#A31 INTEGERS 26 (2026)

ON THE SUMSET OF SETS OF SIZE k

Vincent Schinina
Department of Mathematics, CUNY Graduate Center, New York, New York
vschinina@gradcenter.cuny.edu

Received: 7/27/25, Accepted: 1/23/26, Published: 2/20/26

Abstract

The set $\mathcal{R}_G(h,k)$ consists of all possible sizes for the h-fold sumset of sets containing k elements from an additive abelian group G. The exact makeup of this set is still unknown, but there has been progress towards determining which integers are present. We know that $\mathcal{R}_G(h,k) \subseteq \left[hk-h+1, \binom{h+k-1}{h}\right]$, where the right side is an interval of integers that includes the endpoints. These endpoints are known to be attained. We will prove that the integers in $[hk-h+2, hk-1]$ are not possible sizes for the h-fold sumset of a set containing $k \geq 4$ elements of a torsion-free additive abelian group G. Furthermore, we will confirm that this interval cannot be made larger by exhibiting a subset of G whose h-fold sumset has size hk.

1. Introduction

In a paper by Nathanson [5], he posed a problem, labeled Problem 1, about computing $\mathcal{R}_G(h,k)$ for an additive abelian group G for a fixed k as h increases. Of course, the most important case is the additive abelian group of the integers, but many results for the integers use only their ordering and generalize to all ordered abelian groups. In Nathanson's Theorem 7, he confirmed that $hk-h+1$ is the minimum element of $\mathcal{R}_G(h,k)$ and that $hk-h+2$ is not in $\mathcal{R}_G(h,k)$, where G is an ordered additive abelian group. Moreover, from Nathanson's Theorem 2, it follows that the maximum of $\mathcal{R}_G(h,k)$ is $\binom{h+k-1}{h}$. In this paper, we will provide more insight about the structure of $\mathcal{R}_G(h,k)$ in a step towards solving Nathanson's Problem 1 for a torsion-free additive abelian group G. As motivation for the results, $G=\mathbb{Z}$ was examined first. Data retrieved from a computer program appeared to show that the result $hk-h+2$ is not in $\mathcal{R}_{\mathbb{Z}}(h,k)$ could be extended to the result that the inclusive interval of integers $[hk-h+2, hk-1]$ is not in $\mathcal{R}_{\mathbb{Z}}(h,k)$, for all $h>1$ and $k \geq 4$. This was proven in the first version of this paper.[1] From that

DOI: 10.5281/zenodo.18714443

[1] For an alternate method of proving this result, see the papers by Mohan and Pandey [3], and Tang and Xing [7].

proof, it seemed that the result could be extended further to any torsion-free additive abelian group G by using the same method. To do this, we will define all the necessary terminology and then build up some useful preliminary results that will allow us to confirm this observation. Lastly, we will conclude that this interval cannot be enlarged by showing that $hk \in \mathcal{R}_G(h,k)$. For the construction of explicit elements of the sumset size set $\mathcal{R}_{\mathbb{Z}}(h,k)$, see Nathanson [4].

2. Terminology

We will begin by introducing definitions and notation that will be used throughout this paper. The knowledge of groups, group properties, and total orders will be assumed throughout the paper. We denote some common number systems with the following symbols.

$$\text{Natural Numbers: } \mathbb{N} = \{1,2,3,4,5,\dots\},$$
$$\text{Natural Numbers with 0: } \mathbb{N}_0 = \{0,1,2,3,4,\dots\},$$
$$\text{Integers: } \mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}.$$

Also, we write $[a,b]$ to denote the inclusive interval of integers between a and b. That is,

$$[a,b] = \big\{c \in \mathbb{Z} \mid a \leq c \leq b\big\}.$$

Definition 1. A group G with binary operation $*$ is *ordered* if it has a total order $\preceq$ satisfying the condition that if $a \prec b$, then $a * c \prec b * c$ and $c * a \prec c * b$, for all $a,b,c \in G$. Note that any subset A of G inherits this property.

Remark 1. Every ordered abelian group is torsion-free. Furthermore, Levi [2] and [1] proved that an abelian group is ordered if and only if it is torsion-free.

For an ordered group G with identity e and total order $\preceq$, we use the following symbols to denote some important subsets of G.

$$\text{Positive Elements of } G\text{: } G^+ = \big\{g \in G \mid e \prec g\big\},$$
$$\text{Nonnegative Elements of } G\text{: } G_e = \big\{g \in G \mid e \preceq g\big\},$$
$$\text{Negative Elements of } G\text{: } G^- = \big\{g \in G \mid g \prec e\big\}.$$

Definition 2. Let $h \in \mathbb{N}$ with $h > 1$. The *h-fold sumset* of a subset A of an additive abelian group G is

$$hA = \big\{a_1 + \cdots + a_h \mid a_1,\dots,a_h \in A\big\}.$$

Note that $a_1,\dots,a_h \in A$ do not need to be distinct. For a set $A = \{a_1,\dots,a_k\} \subseteq G$, we may also say

$$hA = \Big\{x_1a_1 + \cdots + x_ka_k \;\Big|\; x_1,\dots,x_k \in \mathbb{N}_0 \text{ and } x_1 + \cdots + x_k = h\Big\}.$$

Remark 2. Sumset addition is translation invariant: $\left|h\left(A+t\right)\right| = |hA|$, for all $A \subseteq G$ and $t \in G$.

The set of h-fold sumset sizes for finite sets A with size k contained in an additive abelian group G is denoted by

$$\mathcal{R}_G(h,k) = \left\{|hA| \,\middle|\, A \subseteq G,\, |A| = k\right\}.$$

Definition 3. Let $A = \{a_1, a_2, \ldots, a_k\}$ be a subset of an ordered additive abelian group G with total order $\preceq$ such that $a_1 \prec a_2 \prec \cdots \prec a_k$. An element of hA is called *trivial* if it appears on the increasing list (uses the ordered assumption)

$$\begin{aligned} ha_1 &\prec (h-1)a_1 + a_2 \prec \cdots \prec a_1 + (h-1)a_2 \prec ha_2 \\ &\prec (h-1)a_2 + a_3 \prec \cdots \prec a_2 + (h-1)a_3 \prec ha_3 \\ &\vdots \\ &\prec (h-1)a_{k-1} + a_k \prec \cdots \prec a_{k-1} + (h-1)a_k \prec ha_k. \end{aligned}$$

Otherwise, the element is called *nontrivial*. Note that the above list contains $hk - h + 1$ elements of hA.

3. A Missing Interval from $\mathcal{R}_G(h,k)$

Before we can prove that the integers in $[hk-h+2, hk-1]$ are missing from $\mathcal{R}_G(h,k)$, we will need to confirm some useful preliminary results.

Lemma 1. *Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{a_1, a_2, \ldots, a_{k-1}\}$ be subsets of an ordered additive abelian group G, for some $k \in \mathbb{N}$ with $k > 1$ and $a_1 \prec \cdots \prec a_k$. Then, $hB \subseteq hA$. Furthermore, if c is a nontrivial element of hB, then c is also a nontrivial element of hA.*

Proof. Let $c \in hB$. Then, by Definition 2,

$$c = x_1a_1 + x_2a_2 + \cdots + x_{k-1}a_{k-1},$$

where $x_1, \ldots, x_{k-1} \in \mathbb{N}_0$ and $x_1 + \cdots + x_{k-1} = h$. Since $B \subseteq A$, then this also satisfies the definition of hA. Hence, $c \in hA$, and we have that $hB \subseteq hA$.

To prove the last statement, we will prove its contrapositive. Let $c \in hB$. Suppose that c is a trivial element of hA, which makes sense because $hB \subseteq hA$, as we just showed. Then, by Definition 3,

$$c = (h-i)a_j + ia_{j+1},$$

for some $0 \leq i \leq h$ and $1 \leq j \leq k-1$. If $j = k-1$ and $1 \leq i \leq h$, then, using the ordered property,

$$ha_{k-1} = (h-i)a_{k-1} + ia_{k-1} \prec (h-i)a_{k-1} + ia_k = c,$$

where strict inequality is always ensured because $i \neq 0$ and $a_{k-1} \prec a_k$. Since the largest element of hB is ha_{k-1}, then this implies $c \notin hB$, which is a contradiction. Otherwise, c is a trivial element of hB, by Definition 3, which proves the contrapositive. Therefore, if c is a nontrivial element of hB, then c is also a nontrivial element of hA. $\square$

Theorem 1. *Let $k \geq 4$ and let G be an ordered additive abelian group with respect to $\preceq$. Suppose $0 \prec a_1 \prec a_2 \prec \cdots \prec a_{k-3}$ for $a_i \in G^+$. If*

$$A = \{0, a_1, a_2, \ldots, a_{k-3}, a_{k-3} + b, a_{k-3} + b + c\},$$

for some $b, c \in G^+$, and $c \neq db$, for all $d \in \mathbb{N}$, then hA has at least $h-1$ nontrivial elements.

Proof. As in Definition 3, we can write out the $hk - h + 1$ trivial elements of hA. With simplifications, we are left with

$$\begin{aligned} 0 &\prec a_1 \prec \cdots \prec (h-1)a_1 \prec ha_1 \\ &\prec (h-1)a_1 + a_2 \prec \cdots < a_1 + (h-1)a_2 \prec ha_2 \\ &\vdots \\ &\prec ha_{k-3} + b \prec \cdots \prec ha_{k-3} + (h-1)b \prec h(a_{k-3} + b) \\ &\prec h(a_{k-3} + b) + c \prec \cdots \prec h(a_{k-3} + b) + (h-1)c \prec h(a_{k-3} + b + c). \end{aligned} \tag{1}$$

Observe that for all $e \in [1, h-1]$,

$$ha_{k-3} \prec ha_{k-3} + eb + c \prec h(a_{k-3} + b) + c$$

and

$$ha_{k-3} + eb + c = (h-e)a_{k-3} + (e-1)(a_{k-3} + b) + (a_{k-3} + b + c).$$

Hence, $ha_{k-3} + eb + c \in hA$, for all $e \in [1, h-1]$, and they are among or between the elements seen in the second to last line of inequalities in Equation (1). These are $h-1$ nontrivial elements of hA, otherwise

$$ha_{k-3} + eb + c = ha_{k-3} + fb,$$

for some $f \in [1, h]$, will contradict the assumption that $c \neq db$, for all $d \in \mathbb{N}$. $\square$

Theorem 2. *Let $k \geq 4$ and let G be an ordered additive abelian group with respect to $\preceq$. Suppose $0 \prec a_1 \prec a_2 \prec \cdots \prec a_{k-3}$ for $a_i \in G^+$. If*

$$A = \{0, a_1, a_2, \ldots, a_{k-3}, a_{k-3} + b, a_{k-3} + b + c\}$$

and $c = db$, for some $b, c \in G^+$ and $d \in \mathbb{N}$ with $d > 1$, then hA has at least $h - 1$ nontrivial elements.

Proof. As in Definition 3, we can write out the $hk - h + 1$ trivial elements of hA using $c = db$. With simplifications, we have

$$\begin{aligned} 0 &\prec a_1 \prec \cdots \prec (h-1)a_1 \prec ha_1 \\ &\prec (h-1)a_1 + a_2 \prec \cdots \prec a_1 + (h-1)a_2 \prec ha_2 \\ &\vdots \\ &\prec ha_{k-3} + b \prec \cdots \prec ha_{k-3} + (h-1)b \prec h(a_{k-3} + b) \\ &\prec h(a_{k-3} + b) + db \prec \cdots \prec h(a_{k-3} + b) + (h-1)db \prec h(a_{k-3} + b + db). \end{aligned} \tag{2}$$

Observe that for all $e \in [1, h-1]$,

$$h(a_{k-3} + b) \prec h(a_{k-3} + b) + (ed - 1)b \prec h(a_{k-3} + b + db), \tag{3}$$

where the first inequality needs the fact that $d > 1$. Notice that $h(a_{k-3} + b) + (ed - 1)b \in hA$, for all $e \in [1, h-1]$ because

$$h(a_{k-3} + b) + (ed - 1)b = a_{k-3} + (h - e - 1)(a_{k-3} + b) + e(a_{k-3} + b + db) \in hA,$$

where we used db in place of c as allowed by the assumption that $c = db$.

From the inequality in Equation (3), these elements are among or between the elements in the last line of Equation (2). These are $h-1$ nontrivial elements of hA, otherwise

$$h(a_{k-3} + b) + (ed - 1)b = h(a_{k-3} + b) + fdb,$$

for some $f \in [1, h-1]$, will imply that $d = 1$, which contradicts $d > 1$. □

Theorem 3. *Let G be an ordered additive abelian group with respect to $\preceq$. For every $k \geq 4$, if $A = \{0, a, a + b_1, \ldots, a + b_1 + \cdots + b_{k-2}\} \subseteq G$ with $a, b_1, \ldots, b_{k-2} \in G^+$ and $b_{k-2} = b_{k-3}$, then either A is a k-term arithmetic progression or the set $B = \{0, a, a + b_1, \ldots, a + b_1 + \cdots + b_{k-3}\} \subseteq G$ produces the sumset hB that has at least $h - 1$ nontrivial elements.*

Proof. We will proceed by induction on k. For the base case, suppose $k = 4$. Then, we assume $A = \{0, a, a + b_1, a + b_1 + b_2\}$ with $a, b_1, b_2 \in G^+$ and $b_2 = b_1$. This means we are working with $A = \{0, a, a + b_1, a + 2b_1\}$ and $B = \{0, a, a + b_1\}$. We can list the $2h+1$ trivial elements of hB as seen in Definition 3. With simplifications, we obtain

$$\begin{aligned} &0 \prec a \prec 2a \prec \cdots \prec ha \\ &\quad \prec ha + b_1 \prec ha + 2b_1 \prec \cdots \prec ha + (h-1)b_1 \prec h\,(a+b_1)\,. \end{aligned} \tag{4}$$

Now, we will split this into two cases.

`Case 1:` $b_1 \neq ea$, for all $e \in \mathbb{N}$. Observe that for all $f \in [1, h-1]$,

$$a \prec fa + b_1 \prec ha + b_1 \quad \text{and} \quad fa + b_1 = (h-f)0 + (f-1)a + (a+b_1).$$

So, we have that $fa + b_1 \in hB$ and they are among or between the elements in the first line of Equation (4). These are $h-1$ nontrivial elements of hB, otherwise

$$fa + b_1 = ga,$$

for some $g \in [2, h]$, will contradict the Case 1 assumption.

`Case 2:` $b_1 = ea$, for some $e \in \mathbb{N}$. If $e = 1$, then we have that $b_1 = a$ and $A = \{0, a, a+b_1, a+2b_1\}$ becomes $A = \{0, a, 2a, 3a\}$, which means A is a 4-term arithmetic progression. Otherwise, $e > 1$, which means $A = \{0, a, a+b_1, a+2b_1\}$ becomes $A = \big\{0, a, (e+1)a, (2e+1)a\big\}$, and $B = \big\{0, a, (e+1)a\big\}$. Rewriting Equation (4), we have

$$\begin{aligned} &0 \prec a \prec 2a \prec \cdots \prec ha \\ &\quad \prec (h+e)a \prec (h+2e)a \prec \cdots \prec \big[h + (h-1)e\big]\, a \prec (h+he)a. \end{aligned} \tag{5}$$

Observe that for all $f \in [1, h-1]$,

$$ha \prec (h + ef - 1)\, a \prec (h+he)a, \tag{6}$$

where the first inequality needs the assumption that $e > 1$. So, $(h+ef-1)\,a \in hB$ for all $f \in [1, h-1]$ because

$$(h+ef-1)\,a = 0 + (h-f-1)a + f(e+1)a \in hB.$$

From the inequality in Equation (6), we know that these elements are among or between the elements in the second line of Equation (5). These are $h-1$ nontrivial elements of hB, otherwise

$$(h+ef-1)\,a = (h+ge)a,$$

for some $g \in [1, h-1]$, will imply that $e = 1$, which contradicts $e > 1$.

At this point, we exhausted all necessary cases. Collecting all the results, we see that either A is a 4-term arithmetic progression or hB has at least $h-1$ nontrivial elements. Thus, the base case holds.

For the inductive hypothesis, assume if $A = \{0, a, a+b_1, \ldots, a+b_1+\cdots+b_{k-2}\}$ with $a, b_1, \ldots, b_{k-2} \in G^+$ and $b_{k-2} = b_{k-3}$, then either A is k-term arithmetic progression or the set $B = \{0, a, a+b_1, \ldots, a+b_1+\cdots+b_{k-3}\}$ produces a sumset hB that has at least $h-1$ nontrivial elements. The inductive step requires us to show that if $A = \{0, a, a+b_1, \ldots, a+b_1+\cdots+b_{k-1}\}$ with $a, b_1, \ldots, b_{k-1} \in G^+$ and $b_{k-1} = b_{k-2}$, then either A is a $(k+1)$-term arithmetic progression or the set $B = \{0, a, a+b_1, \ldots, a+b_1+\cdots+b_{k-2}\}$ produces a sumset hB that has at least $h-1$ nontrivial elements. Start by assuming $A = \{0, a, a+b_1, \ldots, a+b_1+\cdots+b_{k-1}\}$ with $a, b_1, \ldots, b_{k-1} \in G^+$ and $b_{k-1} = b_{k-2}$. Then, we are working with

$$A = \{0, a, a+b_1, \ldots, a+b_1+\cdots+b_{k-2}, a+b_1+\cdots+2b_{k-2}\},$$
$$B = \{0, a, a+b_1, \ldots, a+b_1+\cdots+b_{k-2}\}.$$

We will need two cases.

`Case 1:` $b_{k-2} \neq e b_{k-3}$, for all $e \in \mathbb{N}$. By Theorem 1 on B, we have that there are at least $h-1$ nontrivial elements in hB.

`Case 2:` $b_{k-2} = e b_{k-3}$, for some $e \in \mathbb{N}$. If $e > 1$, then, by Theorem 2 on B, there are at least $h-1$ nontrivial elements in hB. Otherwise, $e = 1$ and this means $b_{k-2} = b_{k-3}$. Therefore, by the inductive hypothesis, either B is a k-term arithmetic progression or the set $C = \{0, a, a+b_1, \ldots, a+b_1+\cdots+b_{k-3}\}$ produces the sumset hC that has at least $h-1$ nontrivial elements. In the case that B is a k-term arithmetic progression, then it follows that $b_i = a$, for all $1 \leq i \leq k-2$. Substituting $b_i = a$ into the elements in A, for all $1 \leq i \leq k-2$, gives us

$$A = \left\{0, a, 2a, \ldots, (k-1)a, ka\right\},$$

which indicates that A is a $(k+1)$-term arithmetic progression. In the case that hC has at least $h-1$ nontrivial elements, then, by Lemma 1, these elements are also nontrivial in hB.

At this point, we considered all the necessary cases and it can be seen that either A is a $(k+1)$-term arithmetic progression or hB has at least $h-1$ nontrivial elements. This completes the inductive step.

Thus, we have that for every $k \geq 4$, if $A = \{0, a, a+b_1, \ldots, a+b_1+\cdots+b_{k-2}\}$ with $a, b_1, \ldots, b_{k-2} \in \mathbb{N}$ and $b_{k-2} = b_{k-3}$, then either A is a k-term arithmetic progression or the set $B = \{0, a, a+b_1, \ldots, a+b_1+\cdots+b_{k-3}\}$ produces the sumset hB that has at least $h-1$ nontrivial elements. □

The next theorem is the main result. Most of the heavy lifting occurred in the preceding lemmas and theorems.

Theorem 4. *Let G be a torsion-free additive abelian group. For all $k \geq 4$, $\mathcal{R}_G(h,k) \cap [hk-h+2, hk-1] = \varnothing$.*

Proof. Let $k \geq 4$ be given. By assumption, G is torsion-free. This means that G can be ordered and we will say that it is ordered with respect to $\preceq$. Since a translation does not change the size of the sumset, it suffices to consider the set $A = \{0, a_1, \ldots, a_{k-1}\} \subseteq G$ with $0 \prec a_1 \prec \cdots \prec a_{k-1}$. Based on this inequality, we have that

$$a_{k-2} = a_{k-3} + b \quad \text{and} \quad a_{k-1} = a_{k-3} + b + c$$

for some unique $b, c \in G^+$. This leaves us with

$$A = \{0, a_1, a_2, \ldots, a_{k-3}, a_{k-3} + b, a_{k-3} + b + c\}.$$

We will need two cases.

`Case 1:` $c \neq db$, for all $d \in \mathbb{N}$. By Theorem 1, it follows that hA has at least $h-1$ nontrivial elements. As noted in Definition 3, hA has $hk - h + 1$ trivial elements. So, we must have that

$$|hA| \geq (hk - h + 1) + (h - 1) = hk.$$

`Case 2:` $c = db$, for all $d \in \mathbb{N}$. If $d > 1$, then, by Theorem 2, we know that hA has at least $h-1$ nontrivial elements. As noted in Definition 3, hA has $hk - h + 1$ trivial elements. So, we must have that

$$|hA| \geq (hk - h + 1) + (h - 1) = hk.$$

If $d = 1$, then we have that $c = b$. By Theorem 3, we know that either A is a k-term arithmetic progression or that the set

$$B = \{0, a_1, a_2, \ldots, a_{k-3}, a_{k-3} + b\}$$

produces the sumset hB with at least $h-1$ nontrivial elements. In the case that A is a k-term arithmetic progression, then we know from Theorem 2 in Nathanson's paper [5] that $|hA| = hk - h + 1$. In the case that hB has at least $h-1$ nontrivial elements, it follows by Lemma 1 that hA has at least $h-1$ nontrivial elements. As noted in Definition 3, hA has $hk - h + 1$ trivial elements. Therefore, we must have that

$$|hA| \geq (hk - h + 1) + (h - 1) = hk.$$

At this point, we considered all the required cases and saw that either

$$|hA| = hk - h + 1 \quad \text{or} \quad |hA| \geq hk.$$

Thus, for all $k \geq 4$,

$$\mathcal{R}_G(h, k) \cap [hk - h + 2, hk - 1] = \varnothing,$$

as desired. □

An immediate question to ask is: can this interval be enlarged? We know from Theorem 2 in Nathanson's paper [5] that a k-term arithmetic progression produces an h-fold sumset with size $hk - h + 1$. So, the interval cannot be extended to the left. As for extending it to the right, we prove the following result.

Theorem 5. *Let G be an ordered additive abelian group with respect to $\preceq$ and let $a \in G^+$. For $k \geq 3$, if $A = \{0, a, 2a, \ldots, (k-2)a, ka\}$, then $|hA| = hk$.*

Proof. We will prove that $hA = \Big\{ ja \mid j \in [0, hk] - \{hk-1\} \Big\}$. By assumption, $a \in G^+$, which means that $hA \subseteq \Big\{ ja \mid j \in [0, hk] \Big\}$. For all $c \in hA$ with $c \neq h(ka)$,

$$c = x_1(0) + x_2(a) + x_3(2a) + \cdots + x_{k-1}\left[(k-2)a\right] + x_k(ka),$$

where $x_1, \ldots, x_k \in \mathbb{N}_0$ with $x_1 + \cdots + x_k = h$, by Definition 2. Since $c \neq h(ka)$, then x_i, for some $1 \leq i \leq k-1$, must be nonzero. Since $a \in G^+$, then $0 \prec a \prec 2a \prec \cdots \prec (k-2)a \prec ka$. It follows that

$$c \preceq (k-2)a + (h-1)(ka) = (hk-2)a.$$

Hence, $(hk-1)a \notin hA$, which means $hA \subseteq \Big\{ ja \mid j \in [0, hk] - \{hk-1\} \Big\}$. Now, we will confirm that $\Big\{ ja \mid j \in [0, hk] - \{hk-1\} \Big\} \subseteq hA$. Observe that

$$(hk)a = h(ka) \in hA \quad \text{and} \quad (hk-i)a = (k-i)a + (h-1)(ka) \in hA,$$

for all $2 \leq i \leq k$. It remains to show that

$$\Big\{ ja \mid j \in \big[0, hk-(k+1)\big] \Big\} = \Big\{ ja \mid j \in \big[0, (h-2)k + (k-1)\big] \Big\} \subseteq hA.$$

Let $c \in \Big\{ ja \mid j \in \big[0, (h-2)k + (k-1)\big] \Big\}$. By the division algorithm, there exists $q \in [0, h-2]$ and $r \in [0, k-1]$ such that

$$j = qk + r.$$

For all r, we get

$$c = ja = \begin{cases} (h-q-1)0 + ra + q(ka) & \text{if } 0 \leq r \leq k-2, \\ (h-q-2)0 + a + (k-2)a + q(ka) & \text{if } r = k-1. \end{cases}$$

It now follows that $\Big\{ ja \mid j \in [0, hk] - \{hk-1\} \Big\} \subseteq hA$.

Therefore, with both inclusions, we have that $hA = \Big\{ ja \mid j \in [0, hk] - \{hk-1\} \Big\}$ which has size hk as desired. □

As a consequence of this result, we see that the missing interval cannot be extended to the right. Thus, we obtain the following.

Corollary 1. *Let G be a torsion-free additive abelian group. For all $k \geq 4$, $\mathcal{R}_G(h,k) \cap [hk-h+1, hk] = \{hk-h+1, hk\}$.*

Proof. By a previous remark, we know that G is a torsion-free additive abelian group if and only if G is ordered. By Theorem 2 in Nathanson's paper [5], Theorem 4, and Theorem 5, it follows that

$$\mathcal{R}_G(h,k) \cap [hk-h+1, hk] = \{hk-h+1, hk\},$$

for all $k \geq 4$. □

We conclude by sharing that Isaac Rajagopal generalized Theorem 4 a few months after the original submission of this paper. For the sake of clarity, we provide a modified statement of Theorem 1.3 from Rajagopal's paper [6] below.

Theorem 6 ([6])**.** *For all $h, k \in \mathbb{N}$, we have*

$$\mathcal{R}_G(h,k) \cap \left(\bigcup_{\ell=0}^{\min(h,k)-3} \big[hk-h+1+\ell h+1, hk-h+1+\ell h+(h-2-\ell)\big] \right) = \varnothing.$$

We note that Theorem 4 is a special case of Theorem 6 at $\ell = 0$. See Rajagopal's paper [6] for more details.

Acknowledgements. The author sincerely thanks Melvyn Nathanson for introducing him to this topic, for providing the necessary resources and guidance to confirm these results, and for the helpful feedback on this paper. The author also thanks the referee for their suggestions.

References

[1] F. W. Levi, Ordered groups, *Proceedings of the Indian Academy of Sciences-Section A* **16** (4) (1942), 256–263.

[2] F. W. Levi, Contributions to the theory of ordered groups, *Proceedings of the Indian Academy of Sciences-Section A* **17** (6) (1943), 199–201.

[3] Mohan and R. K. Pandey, Extended inverse theorems for h-fold sumsets in integers, *Contrib. Discrete. Math.* **18** (2) (2023), 129–145.

[4] M. B. Nathanson, Explicit sumset sizes in additive number theory, preprint, `arXiv:2505.05329`.

[5] M. B. Nathanson, Problems in additive number theory. vi: Sizes of sumsets of finite sets, *Acta Math. Hungar.* **176** (2025), 498–521.

[6] I. Rajagopal, Possible sizes of sumsets, preprint, `arXiv:2510.23022`.

[7] M. Tang and Y. Xing, Some inverse results of sumsets, *Bull. Korean Math. Soc.* **58** (2) (2021), 305–313.