (-1)^{i+1}\big(u_i v_{i-2} - u_{i-2} v_i\big)$ $(1 \le i \le n)$*.*

We now return to Example 1 to examine how the conclusions of the preceding corollaries relate to the paths under consideration. For convenience, we set the notation as follows: for any two vertices u/v and r/s in χ_3',

$$\frac{u}{v} \xrightarrow{+} \frac{r}{s} \text{ means } \frac{u}{v} \sim \frac{r}{s} \text{ with } rv - su = 1, \text{ and}$$

$$\frac{u}{v} \xrightarrow{-} \frac{r}{s} \text{ means } \frac{u}{v} \sim \frac{r}{s} \text{ with } rv - su = -1.$$

By Corollary 1, the path

$$x \xrightarrow{+} \frac{x^2+1}{x} \xrightarrow{-} \frac{x^3-x}{x^2+1} \xrightarrow{+} \frac{x^4+1}{x^3-x} \xrightarrow{-} \frac{x^5+x^3}{x^4+1} \tag{4}$$

defines the regular continued fraction

$$\frac{x^5+x^3}{x^4+1} = [x, x, x, x, x].$$

Table 1 displays the vertices of the path in (4) and their associated convergents.

i	Vertex u_i/v_i	Convergent A_i/B_i
0	x	x
1	$\frac{x^2+1}{x}$	$\frac{x^2+1}{x}$
2	$\frac{x^3-x}{x^2+1}$	$\frac{x^3-x}{x^2+1}$
3	$\frac{x^4+1}{x^3-x}$	$\frac{x^4+1}{x^3-x}$
4	$\frac{x^5+x^3}{x^4+1}$	$\frac{x^5+x^3}{x^4+1}$

Table 1: The vertices of the path and their associated convergents of $[x, x, x, x, x]$.

By Corollary 2, the path

$$x \xrightarrow{-} \frac{x^2-1}{x} \xrightarrow{+} \frac{x^3}{x^2+1} \xrightarrow{-} \frac{x^4+x^2-1}{x^3-x} \xrightarrow{+} \frac{-x^5+x}{-x^4-x^2+1} \tag{5}$$

defines the regular continued fraction

$$\frac{-x^5+x}{-x^4-x^2+1} = [x, -x, -x, -x, x].$$

Table 2 displays the vertices of the path in (5) and their associated convergents.

i	Vertex u_i/v_i	Convergent A_i/B_i
0	x	x
1	$\frac{x^2-1}{x}$	$\frac{-x^2+1}{-x}$
2	$\frac{x^3}{x^2+1}$	$\frac{x^3}{x^2+1}$
3	$\frac{x^4+x^2-1}{x^3-x}$	$\frac{-x^4-x^2+1}{-x^3+x}$
4	$\frac{-x^5+x}{-x^4-x^2+1}$	$\frac{-x^5+x}{-x^4-x^2+1}$

Table 2: The vertices of the path and their associated convergents of $[x, -x, -x, -x, x]$.

By Corollary 3, the path

$$x \xrightarrow{+} \frac{x^2+1}{x} \xrightarrow{+} \frac{x^3}{x^2-1} \xrightarrow{+} \frac{x^4-x^2-1}{x^3+x} \xrightarrow{+} \frac{-x^5-x^4+x^2+x+1}{-x^4-x^3+x^2-x+1} \tag{6}$$

defines the regular continued fraction

$$\frac{-x^5 - x^4 + x^2 + x + 1}{-x^4 - x^3 + x^2 - x + 1} = [x, x, -x, x, x + 1].$$

Table 3 displays the vertices of the path in (6) and their associated convergents.

i	Vertex u_i/v_i	Convergent A_i/B_i
0	x	x
1	$\frac{x^2+1}{x}$	$\frac{x^2+1}{x}$
2	$\frac{x^3}{x^2-1}$	$\frac{-x^3}{-x^2+1}$
3	$\frac{x^4-x^2-1}{x^3+x}$	$\frac{-x^4+x^2+1}{-x^3-x}$
4	$\frac{-x^5-x^4+x^2+x+1}{-x^4-x^3+x^2-x+1}$	$\frac{-x^5-x^4+x^2+x+1}{-x^4-x^3+x^2-x+1}$

Table 3: The vertices of the path and their associated convergents of $[x, x, -x, x, x+1]$.

By Corollary 4, the path

$$x \xrightarrow{-} \frac{x^2-1}{x} \xrightarrow{-} \frac{x^3+x}{x^2-1} \xrightarrow{-} \frac{-x^4+x^2+1}{-x^3} \xrightarrow{-} \frac{x^6-x^4-x^3-x^2-x}{x^5-x^2+1} \tag{7}$$

defines the regular continued fraction

$$\frac{x^6 - x^4 - x^3 - x^2 - x}{x^5 - x^2 + 1} = [x, -x, x, x, -x^2].$$

Table 4 displays the vertices of the path in (7) and their associated convergents.

i	Vertex u_i/v_i	Convergent A_i/B_i
0	x	x
1	$\frac{x^2-1}{x}$	$\frac{-x^2+1}{-x}$
2	$\frac{x^3+x}{x^2-1}$	$\frac{-x^3-x}{-x^2+1}$
3	$\frac{-x^4+x^2+1}{-x^3}$	$\frac{-x^4+x^2+1}{-x^3}$
4	$\frac{x^6-x^4-x^3-x^2-x}{x^5-x^2+1}$	$\frac{x^6-x^4-x^3-x^2-x}{x^5-x^2+1}$

Table 4: The vertices of the path and their associated convergents of $[x, -x, x, x, -x^2]$.

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