



**THREE INTEGERS WHOSE SUM, PRODUCT, AND THE SUM OF
THE PRODUCTS OF THE INTEGERS, TAKEN TWO AT A TIME,
ARE PERFECT SQUARES**

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Abstract

Euler had considered the problem of finding three integers whose sum, product, and also the sum of the products of the integers, taken two at a time, are all perfect squares. Euler's methods of solving the problem lead to parametric solutions in terms of polynomials of high degrees and his numerical solutions consisted of very large integers. We obtain, by a new method, several parametric solutions given by polynomials of much smaller degrees and thus, we get a number of numerically small solutions of the problem.

1. Introduction

This paper is concerned with the problem of finding three integers a , b , and c whose sum, product, and also the sum of the products of the integers, taken two at a time, are all perfect squares. The problem was first considered by Euler in 1760. While Euler's original paper is in Latin [3], an English translation of the paper, together with a commentary, has recently been published by Snively and Woodruff in 2022 [7]. Euler has mentioned that he spent "a long time in vain" searching for such numbers and then, "almost unexpectedly" found a solution by "a remarkable method." He noted that any solution of the problem in rational numbers readily yields, on multiplying the rational numbers by a suitable perfect square, a solution of the problem in integers. His method of solving leads to very large numbers. In fact, his method yields parametric solutions in terms of a single rational parameter. While Euler did not carry out the cumbersome computations to find the parametric solutions, we found, using the software MAPLE, that in the simplest parametric solution, obtained by Euler's method, the values of a , b , and c are given by polynomials of degree 132. The smallest numerical solution found by Euler, consisting of

integers with 13 digits, is as follows:

$$1633780814400, \quad 252782198228, \quad 3474741058973.$$

As mentioned by Dickson [2, p. 603], “Euler had expressed his belief that these give the least integers” satisfying the conditions of the problem.

Euler revisited the problem in 1779 [4]. While he again referred to the problem as being “very difficult” and giving rise to “immense numbers,” he obtained, after “some futile attempts,” relatively simpler solutions in smaller integers by imposing the additional condition that all the three integers a, b , and c must be perfect squares. Euler’s new method of solving also leads to parametric solutions. Since Euler did not find the parametric solutions explicitly, we did the computations on MAPLE and found several parametric solutions in terms of univariate polynomials. The simplest such solution may be written as follows:

$$\begin{aligned} a &= (t - 1)^2(t + 1)^2(t^2 + 5)^2(t^4 - 10t^2 + 5)^2, \\ b &= 256t^4(t - 1)^2(t + 1)^2(t^2 - 3)^2, \\ c &= 4t^2(t^2 - 3)^2(t^4 - 10t^2 + 5)^2, \end{aligned}$$

where t is an arbitrary rational parameter. The smallest numerical solution found by Euler is the triad of integers: 81, 784 and 186624. Euler’s son, J. A. Euler, found the following solution (as quoted by Dickson [2, p. 669]) in which a, b , and c are all squares:

$$a = (6p^2 - p^4 - 1)^2(p^2 + 1)^2, \quad b = 16p^2(p^2 - 1)^4, \quad c = 64p^4(p^2 - 1)^2, \quad (1.1)$$

where p is an arbitrary parameter. When $p = 2$, we get $(a, b, c) = (1225, 5184, 9216)$.

In 1899, Fauquembergue [5] obtained the simpler solution

$$(a, b, c) = 4s^4t^2(s^2 + t^2), 4t^4(s^2 + t^2)s^2, (s^4 - t^4)^2$$

which, on taking $(s, t) = (2, 1)$, yields the numerically small solution $(a, b, c) = (320, 80, 225)$. It seems that no other parametric solutions of the problem have been published. Snively and Woodruff [7] have made a brief comment that solutions that are numerically smaller, as compared to Euler’s solutions, have been found recently using “brute force methods” but they have not explicitly mentioned any of these smaller solutions.

In this paper we show that each of the three integers a, b , and c must be a sum of two squares of rational numbers of which one may possibly be zero. This fact readily leads to simple parametric solutions of the problem. Further, it follows from this fact that in any solution, either exactly one of the integers is a square, or all three of them are squares, or none of them is a square. We find parametric solutions of all three types, and we show how infinitely many parametric solutions can be obtained.

Specifically, we obtain four parametric solutions in which one of the three integers a, b , and c is a square — two of these solutions are given by simple polynomials of degree 8 while the remaining two are given by polynomials of degree 20. These four solutions, on suitable specialization, yield four parametric solutions of degrees 12, 16, 36, and 40, respectively, in which all the three integers a, b , and c are squares. We also obtain a parametric solution given by polynomials of degree 32 in which none of the three integers is a square.

While two of our solutions have been found earlier, all other solutions are new and are given by polynomials of much lower degrees as compared to the solutions generated by Euler's methods. Further, the parametric solutions obtained in the paper yield several numerically small solutions of the problem, the following three triads being examples:

$$\{45, 64, 180\}, \quad \{81, 160, 1440\}, \quad \{72, 136, 153\}.$$

While we have proved the existence of infinitely many parametric solutions of the problem by using elliptic curves and the Nagell-Lutz theorem discovered in the twentieth century, the parametric solutions of low degrees as well as the numerically small solutions have been obtained by elementary methods known to Euler but the great master somehow missed the fact that each of the three integers a, b , and c must be a sum of two squares, and had to resort to cumbersome methods that led to the numerically huge solutions mentioned earlier.

2. Preliminary Lemmas and Remarks

In this section we prove a couple of preliminary lemmas.

Lemma 1. *If two nonzero rational numbers α and β are both expressible as sums of two squares of rational numbers, then their ratio α/β is also expressible as a sum of two squares of rational numbers.*

Proof. Let $\alpha = \alpha_1^2 + \alpha_2^2$ and $\beta = \beta_1^2 + \beta_2^2$ where $\alpha_i, \beta_i, i = 1, 2$ are all rational numbers. Since α and β are both nonzero, we may define two rational numbers m and n as follows:

$$m = (\alpha_1\beta_1 + \alpha_2\beta_2)/(\beta_1^2 + \beta_2^2), \quad n = (\alpha_1\beta_2 - \alpha_2\beta_1)/(\beta_1^2 + \beta_2^2).$$

It is readily verified that $m^2 + n^2 = (\alpha_1^2 + \alpha_2^2)/(\beta_1^2 + \beta_2^2) = \alpha/\beta$. This proves the lemma. \square

We note that when $\alpha_1/\beta_1 = \alpha_2/\beta_2$, then $n = 0$, that is, one of the two squares becomes zero, and α/β simply becomes m^2 . We will apply Lemma 1 to show that the three integers that we are seeking must all be sums of two squares of rational

numbers (see Lemma 3 for a proof). We will then use this fact to obtain several parametric solutions of our problem.

Lemma 2. *If all three elementary symmetric functions of three integers a, b , and c are squares, then all three elementary symmetric functions of the three integers ab, bc , and ca are also squares.*

Proof. If $a + b + c$, $ab + bc + ca$, and abc are all squares, then it immediately follows that the sum and product of the integers ab, bc , and ca , namely $ab + bc + ca$ and $a^2b^2c^2$, are both squares. Further, the sum of the products of the integers ab, bc , and ca , taken two at a time, is $abc(a + b + c)$, which is also a square. Thus, all three elementary symmetric functions of the three integers ab, bc , and ca are squares. \square

If we begin with a triad of integers a, b , and c whose elementary symmetric functions are all squares, and apply Lemma 2 twice, we revert, after removing the common squared factor abc , to the original triad of integers. We also note that the converse of Lemma 2 is not true. This is seen from the counterexample of the three integers $(a, b, c) = (8, 9, 17)$ whose sum is 34, and hence the three elementary symmetric functions of the integers a, b , and c are not squares. However, it is readily seen that the elementary symmetric functions of the integers ab, bc , and ca , that is, the integers 72, 153, and 136, are all squares. In fact, we have already mentioned in the introduction that the triad $(72, 153, 136)$ is one of the numerically small solutions of the problem obtained in this paper.

3. Three Integers Whose Sum, Product, and the Sum of the Products, Taken Two at a Time, are Perfect Squares

We now consider the problem of finding three rational numbers a, b , and c that satisfy the conditions of our problem, that is, there must exist rational numbers f, g , and h such that

$$\begin{aligned} a + b + c &= f^2, \\ ab + bc + ca &= g^2, \\ abc &= h^2. \end{aligned} \tag{3.1}$$

Thus, the three elementary symmetric functions of a, b , and c are all squares, and a, b , and c are the roots of the following cubic equation:

$$x^3 - f^2x^2 + g^2x - h^2 = 0. \tag{3.2}$$

As already noted by Euler, any solution of our problem in rational numbers yields, on appropriate scaling, a solution in integers. We now prove, in the following lemma, an important property of the numbers a, b , and c .

Lemma 3. *If there exists a triad $\{a, b, c\}$ of rational numbers whose elementary symmetric functions $a + b + c$, $ab + bc + ca$, and abc are all squares, then each of the three numbers a, b , and c is a sum of two squares of rational numbers. Further, on appropriate scaling, each such triad of rational numbers yields a triad of integers whose elementary symmetric functions are all perfect squares and such that each of the three integers is expressible as a sum of two squares of integers.*

Proof. As already noted, the three rational numbers a, b , and c are the roots of the cubic Equation (3.2) which may be written as

$$x = (f^2x^2 + h^2)/(x^2 + g^2).$$

It follows that any rational root of Equation (3.2) is expressible as a ratio of two rational numbers both of which are sums of squares of two rational numbers. It now follows from Lemma 1 that every rational root of Equation (3.2) is expressible as a sum of two squares of rational numbers. We may thus write the rational numbers a, b , and c as $(a_1^2 + a_2^2)/d_1^2$, $(b_1^2 + b_2^2)/d_2^2$, and $(c_1^2 + c_2^2)/d_3^2$, respectively, where a_i, b_i, c_i , and d_i are all integers. Now, on multiplying each of the rational numbers a, b , and c by $(d_1d_2d_3)^2$ we get three integers, each of which is expressible as a sum of two squares of integers, and such that all the three elementary symmetric functions of the three integers are perfect squares. This proves the lemma. \square

We note that since the three rational numbers a, b , and c are sums of two squares of rational numbers one of which could possibly be zero, it readily follows that we will obtain, on appropriate scaling, triads of integers such that either one of the three integers is a square, or all three are squares, or all the three integers are sums of two nonzero squares. In the next three subsections, we obtain examples of these three types of solutions.

3.1. Solutions in Which One of the Three Integers is a Square

The following theorem gives four parametric solutions that yield integer solutions of our problem such that one of the three integers is a square.

Theorem 1. *If a, b , and c are defined in terms of arbitrary integer parameters s and t by*

$$(a, b, c) = (t^2(s^2 - t^2)^2(s^2 + t^2), s^2(s^2 - t^2)^2(s^2 + t^2), 4s^4t^4), \tag{3.3}$$

$$(a, b, c) = (4s^4t^2(s^2 + t^2), 4s^2t^4(s^2 + t^2), (s^4 - t^4)^2), \tag{3.4}$$

$$\begin{aligned} a &= 4s^4t^2(s^2 + t^2)(s^6 - s^4t^2 - 5s^2t^4 + t^6)^2, \\ b &= (s^4 - t^4)^2(s^6 - s^4t^2 - 5s^2t^4 + t^6)^2, \\ c &= 4s^2t^4(s^2 + t^2)(3s^6 + s^4t^2 + s^2t^4 - t^6)^2, \end{aligned} \tag{3.5}$$

or

$$\begin{aligned} a &= t^2(s^2 - t^2)^2(s^2 + t^2)(3s^6 + s^4t^2 + s^2t^4 - t^6)^2, \\ b &= 4s^4t^4(3s^6 + s^4t^2 + s^2t^4 - t^6)^2, \\ c &= s^2(s^2 - t^2)^2(s^2 + t^2)(s^6 - s^4t^2 - 5s^2t^4 + t^6)^2, \end{aligned} \tag{3.6}$$

then, in each case, all the three elementary symmetric functions of a, b , and c are squares. Further, there exist infinitely many ways in which the values of a, b , and c may be defined, in parametric terms, such that the three elementary symmetric functions of a, b , and c are squares, and such values of a, b , and c can be effectively determined.

Proof. We will obtain the triads $\{a, b, c\}$ by solving Equation (3.2) to obtain three rational roots. Since each such root must be a sum of two squares, we take $x = p^2 + q^2$ as one of the three roots where p and q are some rational numbers. On substituting $x = p^2 + q^2$ in Equation (3.2) and using the identity

$$(\alpha_1^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2) = (\alpha_1\beta_1 + \alpha_2\beta_2)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2,$$

we may write $x^3 + g^2x = x(x^2 + g^2)$ as a sum of two squares, and accordingly, Equation (3.2) may be written as

$$(p^3 + pq^2 + qg)^2 + (p^2q + q^3 - pg)^2 - f^2(p^2 + q^2)^2 - h^2 = 0. \tag{3.7}$$

After suitable transposition of terms and factorization, Equation (3.7) may be written as follows:

$$\begin{aligned} (p^3 - p^2f + pq^2 - q^2f + qg)(p^3 + p^2f + pq^2 + q^2f + qg) \\ = (h - p^2q - q^3 + pg)(h + p^2q + q^3 - pg). \end{aligned} \tag{3.8}$$

Assuming that both sides of Equation (3.8) are not equal to 0, there exists a nonzero rational number m such that

$$p^3 - p^2f + pq^2 - q^2f + qg = m(h - p^2q - q^3 + pg), \tag{3.9}$$

and now Equation (3.8) reduces to

$$m(p^3 + p^2f + pq^2 + q^2f + qg) = h + p^2q + q^3 - pg. \tag{3.10}$$

Equations (3.9) and (3.10) may be considered as two linear equations in the variables f and g , and on solving them, we get

$$\begin{aligned} f &= -((p^4 + 2p^2q^2 + q^4 - qh)m^2 - 2mph + p^4 + 2p^2q^2 + q^4 + qh) \\ &\quad \times ((m^2p - 2mq - p)(p^2 + q^2))^{-1}, \\ g &= ((p^2q + q^3 - h)m^2 + (2p^3 + 2pq^2)m - p^2q - q^3 - h)/(m^2p - 2mq - p). \end{aligned} \tag{3.11}$$

With the values of f and g given by (3.11), Equation (3.2) has a root $x = p^2 + q^2$, and indeed, Equation (3.2) now reduces to

$$\begin{aligned} &(x - p^2 - q^2)\{(m^2p - 2mq - p)^2(p^2 + q^2)^2x^2 - ((m^2q + 2pm - q)^2h^2 \\ &- 2(p^2 + q^2)^2(m^2 + 1)(m^2q + 2pm - q)h + (p^2 + q^2)^3(m^2q + 2pm - q)^2\}x \\ &+ h^2(p^2 + q^2)(m^2p - 2mq - p)^2\} = 0. \end{aligned} \tag{3.12}$$

We remove the first factor $x - p^2 - q^2$, and write

$$\begin{aligned} p &= (m^2t + 2ms - t)/(m^2 + 1), \\ q &= (m^2s - 2mt - s)/(m^2 + 1), \\ h &= u(s^2 + t^2), \end{aligned} \tag{3.13}$$

where s, t , and u are rational parameters, and now Equation (3.12) reduces, after removing the factors $(s^2 + t^2)^2(m^2 + 1)^2$, to the following quadratic equation in x :

$$t^2x^2 - s(s^3 - 2s^2u + st^2 + su^2 - 2t^2u)x + u^2t^2(s^2 + t^2) = 0. \tag{3.14}$$

The discriminant of Equation (3.14) with respect to x is $s^4\phi(s, t, u)$, where

$$\begin{aligned} \phi(s, t, u) &= u^4 - 4(s^2 + t^2)u^3/s + 2(s^2 + t^2)(3s^4 + 2s^2t^2 - 2t^4)u^2/s^4 \\ &- 4(s^2 + t^2)^2u/s + (s^2 + t^2)^2. \end{aligned} \tag{3.15}$$

Equation (3.14) will have two rational roots if its discriminant, and hence also $\phi(s, t, u)$, is a perfect square, that is, there must exist a rational number v such that the following equation has a solution in rational numbers:

$$v^2 = u^4 + a_1u^3 + a_2u^2 + a_3u + a_4, \tag{3.16}$$

where

$$\begin{aligned} a_1 &= -4(s^2 + t^2)/s, & a_2 &= 2(s^2 + t^2)(3s^4 + 2s^2t^2 - 2t^4)/s^4, \\ a_3 &= -4(s^2 + t^2)^2/s, & a_4 &= (s^2 + t^2)^2. \end{aligned} \tag{3.17}$$

Each solution of Equation (3.16) yields a value of u which makes the discriminant of Equation (3.14) a perfect square, and hence Equation (3.16) has two rational roots. Thus, Equation (3.2) has three rational roots and we will obtain the desired numbers a, b , and c .

The right-hand side of Equation (3.16) is a quartic function of u in which the coefficients of both u^4 and u^0 are perfect squares. Fermat had described a method (as quoted by Dickson [2, p. 639]) of making such a quartic function a perfect square, and using Fermat's method, we readily find the following two solutions of Equation (3.16):

$$(u, v) = (2s^3/(s^2 - t^2), -(s^6 - s^4t^2 - 5s^2t^4 + t^6)/(s^2 - t^2)^2), \tag{3.18}$$

$$(u, v) = ((s^4 - t^4)/(2s^3), (s^2 + t^2)(s^6 - s^4t^2 - 5s^2t^4 + t^6)/(4s^6)). \tag{3.19}$$

The value of u given by each of the two solutions (3.18) and (3.19) makes the discriminant of Equation (3.14) a perfect square, and hence this equation has two rational roots, and hence also Equation (3.2) has three rational roots. Using the relations (3.13), these three roots may be expressed as rational functions of s and t , and on appropriate scaling, we get the values of a, b , and c in terms of polynomials in s and t . Specifically, we note that, on taking $u = 2s^3/(s^2 - t^2)$, as in the solution (3.18), Equation (3.14) has the two rational roots,

$$x = s^2(s^2 + t^2)/t^2 \quad \text{and} \quad x = 4s^4t^2/(s^2 - t^2)^2, \tag{3.20}$$

and we thus obtain, on appropriate scaling, the triad $\{a, b, c\}$ of integers given by (3.3) such that the elementary symmetric functions of a, b , and c are all perfect squares.

Similarly, the value of u given by the solution (3.19) of Equation (3.16) yields two rational solutions of Equation (3.14) and, as before, we obtain, on appropriate scaling, the triad a, b , and c defined by (3.4). The solution (3.4) is the same as Fauquembergue’s solution mentioned in the Introduction.

We will now obtain the solutions (3.5) and (3.6). According to a theorem of Choudhry [1, Theorem 4.1, pp. 789-790], if (u_1, v_1) and (u_2, v_2) are two rational solutions of Equation (3.16) with $u_1 \neq u_2$, a new rational solution of Equation (3.16) is given by (u_{12}, v_{12}) where

$$\begin{aligned} u_{12} = & \{-2v_1v_2 + 2(u_1 - u_2)(u_2v_1 - u_1v_2) + a_1(u_1 + u_2)u_1u_2 \\ & + 2a_2u_1u_2 + a_3(u_1 + u_2) + 2a_4 + 2(u_1^2 - u_1u_2 + u_2^2)u_1u_2\} \\ & \times \{(u_1 - u_2)(2v_1 - 2v_2 + a_1(u_1 - u_2) + 2u_1^2 - 2u_2^2)\}^{-1}, \end{aligned} \tag{3.21}$$

provided the denominator on the right-hand side of (3.21) $\neq 0$. We omit the value of v_{12} given in [1, Theorem 4.1, pp. 789-790] as it is cumbersome to write and is not needed for our computations.

Applying Choudhry’s theorem to Equation (3.16), taking (u_1, v_1) as the obvious solution $(0, s^2 + t^2)$ and (u_2, v_2) as the solution (3.18), we find a solution of Equation (3.16) in which

$$u = (s^4 - t^4)(3s^6 + s^4t^2 + s^2t^4 - t^6)/(2s^3(s^6 - s^4t^2 - 5s^2t^4 + t^6)), \tag{3.22}$$

and this yields the solution (3.5) stated in the theorem. Similarly, on applying Choudhry’s theorem to Equation (3.16), again taking (u_1, v_1) as the solution $(0, s^2 + t^2)$ and (u_2, v_2) as the solution (3.19), we find a solution of Equation (3.16) in which

$$u = 2s^3(s^6 - s^4t^2 - 5s^2t^4 + t^6)/((3s^6 + s^4t^2 + s^2t^4 - t^6)(s^2 - t^2)), \tag{3.23}$$

and this yields the solution (3.6).

We will now show that Equation (3.16) has infinitely many rational solutions. We first write,

$$t = ms, \quad u = sU, \quad v = s^2V, \tag{3.24}$$

when Equation (3.16) reduces to

$$V^2 = U^4 - 4(m^2 + 1)U^3 - 2(m^2 + 1)(2m^4 - 2m^2 - 3)U^2 - 4(m^2 + 1)^2U + (m^2 + 1)^2. \quad (3.25)$$

While Equation (3.25) is in three variables, m, U , and V , and hence represents an algebraic surface over the field \mathbb{Q} , we may regard it as representing a quartic model of an elliptic curve over the function field $\mathbb{Q}(m)$. Using APECS, a package written in MAPLE for working with elliptic curves, we found that the birational transformation defined by

$$\begin{aligned} U &= (6(m^2 + 1)X + mY + 72m^6 + 72m^4 - 72m^2 - 72) \\ &\quad \times (6(X - 24m^4 - 36m^2 - 12))^{-1}, \\ V &= (2m^2X^3 - 36m^2(2m^2 + 1)(m^2 + 1)X^2 - m^2Y^2 \\ &\quad - 432m^3(m^2 + 1)^2Y + 1728m^2(m^2 + 1)^3(8m^6 - 15m^4 - 21m^2 + 1)) \\ &\quad \times (6(X - 24m^4 - 36m^2 - 12))^{-2}, \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} X &= 6(3U^2 - (6m^2 + 6)U + 3V - (m^2 + 1)(2m^4 - 2m^2 - 3))/m^2, \\ Y &= 108(U^3 - (3m^2 + 3)U^2 + UV - (m^2 + 1)(2m^4 - 2m^2 - 3)U \\ &\quad - (m^2 + 1)V - (m^2 + 1)^2)/m^3, \end{aligned} \quad (3.27)$$

reduces Equation (3.25) to the following cubic equation:

$$Y^2 = X^3 - 432(m^4 - 2m^2 - 2)(m^2 + 1)^2X - 1728(m^2 - 1)(m^2 + 1)^3(2m^4 - 4m^2 - 7). \quad (3.28)$$

Equation (3.28) also represents an algebraic surface over \mathbb{Q} , but as in the case of Equation (3.25), we may regard it as the Weierstrass model of an elliptic curve over the function field $\mathbb{Q}(m)$. Using the solution (3.18) of Equation (3.16) and the relations (3.24) and (3.27), we readily find that a rational point P on the elliptic curve (3.28) is given by

$$X = -12(m^6 - 4m^2 - 3)/m^2, \quad Y = 216(m^2 + 1)^2/m^3.$$

When $m = 4$, the curve (3.28) reduces to

$$Y^2 = X^3 - 27716256X - 56159127360, \quad (3.29)$$

and, corresponding to the point P on the curve (3.28), a rational point on the curve (3.29) is $(-12087/4, 7803/8)$. Since this rational point on the elliptic curve (3.29) does not have integer coordinates, it follows from the Nagell-Lutz theorem [6, p. 56]

on elliptic curves that the point $(-12087/4, 7803/8)$ is not a point of finite order. We can thus find infinitely many rational points on the curve (3.29) using the group law.

Since in the special case $m = 4$, the point on the curve (3.29) corresponding to the point P is not of finite order, it follows that the point P on the curve (3.28) cannot be a point of finite order. We can thus generate infinitely many rational points on the curve (3.28) using the group law. Each of these rational points will yield a corresponding rational point on the curve (3.25) and using the relations (3.24), we can find infinitely many parametric solutions of the diophantine Equation (3.16). We can thus obtain infinitely many parametric solutions of our problem. \square

We note that all the solutions given by Theorem 1 have been obtained by assuming that both sides of Equation (3.8) are not equal to 0. If we take both sides of Equation (3.8) as 0 and then solve Equation (3.2), we do not get any new solutions of our problem.

We also note that on applying Lemma 2 to the triad of integers given by (3.3), we get, after removing common squared factors, the triad of integers given by (3.4) and vice versa. Similarly, on applying Lemma 2, the triad of integers (3.5) yields the triad (3.6) and vice versa. Thus, applying Lemma 2 to the triads given by Theorem 1 does not yield any new solutions of our problem.

3.2. Solutions in Which All Three Integers are Squares

As mentioned earlier, Euler found three square numbers whose elementary symmetric functions are squares after several attempts. We will derive three such square numbers from each of the four parametric solutions (3.3), (3.4), (3.5), and (3.6) given in Theorem 1 by choosing the parameters s and t suitably. While the results thus obtained appear here as a corollary to Theorem 1, in view of their significance, we state them below as a theorem.

Theorem 2. *If $a, b,$ and c are defined in terms of arbitrary integer parameters m and n by*

$$\begin{aligned} a &= (m^4 - n^4)^2(m^4 - 6m^2n^2 + n^4)^2, \\ b &= 4m^2n^2(m^2 + n^2)^2(m^4 - 6m^2n^2 + n^4)^2, \\ c &= 64m^4n^4(m^2 - n^2)^4, \end{aligned} \tag{3.30}$$

$$\begin{aligned} a &= 64m^4n^4(m^2 - n^2)^2, \\ b &= 16m^2n^2(m^2 - n^2)^4, \\ c &= (m^2 + n^2)^2(m^4 - 6m^2n^2 + n^4)^2, \end{aligned} \tag{3.31}$$

$$\begin{aligned}
 a &= 64m^4n^4(m^2 - n^2)^2(m^{12} - 26m^{10}n^2 + 79m^8n^4 \\
 &\quad - 44m^6n^6 + 79m^4n^8 - 26m^2n^{10} + n^{12})^2, \\
 b &= (m^2 + n^2)^2(m^4 - 6m^2n^2 + n^4)^2(m^{12} - 26m^{10}n^2 + 79m^8n^4 \\
 &\quad - 44m^6n^6 + 79m^4n^8 - 26m^2n^{10} + n^{12})^2, \\
 c &= 16m^2n^2(m^2 - n^2)^4(m^{12} - 10m^{10}n^2 + 15m^8n^4 \\
 &\quad - 204m^6n^6 + 15m^4n^8 - 10m^2n^{10} + n^{12})^2,
 \end{aligned} \tag{3.32}$$

or

$$\begin{aligned}
 a &= (m^4 - n^4)^2(m^4 - 6m^2n^2 + n^4)^2(m^{12} - 10m^{10}n^2 \\
 &\quad + 15m^8n^4 - 204m^6n^6 + 15m^4n^8 - 10m^2n^{10} + n^{12})^2, \\
 b &= 64m^4n^4(m^2 - n^2)^4(m^{12} - 10m^{10}n^2 + 15m^8n^4 \\
 &\quad - 204m^6n^6 + 15m^4n^8 - 10m^2n^{10} + n^{12})^2, \\
 c &= 4m^2n^2(m^2 + n^2)^2(m^4 - 6m^2n^2 + n^4)^2(m^{12} - 26m^{10}n^2 \\
 &\quad + 79m^8n^4 - 44m^6n^6 + 79m^4n^8 - 26m^2n^{10} + n^{12})^2,
 \end{aligned} \tag{3.33}$$

then, in each case, all the three elementary symmetric functions of a, b , and c are squares.

Proof. It is easy to observe that if we choose the parameters s and t such that $s^2 + t^2$ is a square, all the three integers a, b , and c given by each of the four solutions (3.3), (3.4), (3.5), and (3.6) become squares. Accordingly, we take $s = 2mn$ and $t = m^2 + n^2$ in each of the four aforementioned solutions, and thus obtain the four triads of squares (3.30), (3.31), (3.32), and (3.33) given in the theorem. Since all these four triads $\{a, b, c\}$ have been derived from the parametric solutions obtained in Theorem 1 simply by assigning special values to the parameters, it immediately follows that, in each case, the three elementary symmetric functions of a, b , and c are squares. \square

As in the case of the triads given by Theorem 1, applying Lemma 2 to the triads given by Theorem 2 also does not yield any new solutions of our problem. We note that the solution (1.1) found by J. A. Euler yields, on writing $p = m/n$ and homogenizing, a solution equivalent to the solution (3.31). Thus, we cannot consider the solution (3.31) as a new solution.

3.3. Solutions in Which All Three Integers are Sums of Two Nonzero Squares

The following theorem gives three integers, each expressible as a sum of two nonzero squares, such that their elementary symmetric functions are all squares.

Theorem 3. *If $a, b,$ and c are defined in terms of arbitrary integer parameters r and s by*

$$\begin{aligned}
 a &= r^2(r^2 + s^2)(r^8 + 2r^6s^2 - 12r^4s^4 - 6r^2s^6 - s^8)^2 \\
 &\quad \times (r^{12} + 6r^{10}s^2 + 87r^8s^4 + 108r^6s^6 + 55r^4s^8 + 14r^2s^{10} + s^{12}), \\
 b &= s^2(r^2 + s^2)(5r^6 + 3r^4s^2 + 3r^2s^4 + s^6)^2 \\
 &\quad \times (r^8 + 2r^6s^2 - 12r^4s^4 - 6r^2s^6 - s^8)^2, \\
 c &= 4r^4s^4(5r^6 + 3r^4s^2 + 3r^2s^4 + s^6)^2 \\
 &\quad \times (r^{12} + 6r^{10}s^2 + 87r^8s^4 + 108r^6s^6 + 55r^4s^8 + 14r^2s^{10} + s^{12}),
 \end{aligned}
 \tag{3.34}$$

all the three elementary symmetric functions of $a, b,$ and c are squares.

Proof. To obtain the three desired integers, we must choose the parameters $f, g,$ and h such that all the three roots of the cubic Equation (3.2) are expressible as sums of two nonzero squares. With f and g defined by (3.11), one root of Equation (3.2) is $p^2 + q^2,$ as desired, and after removing the factor $x - p^2 - q^2,$ we had reduced Equation (3.2) to the quadratic Equation (3.14). Thus, we need to choose the parameters $s, t,$ and u such that both roots of Equation (3.14) are expressible as sums of two nonzero squares. Since the product of the two roots of Equation (3.14) is $u^2(s^2 + t^2),$ it follows from Lemma 1, that if one of the two roots is a sum of two nonzero squares and this root is not expressible as $k^2(s^2 + t^2)$ where k is a nonzero rational number, then the second root of Equation (3.14) will also be a sum of two nonzero squares.

Now Equation (3.14) may be considered as a quadratic equation in u and it will be satisfied by a rational value of u if its discriminant $4t^2\psi(s, t, x)$ with respect to u is a square, where

$$\psi(s, t, x) = x(s^2 - x)t^4 + 2s^4t^2x + s^2(s^4 + s^2x + x^2)x.
 \tag{3.35}$$

As we have already noted, the rational value of u given by (3.18) yields two roots of Equation (3.14) given by (3.20), the first root being $x = s^2(s^2 + t^2)/t^2.$ It follows that, with this value of $x,$ the discriminant of Equation (3.14) with respect to $u,$ and hence also $\psi(s, t, x),$ must be a square, and further, if we take

$$x = s^2(r^2 + s^2)/r^2,
 \tag{3.36}$$

where r is an arbitrary rational parameter, then $\psi(s, t, x)$ must become a perfect square when $t = r.$ This can also be verified by direct computation.

Thus when we take x as defined by (3.36), then $\psi(s, t, x)$ is a quartic function of t which becomes a square when $t = r.$ Using this known value of $t,$ we apply the aforementioned method of Fermat to find the following value of t which also makes $\psi(s, t, x)$ a square:

$$t = r(r^6 - 9r^4s^2 - 9r^2s^4 - 3s^6)/(5r^6 + 3r^4s^2 + 3r^2s^4 + s^6).
 \tag{3.37}$$

Since $\psi(s, t, x)$ is now a square, Equation (3.14) has two rational solutions for u , one of which is given below:

$$u = 2s^3(5r^6 + 3r^4s^2 + 3r^2s^4 + s^6)/(r^8 + 2r^6s^2 - 12r^4s^4 - 6r^2s^6 - s^8). \quad (3.38)$$

We omit writing the second rational value of u that satisfies Equation (3.14) as it is cumbersome to write. We note that the values of x , t , and u given by (3.36), (3.37), and (3.38), respectively, give a rational solution of Equation (3.14).

In Equation (3.14), we now take t and u as defined by (3.37) and (3.38), respectively, and consider it as a quadratic equation in x . Since (3.36) gives one rational solution of this equation, the second solution is also rational, and is given by

$$x = 4r^2s^4(r^{12} + 6r^{10}s^2 + 87r^8s^4 + 108r^6s^6 + 55r^4s^8 + 14r^2s^{10} + s^{12}) \times (r^8 + 2r^6s^2 - 12r^4s^4 - 6r^2s^6 - s^8)^{-2}. \quad (3.39)$$

We have thus obtained two rational values of x that are roots of Equation (3.14) and both of them are expressible as sums of two nonzero squares. Thus, Equation (3.2) has three rational roots, and hence, as before, on appropriate scaling, we get the three integers a, b , and c defined by (3.34) such that all the three elementary symmetric functions of a, b , and c are squares. \square

We now indicate how more triads $\{a, b, c\}$ may be obtained, in parametric terms, such that a, b , and c are all expressible as sums of two nonzero squares and their three elementary symmetric functions are squares. We first note that if we apply Lemma 2 to the triad of integers given by (3.34), we obtain, after removing common squared factors, another solution of our problem given by polynomials of degree 32 in terms of arbitrary parameters r and s . We do not give this solution explicitly.

Secondly, when we had mentioned the value of u given by (3.38) as a rational solution of Equation (3.14), we had omitted writing the second rational value of u that satisfies Equation (3.14). We can use this second solution, and proceeding exactly as before, we get a second parametric solution, in terms of polynomials of degree 52, for the three integers a, b , and c whose elementary symmetric functions are all squares. As this solution is cumbersome to write, we do not give it explicitly.

Finally, we note that more such solutions can be found by making $\psi(s, t, x)$ a square where x is defined by (3.36). As already noted, $\psi(s, t, x)$ is a quartic function of t which becomes a square when $t = r$ and also when t is given by (3.37). Just as we proved in Section 3.1 that there exist infinitely many values of u that make the quartic function $\phi(s, t, u)$ a square, we can now prove that there exist infinitely many values of t that make the function $\psi(s, t, x)$ a square, and these values of t yield infinitely many parametric solutions of our problem in which the three integers a, b , and c are sums of two nonzero squares.

a	b	c	Remarks
180	45	64	Solution (3.3) with $(s, t) = (1, 2)$
1440	160	81	Solution (3.3) with $(s, t) = (1, 3)$
61200	3825	1024	Solution (3.3) with $(s, t) = (1, 4)$
2925	1300	5184	Solution (3.3) with $(s, t) = (2, 3)$
93600	3744	625	Solution (3.3) with $(s, t) = (1, 5)$
319725	51156	40000	Solution (3.3) with $(s, t) = (2, 5)$
54400	19584	50625	Solution (3.3) with $(s, t) = (3, 5)$
83025	53136	640000	Solution (3.3) with $(s, t) = (4, 5)$
80	320	225	Solution (3.4) with $(s, t) = (1, 2)$
90	810	1600	Solution (3.4) with $(s, t) = (1, 3)$
1088	17408	65025	Solution (3.4) with $(s, t) = (1, 4)$
7488	16848	4225	Solution (3.4) with $(s, t) = (2, 3)$
650	16250	97344	Solution (3.4) with $(s, t) = (1, 5)$
46400	290000	370881	Solution (3.4) with $(s, t) = (2, 5)$
98	4802	57600	Solution (3.4) with $(s, t) = (1, 7)$
68850	191250	73984	Solution (3.4) with $(s, t) = (3, 5)$
28880	81225	537920	Solution (3.5) with $(s, t) = (1, 2)$
302580	107584	16245	Solution (3.6) with $(s, t) = (1, 2)$
11025	19600	82944	Solution (3.30) with $(m, n) = (1, 2)$
9216	5184	1225	Solution (3.31) with $(m, n) = (1, 2)$
136	72	153	Solution (3.34) with $(r, s) = (1, 1)$

Table 1: Triads of integers a, b , and c whose elementary symmetric functions are squares.

3.4. Numerical Examples

By assigning small numerical values to the parameters in the parametric solutions obtained in Sections 3.1, 3.2, and 3.3, we obtained 21 numerical examples of triads of integers smaller than 10^6 such that their elementary symmetric functions are all squares. In fact, seven of these examples consist of integers smaller than 10000. Table 1 lists these numerical examples — in the last column we have indicated the specific parametric solution and the values of the parameters that have yielded the numerical example. In a few cases, the triad given in Table 1 has been obtained after factoring out any squared factor common to the three integers obtained from the parametric solution.

We also performed computer trials to find small numerical solutions of the problem. It is clear from Lemma 3 that all the three integers a, b , and c are necessarily positive. Trials over the range $a + b + c \leq 500$ yielded only two numerical solutions, namely $(45, 64, 180)$ and $(72, 136, 153)$. These are the two smallest solutions of the problem, and they are included as the first and last entries of Table 1.

4. Concluding Remarks

In this paper we reconsidered Euler's diophantine problem of finding three integers whose elementary symmetric functions are all squares. Euler had found this to be a very difficult problem and had obtained solutions in very large integers. We first showed that each of the three integers must be a sum of two rational squares, and using this fact, we were able to find several new parametric solutions that are much simpler than the solutions obtained by Euler's methods. These parametric solutions yielded a number of numerical examples of our problem in small integers.

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