



**AN EXTENSION OF WOLSTENHOLME'S THEOREM BY
BINOMIAL COEFFICIENTS**

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Abstract

In this paper, we prove an extended version of the p -adic mean value theorem, then apply it several times to refine Wolstenholme's theorem up to modulo p^7 . We also introduce a new family of congruences among binomial coefficients.

1. Introduction and Preliminaries

Wolstenholme's well-known theorem states that for any prime $p \geq 5$, we have

$$\binom{2p-1}{p-1} = \frac{1}{2} \binom{2p}{p} \equiv 1 \pmod{p^3}.$$

This theorem has been generalized in a variety of ways (see [8] for a survey). We recall here only some relatively recent generalizations. J. Zhao [18] demonstrated that

$$\binom{np}{mp} / \binom{n}{m} \equiv 1 + w_p nm(n-m)p^3 \pmod{p^5},$$

where $H_m(n) = \sum_{i=1}^m \frac{1}{i^n}$ denotes the m -th generalized harmonic number of order n , and $w_p < p^2$ is the unique nonnegative integer such that $w_p \equiv \frac{H_{p-1}(1)}{p^2} \pmod{p^2}$. C. Helou and G. Terjanian [4] established that for $p \geq 5$,

$$\binom{np}{mp} \equiv \binom{n}{m} \left(1 - mn(n-m) \left(\frac{p^3}{2} B_{p^3-p^2-2} - \frac{p^5}{6} B_{p-3} + \frac{1}{5} (m^2 - mn + n^2) p^5 B_{p-5} \right) \right) \pmod{p^{6+v_p((n-m)\binom{n}{m})}},$$

where B_n denotes the n -th Bernoulli number and v_p is the normalized p -adic valuation with $|p|_p = p^{-v_p(p)} = p^{-1}$. R. Meštrović [7] showed that

$$\binom{2p-1}{p-1} \equiv 1 - 2pH_{p-1}(1) + 4p^2H_{p-1}(1,1) \pmod{p^7},$$

where $H_m(1, 1) = \sum_{1 \leq i < j \leq m} \frac{1}{ij}$, which is an example of a multiple harmonic sum. More generally, for a k -tuple $(n_1, \dots, n_k) \in \mathbb{N}^k$ and a positive integer m , these sums are defined by

$$H_m(n_1, \dots, n_k) := \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq m} \frac{1}{j_1^{n_1} \dots j_k^{n_k}},$$

and $H_m(n_1, \dots, n_k) = 0$ if $m < k$. J. Rosen [15] extended and optimized congruences for $\binom{kp-1}{p-1}$ modulo p^{2n+3} and gave a general family of polynomials $\{b_{j,n}(k)\}_{0 \leq j \leq n}$ (called extremal polynomials) such that

$$\binom{kp-1}{p-1} \equiv \sum_{j=0}^n b_{j,n}(k) p^j H_{p-1}(\{1\}^j) \pmod{p^{2n+3}},$$

where $H_{p-1}(\{1\}^j) = H_{p-1}(\underbrace{1, \dots, 1}_j)$.

Most of these extensions add extra terms to the congruence, typically involving multiple harmonic sums or Bernoulli numbers. However, in this paper, we expand these congruences in terms of binomial coefficients of the form $\binom{ap}{bp}$ up to modulo p^7 . The general method remains applicable for higher orders, but the computations become increasingly intricate. In the course of the proof, we uncover a new family of congruences among binomial coefficients.

There are numerous congruences that arise from p -adic analysis or p -adic related functions, such as the p -adic L -function and the p -adic gamma function (see, for example, [12, 14, 17]). Many of these congruences are concerned with multiple harmonic sums and (generalized) Bernoulli numbers.

We briefly recall some basic facts about p -adic numbers (for comprehensive treatments see [2, 10, 12, 16]). The field \mathbb{Q}_p of p -adic numbers is the completion of \mathbb{Q} with respect to the p -adic norm. Since \mathbb{Q}_p is not algebraically closed, one considers its algebraic closure $\overline{\mathbb{Q}_p}$, whose completion is denoted by \mathbb{C}_p . The field \mathbb{C}_p is both algebraically closed and complete. Moreover,

$$v_p(\mathbb{Q}_p^\times) = \{v_p(x) : x \in \mathbb{Q}_p^\times\} = \mathbb{Z} \text{ and } v_p(\mathbb{C}_p^\times) = \{v_p(x) : x \in \mathbb{C}_p^\times\} = \mathbb{Q}.$$

The set

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : v_p(x) \geq 0\} = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} = \overline{B}(0, 1)$$

is a subring of \mathbb{Q}_p , known as the ring of p -adic integers. We write $|\cdot|$ for $|\cdot|_p$ when the choice of prime is clear from the context. Each $x \in \mathbb{Z}_p$ admits a unique p -adic expansion $x = \sum_{k=0}^\infty a_k p^k$, where $a_k \in \{0, 1, \dots, p-1\}$. Define $x_n = \sum_{k=0}^{n-1} a_k p^k \in \mathbb{N}$. Then $|x - x_n|_p \leq p^{-n} \rightarrow 0$ as $n \rightarrow \infty$, so $x_n \rightarrow x$ in \mathbb{Z}_p . Hence, every p -adic integer is the limit of a sequence of natural numbers. In other words, \mathbb{N} is dense in \mathbb{Z}_p .

Throughout this paper, we take \mathbb{K} to be either \mathbb{Q}_p , \mathbb{C}_p , or any intermediate extension. For $v \in \mathbb{Q}$, we generalize the notion of congruence in \mathbb{K} by declaring

$$x \equiv y \pmod{p^v} \text{ if and only if } x - y \in p^v \mathcal{O}_{\mathbb{K}} \text{ if and only if } |x - y| \leq p^{-v},$$

where $\mathcal{O}_{\mathbb{K}} = \{x \in \mathbb{K} : |x| \leq 1\}$ is the valuation ring of \mathbb{K} .

For a power series $f = \sum_{n \geq 0} a_n x^n \in \mathbb{K}[[x]]$, we define its *sup norm* to be $\|f\| = \sup_{n \geq 0} |a_n|$. When $|t| \leq 1$, we clearly have $\|f(t)\| \leq \|f\|$. The subspace

$$\mathbb{K}\{x\} = \left\{ f = \sum_{n \geq 0} a_n x^n \in \mathbb{K}[[x]] : a_n \rightarrow 0 \right\}$$

is the completion of the polynomial space under the sup norm, and we have $\mathbb{K}[x] \subset \mathbb{K}\{x\} \subset \mathbb{K}[[x]]$ (see [12, p. 233] for further details).

1.1. Higher p -adic Mean Value Theorem

The Mean Value Theorem is a fundamental result in real analysis; however, it does not generally hold in p -adic analysis, even for polynomial functions (consider, for example, $f(t) = t^p$). Nevertheless, A. Robert [12, 13] demonstrated that there exists a universal bound $r_p = |p|^{\frac{1}{p-1}}$ such that for all $|h| \leq |p|^{\frac{1}{p-1}}$ and f a vector-valued function over a complete field \mathbb{K} , the following inequality is satisfied:

$$\|f(t+h) - f(t)\| \leq |h| \|f'\|.$$

For $p \geq 3$ and $|h| \leq |p|^{\frac{1}{p-2}}$, he gave a second-order estimate as

$$\left\| f(t+h) - f(t) - hf'(t) \right\| \leq \left| \frac{h^2}{2} \right\| \|f''\|.$$

In [5, 12, 13, 19], the authors gave many applications of this theorem. Following A. Robert’s approach and introducing a minor modification that produces a higher order estimate, we establish an improved version of the theorem and then apply it several times to binomial coefficients.

Lemma 1. *Let $(\mathbb{B}, \|\cdot\|)$ be an ultrametric Banach space over \mathbb{K} . Let*

$$\mathbb{B}\{t\} = \left\{ f = \sum_{n \geq 0} a_n t^n \in \mathbb{B}[[t]] : a_n \rightarrow 0 \right\}$$

be equipped with the sup norm $\|f\| := \sup_{k \geq 0} \|a_k\|$. Let D be the formal differentiation operator $Df = f'$. Then, for every $n \in \mathbb{N}$, we have

$$\left\| \frac{D^n}{n!} \right\| \leq 1.$$

Proof. Let $f(t) = \sum_{k \geq 0} a_k t^k \in \mathbb{B}\{t\}$. Then

$$D^n f(t) = f^{(n)}(t) = \sum_{k \geq n} k(k-1) \cdots (k-n+1) a_k t^{k-n}.$$

By definition of the norm operator, we get

$$\begin{aligned} \left\| \frac{D^n f}{n!} \right\| &= \left\| \frac{f^{(n)}}{n!} \right\| = \sup_{k \geq n} \left\| \frac{k(k-1) \cdots (k-n+1)}{n!} a_k \right\| \\ &= \sup_{k \geq n} \left\| \binom{k}{n} a_k \right\| \leq \sup_{k \geq n} \|a_k\| \leq \sup_{k \geq 0} \|a_k\| = \|f\|, \end{aligned}$$

which means that $\left\| \frac{D^n f}{n!} \right\| \leq \|f\|$ and $\left\| \frac{D^n}{n!} \right\| \leq 1$. □

Theorem 1. *Let $(\mathbb{B}, \|\cdot\|)$ be an ultrametric Banach space over \mathbb{K} , and let $f(t) \in \mathbb{B}[t]$ be a polynomial with coefficients in \mathbb{B} . Suppose that $t, h \in \mathbb{K}$ with $|t| \leq 1$. Then*

$$\left\| f(t+h) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t)}{k!} h^k \right\| \leq \left| \frac{h^m}{m} \right| \|f^{(m)}\|,$$

provided that

$$|h| \leq |p|^{\frac{1}{p-m}} \quad \text{if } 1 \leq m < p, \quad \text{and} \quad |h| \leq |p|^{\frac{1}{p}} \quad \text{if } m = p.$$

Proof. We first consider the case $1 \leq m < p$. Using the Taylor expansion, we can write

$$f(t+h) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t)}{k!} h^k = h^m \sum_{k \geq m} \frac{h^{k-m}}{k(k-1) \cdots (k-m+1)} \frac{D^{k-m} f^{(m)}(t)}{(k-m)!}.$$

As $|t| \leq 1$, the ultrametric inequality implies

$$\left\| f(t+h) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t)}{k!} h^k \right\| \leq |h^m| \sup_{k \geq m} \left| \frac{h^{k-m}}{k(k-1) \cdots (k-m+1)} \right| \left\| \frac{D^{k-m} f^{(m)}}{(k-m)!} \right\|.$$

Since $\gcd(m, p) = 1$, we have $|m| = 1$ and therefore $\left| \frac{h^m}{m} \right| = |h^m|$. Thus, this inequality can be written as

$$\left\| f(t+h) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t)}{k!} h^k \right\| \leq \left| \frac{h^m}{m} \right| \sup_{k \geq m} \left| \frac{h^{k-m}}{k(k-1) \cdots (k-m+1)} \right| \left\| \frac{D^{k-m} f^{(m)}}{(k-m)!} \right\|.$$

Now, according to Lemma 1,

$$\left\| \frac{D^{k-m} f^{(m)}}{(k-m)!} \right\| \leq \|f^{(m)}\| \quad \text{and} \quad \left\| \frac{D^{k-m} f^{(m)}}{(k-m)!} \right\| \rightarrow 0 \quad (\text{when } k \rightarrow \infty).$$

To establish the expected statement, it suffices to show that

$$\left| \frac{h^{k-m}}{k(k-1)\cdots(k-m+1)} \right| \leq 1 \quad (\text{for } k \geq m).$$

Notice that the prime number p divides at most one of the m consecutive positive integers $k, k-1, \dots, k-m+1$. If $v_p(k-l) = v \geq 1$ for some $l \in \{0, \dots, m-1\}$, then for all $l' \neq l$, one has $v_p(k-l') = 0$. Hence,

$$v_p(k(k-1)\cdots(k-m+1)) = v, \quad |k(k-1)\cdots(k-m+1)| = |p|^v.$$

Since $p^v \mid (k-l)$, we have $k \geq p^v + l \geq p^v$. Hence, using the condition $|h| \leq |p|^{\frac{1}{p-m}}$ we obtain

$$\left| \frac{h^{k-m}}{k(k-1)\cdots(k-m+1)} \right| \leq \frac{|p|^{\frac{k-m}{p-m}}}{|p|^v} = |p|^{\frac{p^v-m}{p-m}-v}.$$

Define $e = \frac{p^v-m}{p-m} - v$. The function $s \mapsto \frac{p^v-s}{p-s}$ is increasing on the interval $[0, p)$, so

$$e \geq \frac{p^v-1}{p-1} - v = (1+p+\cdots+p^{v-1}) - v \geq 0,$$

and thus

$$\left| \frac{h^{k-m}}{k(k-1)\cdots(k-m+1)} \right| \leq |p|^e \leq 1.$$

Now, we consider the case $m = p$. We write

$$f(t+h) - \sum_{k=0}^{p-1} \frac{f^{(k)}(t)}{k!} h^k = \frac{h^p}{p} \sum_{k \geq p} \frac{ph^{k-p}}{k(k-1)\cdots(k-p+1)} \frac{D^{k-p} f^{(p)}(t)}{(k-p)!},$$

and as in the previous case, it suffices to show that

$$\left| \frac{ph^{k-p}}{k(k-1)\cdots(k-p+1)} \right| \leq 1.$$

Among the consecutive integers $k, k-1, \dots, k-p+1$, exactly one is divisible by p . If $v_p(k-l) = v \geq 1$ for some $l \in \{0, \dots, p-1\}$, then $|k(k-1)\cdots(k-p+1)| = |p|^v$ and $k \geq p^v + l \geq p^v$. Under the condition $|h| \leq |p|^{\frac{1}{p}}$, we obtain

$$\frac{|ph^{k-p}|}{|p|^v} = \frac{|h|^{k-p}}{|p|^{v-1}} \leq \frac{|p|^{\frac{p^v-p}{p}}}{|p|^{v-1}} = |p|^{p^{v-1}-v}.$$

Since $p^{v-1} - v \geq 0$ for all $v \geq 1$, the claim follows. □

In terms of valuations, this theorem can be stated as follows.

Theorem 2 (Congruence form). *Let $f(t) = \sum_{k \geq 0} a_k t^k \in \mathbb{B}[t]$. Then*

$$f(t+h) \equiv \sum_{k=0}^{m-1} \frac{f^{(k)}(t)}{k!} h^k \pmod{p^{mv_p(h)+v_p(\|f^{(m)}\|)}} \quad \text{if } 1 \leq m < p,$$

$$f(t+h) \equiv \sum_{k=0}^{p-1} \frac{f^{(k)}(t)}{k!} h^k \pmod{p^{pv_p(h)+v_p(\|f^{(p)}\|)-1}} \quad \text{if } m = p,$$

for all $t, h \in \mathbb{K}$ such that $v_p(t) \geq 0$, and

$$v_p(h) \geq \frac{1}{p-m} \quad \text{if } 1 \leq m < p \quad \text{and} \quad v_p(h) \geq \frac{1}{p} \quad \text{if } m = p.$$

Remark 1. To avoid ambiguity, we recall what is meant by congruences between coefficients in an ultrametric Banach space. Denote by

$$\mathbb{B}^\circ := \overline{B}(0, 1) = \{x \in \mathbb{B} : \|x\| \leq 1\}$$

the closed unit ball of \mathbb{B} . This is naturally an $\mathcal{O}_{\mathbb{K}}$ -module. For any $v \in \mathbb{Q}$, we set

$$p^v \mathbb{B}^\circ := \{p^v x : x \in \mathbb{B}^\circ\} = \overline{B}(0, p^{-v}),$$

which is again an $\mathcal{O}_{\mathbb{K}}$ -module. For $x, y \in \mathbb{B}$, we write

$$x \equiv y \pmod{p^v} \quad \text{if and only if} \quad x - y \in p^v \mathbb{B}^\circ \quad \text{if and only if} \quad \|x - y\| \leq p^{-v}.$$

As a concrete example, take $\mathbb{B} = \mathbb{Q}_p[x]$ endowed with the sup norm $\left\| \sum_{k=0}^m a_k x^k \right\| = \sup_{0 \leq k \leq m} |a_k|$. By definition, we have $\mathbb{Q}_p[x]^\circ = \mathbb{Z}_p[x]$. Now, for $A(t) = \sum_{k \geq 0} a_k(x) t^k$ and $B(t) = \sum_{k \geq 0} b_k(x) t^k$ in $\mathbb{Q}_p[x][t]$, the congruence $A(t) \equiv B(t) \pmod{p^v}$ means precisely that $a_k(x) \equiv b_k(x) \pmod{p^v \mathbb{Z}_p[x]}$ for all $k \geq 0$. In turn, this induces congruences in \mathbb{Z}_p between the coefficients of $a_k(x)$ and $b_k(x)$, as will be illustrated below.

2. Application to Binomial Coefficients

Before presenting the proof of the general formula which involves lengthy and intricate expressions, we first review an application of the second-order estimate treated by A. Junod (see [5, pp. 6–7]). In this setting, with $m = 2$ and $p \geq 3$, the higher mean value theorem of order two is applied with the following ingredients. Define

$$f(t) = (r(x)t + x^p + 1)^n \quad \text{where} \quad r(x) = \frac{(x+1)^p - (x^p + 1)}{p}.$$

It is clear that every coefficient of $f(t)$ belongs to

$$\mathbb{B} = \{g(x) \in \mathbb{Q}_p[x] : \deg(g(x)) \leq np\}.$$

This ultrametric Banach space is also endowed with the sup norm defined by $\|\sum_k a_k x^k\| = \sup_k |a_k|$. Note that for $1 \leq i \leq p - 1$, the prime p divides $\binom{p}{i}$ exactly once. Using the equality

$$r(x) = \frac{(x + 1)^p - (x^p + 1)}{p} = \frac{\sum_{i=0}^p \binom{p}{i} x^i - x^p - 1}{p} = \frac{1}{p} \sum_{i=1}^{p-1} \binom{p}{i} x^i,$$

we find that $\|r(x)\| = 1$. Setting $t = 0$ and $h = p$ yields $f(p) = (x + 1)^{np}$ and $f(0) = (x^p + 1)^n$. Differentiating with respect to t , we get $f'(0) = nr(x)(x^p + 1)^{n-1}$. Because $\|r(x)\| = 1$ and $\|x^p + 1\| = 1$, differentiating twice gives $\|f^{(2)}\| \leq |n(n-1)|$. It follows that

$$\begin{aligned} \left\| f(p) - f(0) - \frac{f'(0)}{1!} p \right\| &= \left\| (x + 1)^{np} - (x^p + 1)^n - npr(x)(x^p + 1)^{n-1} \right\| \\ &= \left\| (x + 1)^{np} - (x^p + 1)^n - n(x^p + 1)^{n-1}(x + 1)^p \right. \\ &\quad \left. + n(x^p + 1)^n \right\| \\ &\leq \left| \frac{p^2}{2} \right| \|f^{(2)}\| \leq |p^2| |n(n-1)| = |p^2| \left| \binom{n}{2} \right|. \end{aligned}$$

In other words,

$$(x + 1)^{np} - (x^p + 1)^n \equiv n(x^p + 1)^{n-1}(x + 1)^p - n(x^p + 1)^n \pmod{p^{2+v_p\left(\binom{n}{2}\right)} \mathbb{Z}_p[x]}.$$

Hence, the coefficients of x^{kp} on both sides must be congruent modulo $p^{2+v_p\left(\binom{n}{2}\right)}$. The coefficient of x^{kp} in $n(x^p + 1)^{n-1}(x + 1)^p - n(x^p + 1)^n$ is $n\left(\binom{n-1}{k} + \binom{n-1}{k-1} - \binom{n}{k}\right) = 0$. Therefore

$$\binom{np}{kp} \equiv \binom{n}{k} \pmod{p^{2+v_p\left(\binom{n}{2}\right)} \mathbb{Z}_p}.$$

We now apply the mean value theorem of order $m + 1 < p$. Differentiating $f(t)$ l times with respect to t gives

$$f^{(l)}(t) = (n(n-1) \cdots (n-l+1))r(x)^l(r(x)t + x^p + 1)^{n-l}.$$

Thus, $\|f^{(m+1)}\| \leq |n(n-1) \cdots (n-m+1)(n-m)| = \left| \binom{n}{m+1} \right|$ (because $\gcd(m+1, p) = 1$). Evaluating at $t = 0$ yields

$$\frac{p^l}{l!} f^{(l)}(0) = p^l \binom{n}{l} r(x)^l (x^p + 1)^{n-l} = \binom{n}{l} ((x + 1)^p - (x^p + 1))^l (x^p + 1)^{n-l}.$$

When applying Theorem 1, we obtain

$$\left\| f(p) - \sum_{l=0}^m \frac{p^l}{l!} f^{(l)}(0) \right\| \leq |p^{m+1}| \|f^{(m+1)}\| \leq |p^{m+1}| \left| \binom{n}{m+1} \right|.$$

In congruence form, we can write

$$f(p) - f(0) \equiv \sum_{l=1}^m \frac{p^l}{l!} f^{(l)}(0) \pmod{p^{m+1+v_p\left(\binom{n}{m+1}\right)} \mathbb{Z}_p[x]}.$$

To extend Wolstenholme’s congruence to a higher modulus, we need to find a way to extract the coefficient of x^{kp} from the terms $\frac{p^l}{l!} f^{(l)}(0)$. To achieve this, we introduce in Theorem 3 the sequence $d(m, n, k)$, which computes precisely the coefficient of x^{kp} from $\frac{p^l}{l!} f^{(l)}(0)$. After determining these coefficients for the different values $l = 0, \dots, m$, we then collect and combine them to find a closed formula for the coefficients of $\binom{jp}{ip}$ that might arise in the congruence, as established in Theorem 4 and Corollary 1.

Lemma 2. *The coefficient of x^{kp} in the product $(x + 1)^{jp}(x^p + 1)^{n-j}$ is*

$$c_j(n, k) := \sum_{s=0}^j \binom{jp}{sp} \binom{n-j}{k-s}.$$

Proof. Expanding the product using the binomial theorem gives

$$(x + 1)^{jp}(x^p + 1)^{n-j} = \left(\sum_{i=0}^{jp} \binom{jp}{i} x^i \right) \left(\sum_{r=0}^{n-j} \binom{n-j}{r} x^{rp} \right).$$

Extracting the coefficient of x^{kp} requires collecting terms where the exponent of x in the first sum is a multiple of p . This immediately yields

$$c_j(n, k) = \sum_{s=0}^j \binom{jp}{sp} \binom{n-j}{k-s}.$$

□

Theorem 3. *For $m, k \geq 0$ and $n \in \mathbb{Z}$, we introduce the sequence*

$$d(m, n, k) := \binom{n}{m} \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} c_j(n, k).$$

With this notation, the following congruence holds:

$$\binom{np}{kp} - \binom{n}{k} \equiv d(1, n, k) + d(2, n, k) + \dots + d(m, n, k) \pmod{p^{m+1+v_p\left(\binom{n}{m+1}\right)}}. \quad (1)$$

Proof. From the foregoing discussion, we have

$$\begin{aligned} \frac{p^l}{l!} f^{(l)}(0) &= \binom{n}{l} ((x+1)^p - (x^p+1))^l (x^p+1)^{n-l} \\ &= \binom{n}{l} \left(\sum_{j=0}^l \binom{l}{j} (x+1)^{jp} (-1)^{l-j} (x^p+1)^{l-j} \right) (x^p+1)^{n-l} \\ &= \binom{n}{l} \sum_{j=0}^l (-1)^{l-j} \binom{l}{j} (x+1)^{jp} (x^p+1)^{n-j}. \end{aligned}$$

By the previous lemma, the coefficient of x^{kp} in $\frac{p^l}{l!} f^{(l)}(0)$ must be exactly the expression

$$\binom{n}{l} \sum_{j=0}^l (-1)^{l-j} \binom{l}{j} c_j(n, k).$$

From

$$f(p) - f(0) \equiv \sum_{l=1}^m \frac{p^l}{l!} f^{(l)}(0) \pmod{p^{m+1+v_p\left(\binom{n}{m+1}\right)} \mathbb{Z}_p[x]},$$

we identify the coefficient of x^{kp} in each term and obtain

$$\begin{aligned} \binom{np}{kp} - \binom{n}{k} &\equiv \sum_{l=1}^m \binom{n}{l} \sum_{j=0}^l (-1)^{l-j} \binom{l}{j} c_j(n, k) \\ &= \sum_{l=1}^m d(l, n, k) \pmod{p^{m+1+v_p\left(\binom{n}{m+1}\right)}}. \end{aligned}$$

□

Remark 2. Note that the expressions for $d(m, n, k)$ are rather lengthy, and the computations become increasingly involved as m grows. Therefore, it is convenient to use a computer algebra system such as MAPLE to implement $d(m, n, k)$.

Example 1. With a straightforward implementation of $d(m, n, k)$ and simplification using Vandermonde’s convolution identity (see [11, p. 50]),

$$\binom{n}{k} = \sum_{j=0}^k \binom{m}{j} \binom{n-m}{k-j} = \sum_{j=0}^{\min(k,m)} \binom{m}{j} \binom{n-m}{k-j} = \sum_{j=0}^m \binom{m}{j} \binom{n-m}{k-j}, \quad 0 \leq m \leq n.$$

We obtain for $m = 1, 2, 3$:

$$\begin{aligned} d(1, n, k) &= \binom{n}{1} \left(\binom{n}{k} - \binom{n}{k} \right) = 0, \\ d(2, n, k) &= \binom{n}{2} \binom{n-2}{k-1} \left(\binom{2p}{p} - 2 \right), \\ d(3, n, k) &= \binom{n}{3} \binom{n-2}{k-1} \left(\binom{3p}{p} - 3 \binom{2p}{p} + 3 \right). \end{aligned}$$

From (1), we deduce that $d(m, n, k) \equiv 0 \pmod{p^m}$. For example, when $(m, n, k) = (3, 3, 1), (4, 4, 1), (4, 4, 2)$, we find

$$\begin{aligned} d(3, 3, 1) &= \binom{3p}{p} - 3\binom{2p}{p} + 3 \equiv 0 \pmod{p^3}, \\ d(4, 4, 1) &= \binom{4p}{p} - 4\binom{3p}{p} + 6\binom{2p}{p} - 4 \equiv 0 \pmod{p^4}, \\ d(4, 4, 2) &= \binom{4p}{2p} - 8\binom{3p}{p} + 12\binom{2p}{p} - 6 \equiv 0 \pmod{p^4}. \end{aligned}$$

It can also be noted that $d(m, n, k) = 0$ if $m > n$ or $k > n$.

Before deriving the closed formula for the coefficient of $\binom{np}{kp}$, we introduce the following technical lemma. For background on binomial coefficients, falling factorials, and related identities, we refer the reader to [3, 9, 11].

Lemma 3. *Let $m, n \in \mathbb{N}$. Then, we have the following identities:*

$$\sum_{s=j}^m (-1)^{s-j} \binom{n}{s} \binom{s}{j} = (-1)^{m-j} \binom{n}{j} \binom{n-j-1}{m-j} = \binom{n}{j} \binom{m-n}{m-j}. \tag{2}$$

$$\sum_{j=0}^m (-1)^{m-j} \binom{n}{j} \binom{n-j-1}{m-j} \binom{n-j}{k} = \binom{n}{k} \binom{m-k}{m}. \tag{3}$$

$$\sum_{j=0}^m (-1)^{m-j} \binom{n}{j} \binom{n-j-1}{m-j} \binom{n-j}{k-j} = \binom{n}{k} \binom{k+m-n}{m}. \tag{4}$$

Proof. 1. Using the identity $\sum_{j=0}^m (-1)^j \binom{n}{j} = (-1)^m \binom{n-1}{m}$ (see [3, p. 165]), we have

$$\begin{aligned} \sum_{s=j}^m (-1)^{s-j} \binom{n}{s} \binom{s}{j} &= \sum_{s=j}^m (-1)^{s-j} \frac{n!}{s!(n-s)!} \frac{s!}{j!(s-j)!} \\ &= \sum_{s=j}^m (-1)^{s-j} \frac{n!}{j!(n-j)!} \frac{(n-j)!}{(s-j)!(n-s)!} \\ &= \binom{n}{j} \sum_{s=j}^m (-1)^{s-j} \binom{n-j}{s-j} = \binom{n}{j} \sum_{l=0}^{m-j} (-1)^l \binom{n-j}{l} \\ &= (-1)^{m-j} \binom{n}{j} \binom{n-j-1}{m-j} = \binom{n}{j} \binom{m-n}{m-j}. \end{aligned}$$

In the last line, we used the formula $\binom{y}{l} = (-1)^l \binom{l-y-1}{l}$ (see [3, p. 164]) with $y = n - j - 1$ and $l = m - j$.

2. By Vandermonde's identity $\binom{x+y}{m} = \sum_{j=0}^m \binom{x}{j} \binom{y}{m-j}$, we find

$$\begin{aligned} & \sum_{j=0}^m (-1)^{m-j} \binom{n}{j} \binom{n-j-1}{m-j} \binom{n-j}{k} \\ &= \sum_{j=0}^m (-1)^{m-j} \frac{n!(n-j)!}{j!(n-j)!k!(n-k-j)!} \binom{n-j-1}{m-j} \\ &= \sum_{j=0}^m (-1)^{m-j} \frac{n!}{k!(n-k)!} \frac{(n-k)!}{j!(n-k-j)!} \binom{n-j-1}{m-j} \\ &= \binom{n}{k} \sum_{j=0}^m (-1)^{m-j} \binom{n-k}{j} \binom{n-j-1}{m-j} \\ &= \binom{n}{k} \sum_{j=0}^m (-1)^{m-j} \binom{n-k}{j} (-1)^{m-j} \binom{m-n}{m-j} \\ &= \binom{n}{k} \binom{m-k}{m}. \end{aligned}$$

3. Again by Vandermonde's identity, we have

$$\begin{aligned} & \sum_{j=0}^m (-1)^{m-j} \binom{n}{j} \binom{n-j-1}{m-j} \binom{n-j}{k-j} \\ &= \sum_{j=0}^m (-1)^{m-j} \frac{n!(n-j)!}{j!(n-j)!(k-j)!(n-k)!} \binom{n-j-1}{m-j} \\ &= \sum_{j=0}^m (-1)^{m-j} \frac{n!}{k!(n-k)!} \frac{k!}{j!(k-j)!} \binom{n-j-1}{m-j} \\ &= \binom{n}{k} \sum_{j=0}^m (-1)^{m-j} \binom{k}{j} \binom{n-j-1}{m-j} \\ &= \binom{n}{k} \sum_{j=0}^m (-1)^{m-j} \binom{k}{j} (-1)^{m-j} \binom{m-n}{m-j} \\ &= \binom{n}{k} \binom{k+m-n}{m}. \end{aligned}$$

□

Lemma 4. For $m, k \geq 0$ with $m + 1 < p$ and $n \in \mathbb{Z}_p$, we have

$$\binom{np}{kp} - \binom{n}{k} \equiv \bar{C}_{00}(m, n, k) + \sum_{1 \leq i < j \leq m} \bar{C}_{ji}(m, n, k) \binom{jp}{ip} \pmod{p^{m+1+v_p\left(\binom{n}{m+1}\right)}},$$

where

$$\begin{aligned} \bar{C}_{00}(m, n, k) &= \left(\binom{m-k}{m} + \binom{k+m-n}{m} + (-1)^{m+1} \binom{n-1}{m} - 1 \right) \binom{n}{k}, \\ \bar{C}_{ji}(m, n, k) &= \binom{n}{j} \binom{m-n}{m-j} \binom{n-j}{k-i} = \frac{(n-k)_{j-i} (k)_i}{j!} \binom{m-n}{m-j} \binom{n}{k}. \end{aligned}$$

Proof. From $d(s, n, k)$ where $2 \leq s \leq m$, we extract the coefficient of $\binom{jp}{ip}$ which is equal to

$$\begin{aligned} d(s, n, k) &= \binom{n}{s} \sum_{j=0}^s (-1)^{s-j} \binom{s}{j} c_j(n, k) = \binom{n}{s} \sum_{j=0}^s (-1)^{s-j} \binom{s}{j} \sum_{i=0}^j \binom{jp}{ip} \binom{n-j}{k-i} \\ &= \binom{n}{s} \sum_{i=0}^s \sum_{j=i}^s (-1)^{s-j} \binom{s}{j} \binom{n-j}{k-i} \binom{jp}{ip}. \end{aligned}$$

For $1 \leq i < j \leq m$, the coefficient of $\binom{jp}{ip}$ from $d(s, n, k)$ is equal to

$$(-1)^{s-j} \binom{n}{s} \binom{s}{j} \binom{n-j}{k-i}.$$

Summing over s from j to m using (2), we find $\bar{C}_{ji}(m, n, k) = \binom{n}{j} \binom{m-n}{m-j} \binom{n-j}{k-i}$. For the rightmost equality, we have

$$\begin{aligned} \binom{n}{j} \binom{m-n}{m-j} \binom{n-j}{k-i} &= \frac{n!}{j!(n-j)!} \cdot \frac{(n-j)!}{(k-i)!(n-j-k+i)!} \binom{m-n}{m-j} \\ &= \frac{n!}{k!(n-k)!} \cdot \frac{k!(n-k)!}{j!(k-i)!(n-k-j+i)!} \binom{m-n}{m-j} \\ &= \frac{(n-k)_{j-i} (k)_i}{j!} \binom{m-n}{m-j} \binom{n}{k}. \end{aligned}$$

The coefficient of the constant term occurs only in the cases $i = 0$ and $i = j$. Collecting these contributions and correcting for the overlap at $j = 0$ (where $i = 0$ and $i = j$ coincide) gives

$$\binom{n}{s} \left(\sum_{j=0}^s (-1)^{s-j} \binom{s}{j} \left[\binom{n-j}{k} + \binom{n-j}{k-j} \right] - (-1)^s \binom{n}{k} \right).$$

Summing s from 2 to m and observing that $\sum_{s=2}^m (-1)^s \binom{n}{s} = (-1)^m \binom{n-1}{m} + n - 1$,

we find

$$\begin{aligned} \bar{C}_{00}(m, n, k) &= \sum_{s=2}^m \binom{n}{s} \left(\sum_{j=0}^s (-1)^{s-j} \binom{s}{j} \left[\binom{n-j}{k} + \binom{n-j}{k-j} \right] - (-1)^s \binom{n}{k} \right) \\ &= \sum_{j=0}^m \sum_{s=j}^m (-1)^{s-j} \binom{n}{s} \binom{s}{j} \binom{n-j}{k} \\ &\quad + \sum_{j=0}^m \sum_{s=j}^m (-1)^{s-j} \binom{n}{s} \binom{s}{j} \binom{n-j}{k-j} \\ &\quad - (2-n) \binom{n}{k} - \left((-1)^m \binom{n-1}{m} + n-1 \right) \binom{n}{k}. \end{aligned}$$

Using (2), (3), and (4), we obtain

$$\begin{aligned} \bar{C}_{00}(m, n, k) &= \sum_{j=0}^m (-1)^{m-j} \binom{n}{j} \binom{n-j-1}{m-j} \binom{n-j}{k} \\ &\quad + \sum_{j=0}^m (-1)^{m-j} \binom{n}{j} \binom{n-j-1}{m-j} \binom{n-j}{k-j} \\ &\quad - (2-n) \binom{n}{k} - \left((-1)^m \binom{n-1}{m} + n-1 \right) \binom{n}{k} \\ &= \left(\binom{m-k}{m} + \binom{k+m-n}{m} + (-1)^{m+1} \binom{n-1}{m} - 1 \right) \binom{n}{k}. \end{aligned}$$

□

Theorem 4. For $m, k \geq 0$ with $m + 1 < p$ and $n \in \mathbb{Z}_p$, we have

$$\begin{aligned} \binom{np}{kp} - \binom{n}{k} &\equiv C'_{00}(m, n, k) + \sum_{2 \leq j \leq m, 1 \leq i \leq \lfloor j/2 \rfloor} C'_{ji}(m, n, k) \binom{jp}{ip} \\ &\quad \pmod{p^{m+1+v_p\left(\binom{n}{m+1}\right)}}, \end{aligned}$$

where

$$\begin{aligned} C'_{00}(m, n, k) &= \left(\binom{m-k}{m} + \binom{k+m-n}{m} + (-1)^{m+1} \binom{n-1}{m} - 1 \right) \binom{n}{k}, \\ C'_{ji}(m, n, k) &= \frac{(n-k)_{j-i} (k)_i + (n-k)_i (k)_{j-i}}{j!} \binom{m-n}{m-j} \binom{n}{k}, \end{aligned}$$

for $j \neq 2i$ and $i \geq 1$. For the case $j = 2i, i \geq 1$, the calculated value should be divided by 2.

Proof. Since $\binom{jp}{ip} = \binom{jp}{(j-i)p}$, the terms corresponding to i and $j-i$ in the summation combine, reducing the range of i to $1 \leq i \leq \lfloor j/2 \rfloor$. We define $C'_{ji}(m, n, k) := \overline{C}_{ji}(m, n, k) + \overline{C}_{j-j-i}(m, n, k)$, so that the coefficient of $\binom{jp}{ip}$ equals $C'_{ji}(m, n, k)$ when $i \neq j-i$. If $j = 2i$, then $i = j-i$; in this case, the two summands are identical and must not be double-counted, leading to the division by 2. \square

Proposition 1. 1. For all $m, k \in \mathbb{N}$ and $n \in \mathbb{Z}$, the polynomial part of $C'_{00}(m, n, k)$ is divisible by $\frac{k(n-k)}{2}$ in $\mathbb{Q}[k]$.

2. For all $m, k \in \mathbb{N}$ and $n \in \mathbb{Z}$, the polynomial part of $C'_{ji}(m, n, k)$ is divisible by $\frac{k(n-k)}{2}$ in $\mathbb{Q}[k]$.

Proof. When $k = n$, replacing in the polynomial part of $C'_{00}(m, n, k)$, we find

$$\begin{aligned} & \binom{m-n}{m} + \binom{n+m-n}{m} + (-1)^{m+1} \binom{n-1}{m} - 1 \\ &= \binom{m-n}{m} + 1 + (-1)^{m+1} \binom{n-1}{m} - 1 \\ &= \binom{m-n}{m} - \binom{m-n}{m} = 0. \end{aligned}$$

Also, when $k = 0$, the value in the polynomial part is 0. As $\gcd(k, \frac{n-k}{2}) = 1$ (viewed as elements of the ring $\mathbb{Q}[k]$), the polynomial $\frac{k(n-k)}{2}$ must divide the polynomial part of $C'_{00}(m, n, k)$. For the second statement, the result follows by the same reasoning as in the proof of Statement 1. \square

This proposition motivates the following definition.

Definition 1. We define the *reduced coefficients* $C_{ji}(m, n, k)$ and $C_{00}(m, n, k)$ by the relation

$$\frac{k(n-k)}{2} \binom{n}{k} C_{ji}(m, n, k) = C'_{ji}(m, n, k).$$

Corollary 1. For integers $m, k \geq 0$ such that $m+1 < p$, and for $n \in \mathbb{Z}_p$, we have

$$\begin{aligned} \binom{np}{kp} - \binom{n}{k} &\equiv \frac{k(n-k)}{2} \binom{n}{k} \left(C_{00}(m, n, k) + \sum_{2 \leq j \leq m, 1 \leq i \leq \lfloor j/2 \rfloor} C_{ji}(m, n, k) \binom{jp}{ip} \right) \\ &\pmod{p^{m+1+v_p\left(\binom{n}{m+1}\right)}}. \end{aligned}$$

Remark 3. As previously noted, the explicit expressions for the coefficients C_{ji} or C'_{ji} are quite involved. For this reason, we suggest that the reader compute them with the help of a computer algebra system such as MAPLE or MATHEMATICA, where a straightforward implementation suffices, particularly for the more demanding cases of congruences modulo p^6 and p^7 .

2.1. Modulo p^3 Congruence

Before applying our theorems, we introduce a convenient expression for $\binom{-np}{kp}$, which we will use repeatedly.

Lemma 5. *For $n \in \mathbb{N}$ and integer $k \geq 1$, we have*

$$\binom{-np}{kp} = (-1)^{kp} \frac{n}{k} \binom{(n+k)p-1}{kp-1} = (-1)^{kp} \frac{n}{n+k} \binom{(n+k)p}{kp}. \tag{5}$$

Proof. We recall the identity

$$\binom{-x}{m} = \frac{(-x)(-x-1)\cdots(-x-m+1)}{m!} = (-1)^m \binom{x+m-1}{m}.$$

By the elementary relation $\binom{y}{m} = \frac{y-m+1}{m} \binom{y}{m-1}$, taking $y = x + m - 1$, we obtain

$$(-1)^m \binom{x+m-1}{m} = (-1)^m \frac{x}{m} \binom{x+m-1}{m-1}.$$

Next, using $\binom{y-1}{m-1} = \frac{m}{y} \binom{y}{m}$ with $y = x + m$, it follows that

$$\binom{x+m-1}{m-1} = \frac{m}{x+m} \binom{x+m}{m}.$$

Combining these formulas yields

$$\binom{-x}{m} = (-1)^m \frac{x}{m} \binom{x+m-1}{m-1} = (-1)^m \frac{x}{x+m} \binom{x+m}{m}.$$

Finally, substituting $x = np$ and $m = kp$ gives the desired result. □

Proposition 2. *For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 5$,*

$$\binom{np}{kp} - \binom{n}{k} \equiv \frac{k(n-k)}{2} \binom{n}{k} \left(\binom{2p}{p} - 2 \right) \pmod{p^{3+v_p\left(\binom{n}{3}\right)}}. \tag{6}$$

Proof. Applying Theorem 4, we find after simplification

$$C'_{00}(2, n, k) = k(k-n) \binom{n}{k} = \frac{k(n-k)}{2} \binom{n}{k} (-2),$$

$$C'_{21}(2, n, k) = \frac{k(n-k)}{2} \binom{n}{k}.$$

□

Corollary 2. *For $p \geq 5$ and $n, k \in \mathbb{N}$, the following assertions are equivalent:*

1. $\binom{2p}{p} \equiv 2 \pmod{p^3}$,
2. $\binom{np}{p} \equiv n \pmod{p^3}$,
3. $\binom{np}{kp} \equiv \binom{n}{k} \pmod{p^3}$.

Theorem 5. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 5$,

$$\binom{np}{kp} \equiv \binom{n}{k} \pmod{p^{3+v_p\left(\binom{n}{3}\right)}}. \tag{7}$$

Proof. Setting $n = -1$ and $k = 1$ in (6), we find

$$\binom{-p}{p} - \binom{-1}{1} = -\frac{1}{2}\binom{2p}{p} - \binom{-1}{1} \equiv -\binom{-1}{1} \left(\binom{2p}{p} - 2 \right) \pmod{p^3}.$$

Simplifying yields

$$\frac{3}{2}\binom{2p}{p} \equiv 3 \pmod{p^3}.$$

Hence, when $p > 3$,

$$\frac{k(n-k)}{2}\binom{n}{k} \left(\binom{2p}{p} - 2 \right) \equiv 0 \pmod{p^{3+v_p\left(\binom{n}{3}\right)}}.$$

It is worth noting that Congruence (7) represents a slight variation of a result established by W. Ljunggren and E. Jacobsthal (1952); see [8]. □

2.2. Modulo p^4 Congruence

Proposition 3. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 7$, we have

$$\binom{np}{kp} - \binom{n}{k} \equiv \frac{k(n-k)}{2}\binom{n}{k} \left((n-4) + (3-n)\binom{2p}{p} + \frac{n-2}{3}\binom{3p}{p} \right) \pmod{p^{4+v_p\left(\binom{n}{4}\right)}}. \tag{8}$$

Proof. Applying Theorem 4, we find

$$\begin{aligned} C'_{00}(3, n, k) &= \frac{k(n-k)}{2}\binom{n}{k}(n-4), \\ C'_{21}(3, n, k) &= \frac{k(n-k)}{2}\binom{n}{k}(3-n), \\ C'_{31}(3, n, k) &= \frac{k(n-k)}{2}\binom{n}{k}\frac{(n-2)}{3}. \end{aligned}$$

□

Theorem 6. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 7$,

$$\binom{np}{kp} - \binom{n}{k} \equiv \frac{nk(n-k)}{4} \binom{n}{k} \left(\binom{2p}{p} - 2 \right) \pmod{p^{4+v_p\left(\binom{n}{4}\right)}}.$$

Proof. Substituting $n = -2$ and $k = 1$ into (1) or (8) and performing some calculations, we find that

$$\binom{3p}{p} \equiv \frac{9}{2} \binom{2p}{p} - 6 \pmod{p^4}.$$

Using the value of $\binom{3p}{p}$ in (8) gives the desired result. □

Corollary 3. For $p \geq 7$ and $n, k \in \mathbb{N}$, the following assertions are equivalent:

1. $\binom{2p}{p} \equiv 2 \pmod{p^4}$ (i.e., p is a Wolstenholme prime; see [8]),
2. $\binom{np}{p} \equiv n \pmod{p^4}$,
3. $\binom{np}{kp} \equiv \binom{n}{k} \pmod{p^4}$.

2.3. Modulo p^5 Congruence

Proposition 4. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 11$, we have

$$\binom{np}{kp} - \binom{n}{k} \equiv \frac{k(n-k)}{2} \binom{n}{k} \left(C_{00}(4, n, k) + \sum_{2 \leq j \leq 4, 1 \leq i \leq \lfloor j/2 \rfloor} C_{ji}(4, n, k) \binom{jp}{ip} \right) \pmod{p^{5+v_p\left(\binom{n}{5}\right)}}, \tag{9}$$

where

$$\begin{aligned} C_{00}(4, n, k) &= -\frac{k^2 - kn + 2n^2 - 15n + 35}{6}, \\ C_{21}(4, n, k) &= \frac{(n-4)(n-3)}{2}, \\ C_{31}(4, n, k) &= \frac{(n-2)(4-n)}{3}, \\ C_{41}(4, n, k) &= \frac{2k^2 - 2kn + n^2 - 3n + 4}{12}, \\ C_{42}(4, n, k) &= \frac{(n-k-1)(k-1)}{12}. \end{aligned}$$

Proof. Apply Corollary 1 and simplify. □

Theorem 7. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 11$,

$$\binom{np}{kp} - \binom{n}{k} \equiv \frac{nk(n-k)}{4} \binom{n}{k} \left(\binom{2p}{p} - 2 \right) \pmod{p^{5+v_p\left(\binom{n}{5}\right)}}. \tag{10}$$

Proof. Substituting the pairs $(n, k) = (-2, 2)$, $(-1, 1)$, and $(-2, 1)$ into (1) or (9), we find, respectively,

$$9\binom{4p}{2p} \equiv 360\binom{2p}{p} - 192\binom{3p}{p} + 60\binom{4p}{p} - 330 \pmod{p^5}, \tag{11}$$

$$\binom{4p}{p} \equiv 5\binom{3p}{p} - \frac{21}{2}\binom{2p}{p} + 10 \pmod{p^5}, \tag{12}$$

$$\binom{4p}{p} \equiv \frac{14}{3}\binom{3p}{p} - 9\binom{2p}{p} + 8 \pmod{p^5}. \tag{13}$$

Subtracting (12) from (13), we obtain

$$\binom{3p}{p} \equiv \frac{9}{2}\binom{2p}{p} - 6 \pmod{p^5}.$$

Hence,

$$\binom{4p}{p} \equiv 12\binom{2p}{p} - 20 \pmod{p^5}.$$

Substituting the expressions for $\binom{4p}{p}$ and $\binom{3p}{p}$ into (11), we find

$$\binom{4p}{2p} \equiv 24\binom{2p}{p} - 42 \pmod{p^5}.$$

The constant term and the coefficient of $\binom{2p}{p}$ are extracted by evaluating the following expressions:

$$\begin{aligned} C_{00} - 6C_{31} - 20C_{41} - 42C_{42} &= -n, \\ C_{21} + \frac{9}{2}C_{31} + 12C_{41} + 24C_{42} &= \frac{n}{2}. \end{aligned}$$

□

2.4. Modulo p^6 Congruence

Proposition 5. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 11$, we have

$$\begin{aligned} \binom{np}{kp} - \binom{n}{k} &\equiv \frac{k(n-k)}{2}\binom{n}{k} \left(C_{00}(5, n, k) + \sum_{2 \leq j \leq 5, 1 \leq i \leq \lfloor j/2 \rfloor} C_{ji}(5, n, k) \binom{jp}{ip} \right) \\ &\pmod{p^{6+v_p\left(\binom{n}{6}\right)},} \end{aligned} \tag{14}$$

where

$$\begin{aligned}
 C_{00}(5, n, k) &= \frac{(n - 6)(n^2 - nk - 6n + k^2 + 15)}{12}, \\
 C_{21}(5, n, k) &= \frac{(5 - n)(4 - n)(3 - n)}{6}, \\
 C_{31}(5, n, k) &= \frac{(n - 2)(n - 5)(n - 4)}{6}, \\
 C_{41}(5, n, k) &= \frac{(2k^2 - 2kn + n^2 - 3n + 4)(5 - n)}{12}, \\
 C_{42}(5, n, k) &= \frac{(k + 1 - n)(k - 1)(n - 5)}{12}, \\
 C_{51}(5, n, k) &= \frac{(n - 4)(3k^2 - 3kn + n^2 - 2n + 3)}{60}, \\
 C_{52}(5, n, k) &= \frac{(n - k - 1)(k - 1)(n - 4)}{60}.
 \end{aligned}$$

Proof. Apply Corollary 1 and simplify (a computer algebra system such as MAPLE or MATHEMATICA may be used for convenience). □

Theorem 8. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 11$, we have

$$\binom{np}{kp} - \binom{n}{k} \equiv \frac{nk(n - k)}{4} \binom{n}{k} \left(a(n, k) \binom{2p}{p} + b(n, k) \binom{3p}{p} + c^t(n, k) \right) \pmod{p^{6+v_p\left(\binom{n}{6}\right)}} \tag{15}$$

where

$$\begin{aligned}
 a(n, k) &= -\frac{k^2 - kn + n^2 - 7}{4}, \\
 b(n, k) &= \frac{k^2 - kn + n^2 - 3}{18}, \\
 c^t(n, k) &= \frac{k^2 - kn + n^2 - 9}{3}.
 \end{aligned}$$

Proof. Using (5) and substituting the pairs $(n, k) = (-1, 1), (-2, 1)$ into (1) or (14), we find the following:

$$\binom{5p}{p} \equiv \frac{41}{2} \binom{2p}{p} - 15 \binom{3p}{p} + 6 \binom{4p}{p} - 15 \pmod{p^6}, \tag{16}$$

$$6 \binom{5p}{p} \equiv 105 \binom{2p}{p} - \frac{250}{3} \binom{3p}{p} + 35 \binom{4p}{p} - 70 \pmod{p^6}. \tag{17}$$

Computing $(6 \cdot (16) - (17))$, we find

$$\binom{4p}{p} \equiv -18 \binom{2p}{p} + \frac{20}{3} \binom{3p}{p} + 20 \pmod{p^6}.$$

Substituting in (16), we obtain

$$\binom{5p}{p} \equiv -\frac{175}{2} \binom{2p}{p} + 25 \binom{3p}{p} + 105 \pmod{p^6}.$$

Next, substituting $(-1, 2)$ and $(-2, 2)$ into (14), we obtain

$$\begin{aligned} \binom{5p}{2p} &\equiv -60 \binom{2p}{p} + \frac{134}{3} \binom{3p}{p} - 30 \binom{4p}{p} + 6 \binom{4p}{2p} + 6 \binom{5p}{p} + 50 \pmod{p^6}, \\ \binom{5p}{2p} &\equiv -70 \binom{2p}{p} + 56 \binom{3p}{p} - 35 \binom{4p}{p} + \frac{23}{4} \binom{4p}{2p} + 7 \binom{5p}{p} + \frac{105}{2} \pmod{p^6}. \end{aligned}$$

Therefore

$$\binom{4p}{2p} \equiv -40 \binom{2p}{p} + \frac{136}{3} \binom{3p}{p} - 20 \binom{4p}{p} + 4 \binom{5p}{p} + 10 \pmod{p^6}.$$

Substituting the expressions for $\binom{4p}{p}$ and $\binom{5p}{p}$ into the above, we arrive at

$$\begin{aligned} \binom{4p}{2p} &\equiv -30 \binom{2p}{p} + 12 \binom{3p}{p} + 30 \pmod{p^6}, \\ \binom{5p}{2p} &\equiv -225 \binom{2p}{p} + \frac{200}{3} \binom{3p}{p} + 260 \pmod{p^6}. \end{aligned}$$

The coefficients $c^t(n, k)$, $a(n, k)$, and $b(n, k)$ are determined by the following linear combinations:

$$\begin{aligned} c^t(n, k) &= C_{00} + 20C_{41} + 30C_{42} + 105C_{51} + 260C_{52}, \\ a(n, k) &= C_{21} - 18C_{41} - 30C_{42} - \frac{175}{2}C_{51} - 225C_{52}, \\ b(n, k) &= C_{31} + \frac{20}{3}C_{41} + 12C_{42} + 25C_{51} + \frac{200}{3}C_{52}. \end{aligned}$$

□

2.5. Modulo p^7 Congruence

Proposition 6. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 11$, we have

$$\begin{aligned} \binom{np}{kp} - \binom{n}{k} &\equiv \frac{k(n-k)}{2} \binom{n}{k} \left(C_{00}(6, n, k) + \sum_{2 \leq j \leq 6, 1 \leq i \leq \lfloor j/2 \rfloor} C_{ji}(6, n, k) \binom{jp}{ip} \right) \\ &\pmod{p^{7+v_p\left(\binom{n}{7}\right)}}, \end{aligned} \tag{18}$$

where

$$C_{00}(6, n, k) = -\frac{1}{720} \left(2k^4 - 4k^3n + 11k^2n^2 - 9kn^3 + 6n^4 - 105k^2n + 105kn^2 - 105n^3 + 350k^2 - 350nk + 700n^2 - 2205n + 3248 \right),$$

$$C_{21}(6, n, k) = \frac{1}{24}(6 - n)(5 - n)(4 - n)(3 - n),$$

$$C_{31}(6, n, k) = \frac{1}{18}(n - 2)(6 - n)(5 - n)(4 - n),$$

$$C_{41}(6, n, k) = \frac{1}{24}(6 - n)(5 - n)(2k^2 - 2kn + n^2 - 3n + 4),$$

$$C_{42}(6, n, k) = \frac{1}{24}(n - k - 1)(k - 1)(6 - n)(5 - n),$$

$$C_{51}(6, n, k) = \frac{1}{60}(4 - n)(n - 6)(3k^2 - 3kn + n^2 - 2n + 3),$$

$$C_{52}(6, n, k) = \frac{1}{60}(-n + k + 1)(k - 1)(n - 4)(n - 6),$$

$$C_{61}(6, n, k) = \frac{1}{360} \left(2k^4 - 4k^3n + 6k^2n^2 - 4kn^3 + n^4 - 30k^2n + 30kn^2 - 10n^3 + 70k^2 - 70kn + 35n^2 - 50n + 48 \right),$$

$$C_{62}(6, n, k) = \frac{1}{360}(n - k - 1)(k - 1)(2k^2 - 2kn + n^2 - 5n + 12),$$

$$C_{63}(6, n, k) = \frac{1}{360}(k - n - 1)(k - n - 2)(k - 1)(k - 2).$$

Proof. The proof proceeds similarly to the case of the modulo p^6 congruence. \square

Theorem 9. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 11$, we have

$$\binom{np}{kp} - \binom{n}{k} \equiv \frac{nk(n - k)}{8} \binom{n}{k} \left(a(n, k) \binom{2p}{p} + b(n, k) \binom{3p}{p} + d(n, k) \binom{4p}{2p} + c^t(n, k) \right) \pmod{p^{7+v_p\left(\binom{n}{7}\right)}},$$

where

$$\begin{aligned} a(n, k) &= -\frac{(n + 2)(k - 2)(-n + k + 2)}{2}, \\ b(n, k) &= \frac{9k^2n - 9kn^2 + 14k^2 - 14kn + 14n^2 - 24}{45}, \\ c^t(n, k) &= \frac{3k^2n - 3kn^2 + 7k^2 - 7kn + 7n^2 - 39}{6}, \\ d(n, k) &= -\frac{(-n + k + 1)(n + 1)(k - 1)}{60}. \end{aligned}$$

Proof. Using (5), we substitute the pairs $(n, k) = (-1, 1), (-2, 1), (-3, 1)$, and $(-4, 1)$ into (1) or (18) to obtain

$$\begin{aligned} \binom{6p}{p} &\equiv -\frac{71}{2} \binom{2p}{p} + 35 \binom{3p}{p} - 21 \binom{4p}{p} + 7 \binom{5p}{p} + 21 \pmod{p^7}, \\ 7 \binom{6p}{p} &\equiv -210 \binom{2p}{p} + \frac{670 \binom{3p}{p}}{3} - 140 \binom{4p}{p} + 48 \binom{5p}{p} + 112 \pmod{p^7}, \\ 28 \binom{6p}{p} &\equiv -756 \binom{2p}{p} + 840 \binom{3p}{p} - \frac{2163}{4} \binom{4p}{p} + 189 \binom{5p}{p} + 378 \pmod{p^7}, \\ 84 \binom{6p}{p} &\equiv -2100 \binom{2p}{p} + 2400 \binom{3p}{p} - 1575 \binom{4p}{p} + \frac{2796}{5} \binom{5p}{p} + 1008 \pmod{p^7}. \end{aligned}$$

Now, solving the resulting linear system, we find

$$\begin{aligned} \binom{4p}{p} &\equiv -18 \binom{2p}{p} + \frac{20}{3} \binom{3p}{p} + 20 \pmod{p^7}, \\ \binom{5p}{p} &\equiv -\frac{175}{2} \binom{2p}{p} + 25 \binom{3p}{p} + 105 \pmod{p^7}, \\ \binom{6p}{p} &\equiv -270 \binom{2p}{p} + 70 \binom{3p}{p} + 336 \pmod{p^7}. \end{aligned}$$

Substituting $(n, k) = (-1, 2), (-2, 2)$, and $(-3, 2)$ into (1), then simplifying by (5), we obtain

$$\begin{aligned} \binom{6p}{2p} - 7 \binom{5p}{2p} + 21 \binom{4p}{2p} - 7 \binom{6p}{p} + 42 \binom{5p}{p} \\ - 105 \binom{4p}{p} + \frac{314}{3} \binom{3p}{p} - 105 \binom{2p}{p} &\equiv -77 \pmod{p^7}, \\ 7 \binom{6p}{2p} - 48 \binom{5p}{2p} + \frac{279}{2} \binom{4p}{2p} - 56 \binom{6p}{p} + 336 \binom{5p}{p} \\ - 840 \binom{4p}{p} + 896 \binom{3p}{p} - 840 \binom{2p}{p} &\equiv -546 \pmod{p^7}, \\ 28 \binom{6p}{2p} - \frac{948}{5} \binom{5p}{2p} + 540 \binom{4p}{2p} - 252 \binom{6p}{p} + 1512 \binom{5p}{p} \\ - 3780 \binom{4p}{p} + 4200 \binom{3p}{p} - 3780 \binom{2p}{p} &\equiv -2268 \pmod{p^7}. \end{aligned}$$

Solving this system yields

$$\begin{aligned} \binom{5p}{2p} &\equiv 35 - \frac{70}{3} \binom{3p}{p} + \frac{15}{2} \binom{4p}{2p} \pmod{p^7}, \\ \binom{6p}{2p} &\equiv 210 - 128 \binom{3p}{p} + \frac{63}{2} \binom{4p}{2p} \pmod{p^7}. \end{aligned}$$

Substituting $(n, k) = (-1, 3)$, $(-2, 3)$, and $(-3, 3)$ into (1) or (18), and simplifying, we obtain

$$\begin{aligned}
 & 28 \binom{6p}{p} - 147 \binom{5p}{p} + \frac{1345}{4} \binom{4p}{p} - 210 \binom{3p}{p} + 210 \binom{2p}{p} + \binom{6p}{3p} \\
 & \quad - 7 \binom{6p}{2p} + 35 \binom{5p}{2p} - 105 \binom{4p}{2p} \equiv 203 \pmod{p^7}, \\
 & 252 \binom{6p}{p} - 1344 \binom{5p}{p} + 3080 \binom{4p}{p} - 2240 \binom{3p}{p} + 2100 \binom{2p}{p} + 7 \binom{6p}{3p} \\
 & \quad - 56 \binom{6p}{2p} + \frac{1442}{5} \binom{5p}{2p} - 840 \binom{4p}{2p} \equiv 1736 \pmod{p^7}, \\
 & 1260 \binom{6p}{p} - 6804 \binom{5p}{p} + 15660 \binom{4p}{p} - 12600 \binom{3p}{p} + 11340 \binom{2p}{p} + \frac{57}{2} \binom{6p}{3p} \\
 & \quad - 252 \binom{6p}{2p} + 1323 \binom{5p}{2p} - 3780 \binom{4p}{2p} \equiv 8400 \pmod{p^7}.
 \end{aligned}$$

Solving, we find

$$\binom{6p}{3p} \equiv 56 - 30 \binom{4p}{p} + \frac{42}{5} \binom{5p}{2p} \equiv -250 + 540 \binom{2p}{p} - 396 \binom{3p}{p} + 63 \binom{4p}{2p} \pmod{p^7}.$$

Finally, the coefficients $c^t(n, k)$, $a(n, k)$, $b(n, k)$, and $d(n, k)$ are determined by the following linear combinations:

$$\begin{aligned}
 c^t(n, k) &= C_{00} + 20C_{41} + 105C_{51} + 336C_{61} + 35C_{52} + 210C_{62} - 250C_{63}, \\
 a(n, k) &= C_{21} - 18C_{41} - \frac{175}{2}C_{51} - 270C_{61} + 540C_{63}, \\
 b(n, k) &= C_{31} + \frac{20}{3}C_{41} + 25C_{51} + 70C_{61} - \frac{70}{3}C_{52} - 128C_{62} - 396C_{63}, \\
 d(n, k) &= C_{42} + \frac{15}{2}C_{52} + \frac{63}{2}C_{62} + 63C_{63}.
 \end{aligned}$$

□

2.6. Some Results in Terms of Generalized Harmonic Numbers

Theorem 10. For $p \geq 11$ and $t \in \mathbb{N}$, we have

$$\begin{aligned}
 \binom{tp}{p} &\equiv t + (-t^5 + 2t^4 - t^2)pH_{p-1}(1) + t^3(t-1)^2p^2H_{p-1}(1, 1) \pmod{p^7}, \\
 \binom{tp}{p} &\equiv t + (-t^5 + 2t^4 - t^2)pH_{p-1}(1) - \frac{t^3(t-1)^2}{2}p^2H_{p-1}(2) \pmod{p^6}, \\
 \binom{tp}{p} &\equiv t + t^2(t-1)pH_{p-1}(1) \pmod{p^5}.
 \end{aligned}$$

Proof. For the first statement, see [15]. For the second statement, we use the first congruence, the identity $H_{p-1}(1, 1) = \frac{1}{2}(H_{p-1}(1)^2 - H_{p-1}(2))$ (see [6]), and the fact that $H_{p-1}(1) \equiv 0 \pmod{p^2}$ and $H_{p-1}(2) \equiv 0 \pmod{p}$ (see [1]). Therefore, $p^2 H_{p-1}(1)^2 \equiv 0 \pmod{p^6}$, so this term can be omitted from the congruence. Finally, for the third statement, see [6, 15]. \square

Theorem 11. *We have the following congruences:*

1. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 11$, we have

$$\binom{np}{kp} - \binom{n}{k} \equiv nk(n-k) \binom{n}{k} p H_{p-1}(1) \pmod{p^{5+v_p\left(\binom{n}{5}\right)}},$$

2. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 11$, we have

$$\binom{np}{kp} - \binom{n}{k} \equiv nk(n-k) \binom{n}{k} (a(n, k)p H_{p-1}(1) + b(n, k)p^2 H_{p-1}(2)) \pmod{p^{6+v_p\left(\binom{n}{6}\right)}},$$

where

$$\begin{aligned} a(n, k) &= -k^2 + kn - n^2 + 2, \\ b(n, k) &= \frac{-k^2 + kn - n^2 + 1}{2}. \end{aligned}$$

Proof. 1. By the previous theorem, we find

$$\binom{2p}{p} \equiv 2 + 4p H_{p-1}(1) \pmod{p^5}.$$

Substituting this into (10), we get the desired result.

2. By setting $t = 2$ and $t = 3$ in the previous theorem, we find

$$\begin{aligned} \binom{2p}{p} &\equiv 2 - 4p H_{p-1}(1) - 4p^2 H_{p-1}(2) \pmod{p^6}, \\ \binom{3p}{p} &\equiv 3 - 90p H_{p-1}(1) - 54p^2 H_{p-1}(2) \pmod{p^6}. \end{aligned}$$

Substituting these into (15) and simplifying yields the congruence. \square

Using Theorem 10, we determine $\binom{2p}{p}$, $\binom{3p}{p}$, $\binom{4p}{p}$, $\binom{5p}{p}$, and $\binom{6p}{p}$ modulo p^7 in terms of generalized harmonic numbers. In the same vein, after routine computation, we obtain the following theorem.

Theorem 12. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 11$, we have

$$\binom{np}{kp} - \binom{n}{k} \equiv nk(n-k) \binom{n}{k} \left(a(n,k)pH_{p-1}(1) + b(n,k)p^2H_{p-1}(1,1) + c(n,k) \left(\binom{4p}{2p} - 6 \right) \right) \pmod{p^{7+v_p\left(\binom{n}{7}\right)},} \tag{19}$$

where

$$\begin{aligned} a(n,k) &= -2k^2n + 2kn^2 - 3k^2 + 3kn - 3n^2 + 4, \\ b(n,k) &= \frac{11k^2n - 11kn^2 + 16k^2 - 16kn + 16n^2 - 16}{5}, \\ c(n,k) &= -\frac{(-n+k+1)(n+1)(k-1)}{240}. \end{aligned}$$

In order to find a congruence that depends entirely on generalized harmonic numbers, we must determine $\binom{4p}{2p}$ modulo p^7 . This can be accomplished by the following lemma.

Lemma 6. Let $m, n \in \mathbb{N}$ such that $1 \leq m \leq n$ and let p be a prime number. We have

$$\binom{np}{mp} = \binom{n}{m} \frac{\prod_{j=0}^{m-1} \binom{(n-m+j+1)p-1}{p-1}}{\prod_{j=0}^{m-1} \binom{(j+1)p-1}{p-1}} = \frac{\prod_{j=0}^{m-1} \binom{(n-m+j+1)p}{p}}{\prod_{j=0}^{m-1} \binom{(j+1)p}{p}}.$$

Proof. From [4], we have that

$$\binom{np}{mp} = \binom{n}{m} \prod_{j=0}^{m-1} \prod_{i=1}^{p-1} \left(1 + \frac{(n-m)p}{jp+i} \right).$$

Therefore, we obtain

$$\begin{aligned} \binom{np}{mp} &= \binom{n}{m} \prod_{j=0}^{m-1} \left(\prod_{i=1}^{p-1} \frac{(n-m+j)p+i}{jp+i} \right) \\ &= \binom{n}{m} \prod_{j=0}^{m-1} \left(\prod_{i=1}^{p-1} \frac{(n-m+j)p+i}{i} \frac{i}{jp+i} \right) = \binom{n}{m} \frac{\prod_{j=0}^{m-1} \binom{(n-m+j+1)p-1}{p-1}}{\prod_{j=0}^{m-1} \binom{(j+1)p-1}{p-1}} \\ &= \frac{(n)_m}{m!} \frac{\prod_{j=0}^{m-1} (n-m+j+1) \binom{(n-m+j+1)p-1}{p-1}}{\prod_{j=0}^{m-1} (j+1) \binom{(j+1)p-1}{p-1}} = \frac{\prod_{j=0}^{m-1} \binom{(n-m+j+1)p}{p}}{\prod_{j=0}^{m-1} \binom{(j+1)p}{p}}. \end{aligned}$$

□

For example, we have

$$\binom{4p}{2p} = 6 \frac{\binom{3p-1}{p-1} \binom{4p-1}{p-1}}{\binom{2p-1}{p-1}} = \frac{\binom{3p}{p} \binom{4p}{p}}{\binom{2p}{p}},$$

$$\binom{2p}{p} \binom{3p}{p} \binom{7p}{3p} = \binom{5p}{p} \binom{6p}{p} \binom{7p}{p}.$$

This example shows that $\binom{4p}{2p}$ is expressible in terms of binomial coefficients of the form $\binom{ap-1}{bp-1}$ with $b = 1$. Congruences for this type of binomial coefficient have been studied extensively, and we can utilize some results to determine $\binom{4p}{2p}$ modulo p^7 in terms of multiple harmonic sums.

Theorem 13. For $p \geq 11$ and $\alpha \in \mathbb{Z}_p$, we have

$$\binom{\alpha p - 1}{p - 1} \binom{(\alpha + 1)p - 1}{p - 1} \equiv 1 - (\alpha p)^2 H_{p-1}(2) + \frac{(\alpha p)^4}{2} (H_{p-1}(2)^2 - H_{p-1}(4)) \pmod{p^7},$$

$$\binom{2p - 1}{p - 1}^{-1} \equiv 1 + p^2 H_{p-1}(2) + \frac{p^4}{2} (H_{p-1}(2)^2 + H_{p-1}(4)) \pmod{p^7}.$$

Proof. We refer the reader to [6]. □

From this, we find

$$\binom{4p}{2p} \equiv 6 \left(1 - 9p^2 H_{p-1}(2) + \frac{81p^4}{2} (H_{p-1}(2)^2 - H_{p-1}(4)) \right) \times \left(1 + p^2 H_{p-1}(2) + \frac{p^4}{2} (H_{p-1}(2)^2 + H_{p-1}(4)) \right) \equiv 6 - 48p^2 H_{p-1}(2) + p^4 (192H_{p-1}(2)^2 - 240H_{p-1}(4)) \pmod{p^7}.$$

Substituting this result into (19), we find the following congruence.

Theorem 14. For $n \in \mathbb{Z}_p, k \in \mathbb{N}$, and $p \geq 11$, we have

$$\binom{np}{kp} - \binom{n}{k} \equiv nk(n - k) \binom{n}{k} \left(a(n, k)p H_{p-1}(1) + b(n, k)p^2 H_{p-1}(1, 1) + c(n, k)p^2 (H_{p-1}(2) - p^2(4H_{p-1}(2)^2 + 5H_{p-1}(4))) \right) \pmod{p^{7+v_p\left(\binom{n}{k}\right)}},$$

where

$$\begin{aligned} a(n, k) &= -2k^2n + 2kn^2 - 3k^2 + 3kn - 3n^2 + 4, \\ b(n, k) &= \frac{11k^2n - 11kn^2 + 16k^2 - 16kn + 16n^2 - 16}{5}, \\ c(n, k) &= \frac{(-n + k + 1)(n + 1)(k - 1)}{5}. \end{aligned}$$

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