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ON HYPERSEQUENCES OF AN ARBITRARY SEQUENCE AND THEIR WEIGHTED SUMS II

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Abstract

In a previous paper, we proved that the weighted sums $t_\ell^{(r)}(n) := \sum_{k=0}^{n} k^\ell a_k^{(r)}$, where $\ell, r, n \in \mathbb{N}_0$, are given by $t_\ell^{(r)}(n) = \sum_{m=0}^{\ell} c_{\ell,m}(n) a_n^{(r+m+1)}$. Here $\left(a_n^{(r)}\right)_{n\in\mathbb{N}_0}$ is the hypersequence of the rth generation of the sequence $(a_n)_{n\in\mathbb{N}_0}$ and the infinite lower diagonal matrix $C(n) := \left(c_{\ell,m}(n)\right)_{\ell,m\in\mathbb{N}_0}$ does not depend on r. In this paper, we derive an inverse relation for $r = 0$, i.e., we express the hypersequences $a_n^{(j+1)}$, $j \in \mathbb{N}_0$, in terms of the weighted sums $t_j^{(0)}(n)$, $j \in \mathbb{N}_0$. In proving this theorem we obtain a matrix $B(n)$, whose entries are a signed transformation of the entries of $E(n)$, a matrix of specializations of elementary symmetric polynomials. This theorem enables us to determine the inverses of both $C(n)$ and $B(n)$. Furthermore, we present some properties of the entries of $B(n)$ and $E(n)$. Finally, we find that the recurrences satisfied by the entries of $B(n)$, $E(n)$, and $C(n)$ are special cases of a general linear two-parameter recurrence problem first posed by Graham, Knuth, and Patashnik.

1. Introduction

In [9], we studied the hypersequence of the rth generation of an arbitrary sequence $(a_n)_{n\in\mathbb{N}_0}$ and its weighted sums of the form $t_\ell^{(r)}(n) := \sum_{k=0}^{n} k^\ell a_k^{(r)}$, $r,\, n,\, \ell \in \mathbb{N}_0$, where $\left(a_n^{(r)}\right)_{n\in\mathbb{N}_0}$ is the *hypersequence of the rth generation* of the sequence (of real or complex numbers) $(a_n)_{n\in\mathbb{N}_0}$ defined recursively for all $n,\, r \in \mathbb{N}_0$ as

$$a_n^{(r)} := \sum_{k=0}^{n} a_k^{(r-1)}, \quad \text{and} \quad a_n^{(0)} := a_n.$$

For $r = 1$, we have $a_n^{(1)} = \sum_{k=0}^{n} a_k^{(0)} = \sum_{k=0}^{n} a_k$, which is the sequence of partial sums of the sequence $(a_n)_{n\in\mathbb{N}_0}$; for $r = 2$, $a_n^{(2)} = \sum_{k=0}^{n} a_k^{(1)} = \sum_{k=0}^{n} \left(\sum_{j=0}^{k} a_j\right)$ is

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the sequence of partial sums of $\left(a_n^{(1)}\right)_{n\in\mathbb{N}_0}$; and so on. The hypersequence of the rth generation of the sequence $(a_n)_{n\in\mathbb{N}_0}$ is given by (see [2, Proposition 2] for the special case $a_0^{(i)} = a_0$ for all $i \in \{1, 2, \dots, r\}$)

$$a_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} a_k = \sum_{k=0}^{n} \binom{r+k-1}{k} a_{n-k}. \tag{1.1}$$

In [9, Theorem 9], we proved that the weighted sums are given by

$$t_\ell^{(r)}(n) = \sum_{m=0}^{\ell} c_{\ell,m}(n) a_n^{(r+m+1)}, \tag{1.2}$$

where $r \in \mathbb{N}_0$ and $c_{\ell,m}(n) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} (k+n+1)^\ell$, $\ell, m, n \geq 0$. Equation (1.2) can be written in matrix form as follows:

$$\begin{pmatrix} t_0^{(r)}(n) \\ t_1^{(r)}(n) \\ t_2^{(r)}(n) \\ t_3^{(r)}(n) \\ \vdots \end{pmatrix} = C(n) \cdot \begin{pmatrix} a_n^{(r+1)} \\ a_n^{(r+2)} \\ a_n^{(r+3)} \\ a_n^{(r+4)} \\ \vdots \end{pmatrix}, \tag{1.3}$$

where $C(n) := \left(c_{\ell,m}(n)\right)_{\ell,m\in\mathbb{N}_0}$. The first few entries of the infinite lower triangular matrix $C(n)$ are

$$C(n) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ n+1 & -1 & 0 & 0 & \cdots \\ (n+1)^2 & -(2n+3) & 2 & 0 & \cdots \\ (n+1)^3 & -(3n^2+9n+7) & 6n+12 & -6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.4}$$

Equation (1.3) gives a linear relationship between the weighted sums $t_\ell^{(r)}(n)$, $\ell \in \mathbb{N}_0$, and the hypersequences $a_n^{(r+j+1)}$, $r, j \in \mathbb{N}_0$, by means of the lower triangular matrix $C(n)$. Since all entries of the main diagonal, $(-1)^j j!$, $j \in \mathbb{N}_0$, are non-zero, the matrix $C(n)$ is invertible. Therefore, from (1.3), it follows that

$$\begin{pmatrix} a_n^{(r+1)} \\ a_n^{(r+2)} \\ a_n^{(r+3)} \\ a_n^{(r+4)} \\ \vdots \end{pmatrix} = C^{-1}(n) \cdot \begin{pmatrix} t_0^{(r)}(n) \\ t_1^{(r)}(n) \\ t_2^{(r)}(n) \\ t_3^{(r)}(n) \\ \vdots \end{pmatrix}.$$

In Section 2, we will determine the inverse of the matrix $C(n)$. This will be achieved by expressing the hypersequences $a_n^{(j+1)}$ in terms of the weighted sums $t_j(n) := t_j^{(0)}(n)$, $j, n \in \mathbb{N}_0$, for the case $r = 0$, since $C(n)$ is independent of r. As it turns out, this relationship is represented by the matrix $B(n)$, which is defined as a signed transformation of the entries of $E(n)$. The entries of $E(n)$ are given by a specialization of the elementary symmetric polynomials. Section 3 presents some properties of the entries of $B(n)$ and $E(n)$. Then, in Section 4, we notice that the recurrences satisfied by the entries of $E(n)$, $B(n)$, and $C(n)$ are special cases of a general two-parameter recurrence known in the literature as "Problem 6.94", which was posed by Graham, Knuth, and Patashnik ([4, Problem 6.94, pp. 319 and 564]). Finally, we mention a class of numbers studied in [7, 8] that generalizes the entries of both $E(n)$ and $B(n)$.

2. A Different Approach for the Weighted Sums

The matrix $C(n)$ as defined in (1.4) does not depend on r. Therefore, to determine its inverse matrix it is sufficient to consider the case when $r = 0$. The next theorem shows that, contrary to (1.2), $a_n^{(j+1)}$ can be expressed in terms of $t_j(n)$, $j \in \mathbb{N}_0$. We will use Vieta's formulas to prove this. These formulas relate the coefficients of a polynomial to sums and products of its roots. Note that we now use the index j instead of ℓ.

Theorem 1. *Let $j, n \in \mathbb{N}_0$. Then*

$$j! \cdot a_n^{(j+1)} = (-1)^j \sum_{k=0}^{j} (-1)^k e_{j,k}(n) \cdot t_{j-k}(n) = \sum_{k=0}^{j} (-1)^k e_{j,j-k}(n) \cdot t_k(n), \tag{2.1}$$

where

$$e_{j,0}(n) := 1, \quad e_{j,1}(n) := \sum_{i=1}^{j} x_i(n), \quad e_{j,2}(n) := \sum_{1 \le i_1 < i_2 \le j} x_{i_1}(n) \cdot x_{i_2}(n), \ldots,$$

$$e_{j,j-1}(n) := \sum_{1 \le i_1 < i_2 < \cdots < i_{k-1} \le j} x_{i_1}(n) \cdot x_{i_2}(n) \cdots x_{i_{k-1}}(n), \quad e_{j,j}(n) := \prod_{i=1}^{j} x_i(n)$$

and, in general,

$$e_{j,k}(n) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le j} x_{i_1}(n) \cdot x_{i_2}(n) \cdots x_{i_k}(n) \tag{2.2}$$

with $x_1(n) := n + 1$, $x_2(n) := n + 2, \ldots, x_j(n) := n + j$, $j \in \mathbb{N}$, are a certain specialization of the elementary symmetric polynomials.

Proof. By the first identity of Equation (1.1) and writing $j+1$ instead of r, we have

$$a_n^{(j+1)} = \sum_{k=0}^{n} \binom{n+j-k}{j} a_k,$$

and, by definition,

$$\binom{n+j-k}{j} = \frac{1}{j!}(n+j-k)(n+j-1-k)\cdots(n+1-k) = \frac{(-1)^j}{j!}\prod_{i=1}^{j}\big(k-(n+i)\big).$$

Using Vieta's formulas, we can derive the following equation:

$$\prod_{i=1}^{j}\big(k-(n+i)\big) = e_{j,0}(n)k^j - e_{j,1}(n)k^{j-1} + \cdots + (-1)^{j-1}e_{j,j-1}(n)k + (-1)^j e_{j,j}(n). \tag{2.3}$$

Therefore,

$$\binom{n+j-k}{j} = \frac{(-1)^j}{j!}\big(e_{j,0}(n)k^j - e_{j,1}(n)k^{j-1} + \cdots + (-1)^{j-1}e_{j,j-1}(n)k + (-1)^j e_{j,j}(n)\big).$$

Since $e_{j,0}(n) = 1$, we obtain by summation

$$\begin{aligned}
a_n^{(j+1)} &= \sum_{k=0}^{n} \binom{n+j-k}{j} a_k \\
&= \frac{(-1)^j}{j!}\Big(\sum_{k=0}^{n} k^j a_k - e_{j,1}(n)\sum_{k=0}^{n} k^{j-1}a_k + e_{j,2}(n)\sum_{k=0}^{n} k^{j-2}a_k + \cdots \\
&\qquad + (-1)^{j-1}e_{j,j-1}(n)\sum_{k=0}^{n} k a_k + (-1)^j e_{j,j}(n)\sum_{k=0}^{n} a_k\Big) \\
&= \frac{(-1)^j}{j!}\Big(t_j(n) - e_{j,1}(n)t_{j-1}(n) + e_{j,2}(n)t_{j-2}(n) + \cdots \\
&\qquad + (-1)^{j-1}e_{j,j-1}(n)t_1(n) + (-1)^j e_{j,j}(n)t_0(n)\Big).
\end{aligned}$$

Hence,

$$j! \cdot a_n^{(j+1)} = (-1)^j \sum_{k=0}^{j} (-1)^k e_{j,k}(n) \cdot t_{j-k}(n),$$

which proves the first equation of (2.1). The second equation in (2.1) follows immediately by noting that $k \in \{0, 1, \ldots, j\}$ if and only if $j - k \in \{0, 1, \ldots, j\}$. This completes the proof. $\square$

Notice that replacing k by z, $z \in \mathbb{R}$, in (2.3) leads to

$$\prod_{i=1}^{j} \big(z - (n+i)\big) = \sum_{k=0}^{j} (-1)^k e_{j,k}(n) z^{j-k}. \tag{2.4}$$

We point out that $t_j(n)$ in Theorem 2.1 is given recursively with respect to j. For example, starting with $t_0(n) = \sum_{k=0}^{n} k^0 a_k = \sum_{k=0}^{n} a_k = a_n^{(1)}$, we obtain $t_1(n)$, $t_2(n)$ as follows: for $j = 1$, we have

$$1! \cdot a_n^{(2)} = e_{1,1}(n)t_0(n) - e_{1,0}(n)t_1(n) = (n+1)a_n^{(1)} - t_1(n),$$

that is, $t_1(n) = (n+1)a_n^{(1)} - a_n^{(2)}$. For $j = 2$, we have

$$2! \cdot a_n^{(3)} = e_{2,2}(n)t_0(n) - e_{2,1}(n)t_1(n) + e_{2,0}(n)t_2(n).$$

Solving for $t_2(n)$ and noting that $e_{2,0}(n) = 1,\ e_{2,1}(n) = (n+1) + (n+2) = 2n+3$, and $e_{2,2}(n) = (n+1)(n+2) = n^2 + 3n + 2$, we obtain

$$\begin{aligned} t_2(n) &= e_{2,1}(n)t_1(n) - e_{2,2}(n)t_0(n) + 2a_n^{(3)} \\ &= (2n+3)\big((n+1)a_n^{(1)} - a_n^{(2)}\big) - (n^2+3n+2)a_n^{(1)} + 2a_n^{(3)}, \end{aligned}$$

which simplifies to

$$t_2(n) = (n+1)^2 a_n^{(1)} - (2n+3)a_n^{(2)} + 2a_n^{(3)}.$$

These are the first three rows of (1.3) for $r = 0$. The same applies to $t_j(n)$, $j \geq 3$.

The specialization of the elementary symmetric polynomials define a matrix $E(n) := \big(e_{j,k}(n)\big)_{j,k\in\mathbb{N}_0}$. Because of (2.2), $E(n)$ is an infinite lower triangular matrix with the first few entries

$$E(n) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & n+1 & 0 & 0 & \cdots \\ 1 & 2n+3 & n^2+3n+2 & 0 & \cdots \\ 1 & 3n+6 & 3n^2+12n+11 & n^3+6n^2+11n+6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Moreover, defining

$$b_{j,k}(n) := (-1)^k e_{j,j-k}(n), \quad j,k \in \mathbb{N}_0, \tag{2.5}$$

Equation (2.1) can be written as follows:

$$j! \cdot a_n^{(j+1)} = \sum_{k=0}^{j} b_{j,k}(n) \cdot t_k(n), \tag{2.6}$$

or, in matrix form,

$$\begin{pmatrix} 0!\cdot a_n^{(1)} \\ 1!\cdot a_n^{(2)} \\ 2!\cdot a_n^{(3)} \\ 3!\cdot a_n^{(4)} \\ \vdots \end{pmatrix} = B(n)\cdot \begin{pmatrix} t_0(n) \\ t_1(n) \\ t_2(n) \\ t_3(n) \\ \vdots \end{pmatrix}, \tag{2.7}$$

where $B(n) := \big(b_{j,k}(n)\big)_{j,k\in\mathbb{N}_0}$ is an infinite lower triangular matrix, since for $k > j$, that is, $j - k < 0$, we have, by definition, $e_{j,j-k}(n) = 0$. The first few entries of $B(n)$ are

$$B(n) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ n+1 & -1 & 0 & 0 & \cdots \\ (n+1)(n+2) & -(2n+3) & 1 & 0 & \cdots \\ (n+1)(n+2)(n+3) & -(3n^2+12n+11) & 3n+6 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Using Theorem 2.1 and (2.7), we can now determine the inverse matrix of $C(n)$. Note that D is the infinite diagonal matrix $D := \operatorname{diag}(0!, 1!, 2!, \ldots)$ with the inverse matrix $D^{-1} = \operatorname{diag}\big(\frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \ldots\big)$.

Theorem 2. *Let $n \in \mathbb{N}_0$. Then*

$$C^{-1}(n) = D^{-1}\cdot B(n) = \left(\frac{b_{j,k}(n)}{j!}\right)_{j,k\in\mathbb{N}_0}.$$

Proof. Let $r = 0$. Then, inserting (1.3) into (2.7), we get

$$D\cdot \begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \\ a_n^{(3)} \\ a_n^{(4)} \\ \vdots \end{pmatrix} = B(n)\cdot C(n)\cdot \begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \\ a_n^{(3)} \\ a_n^{(4)} \\ \vdots \end{pmatrix}.$$

From this equation, we can conclude that

$$D = B(n)\cdot C(n), \tag{2.8}$$

and this means that $D^{-1}\cdot B(n) = \left(\frac{b_{j,k}(n)}{j!}\right)_{j,k\in\mathbb{N}_0}$ is the inverse matrix of $C(n)$. □

Corollary 1. *Let $n \in \mathbb{N}_0$. Then*

$$B^{-1}(n) = C(n)\cdot D^{-1} = \left(\frac{c_{\ell,m}(n)}{m!}\right)_{\ell,m\in\mathbb{N}_0}. \tag{2.9}$$

We note that Equation (2.8) can be written in the following way.

Corollary 2. *For all $j, m, n \in \mathbb{N}_0$, we have*

$$\sum_{i=0}^{j} b_{j,i}(n) \cdot c_{i,m}(n) = j!\, \delta_{j,m},$$

where $\delta_{j,m}$ denotes the Kronecker delta*, meaning 1 if $j = m$ and 0 otherwise.*

3. Some Properties of $e_{j,k}(n)$ and $b_{j,k}(n)$

We now list some properties of the entries $e_{j,k}(n)$ of the matrix $E(n)$ and $b_{j,k}(n)$ of the matrix $B(n)$. The recurrence (3.1) expresses a simple property of the elementary symmetric polynomials, while Equation (3.3) gives the row sums, and (3.4) gives the alternating row sums of the matrix $E(n)$. In (3.2) $\left[{j \atop j-k+\ell}\right]$ is an unsigned Stirling number of the first kind, where $\left[{n \atop k}\right]$ counts the number of permutations of n elements with exactly k cycles, where $0 \leq k \leq n$. Furthermore, $n^{\overline{j}} := \prod_{i=1}^{j} \big(n+i-1\big)$ denotes a *rising factorial power of* n.

Proposition 1. *Let $j, k \in \mathbb{N}_0$. Then, for all $n \in \mathbb{N}_0$, we have*

$$e_{0,k}(n) = \delta_{0,k}, \quad e_{j,k}(n) = e_{j-1,k}(n) + (n+j)e_{j-1,k-1}(n), \tag{3.1}$$

with the solution

$$e_{j,k}(n) = \sum_{\ell=0}^{k} \binom{j-k+\ell}{\ell} \left[{j \atop j-k+\ell}\right] (n+1)^{\ell}. \tag{3.2}$$

Furthermore, for all $n \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{j} e_{j,k}(n) = (n+2)^{\overline{j}} \tag{3.3}$$

and

$$\sum_{k=0}^{j} (-1)^k e_{j,k}(n) = (-1)^j n^{\overline{j}}. \tag{3.4}$$

Proof. The proof of the recurrence (3.1) can be found in [3, Proposition 3.1]. We will now show that its solution is given by (3.2). In fact, by [4, Equation (6.13)], we have $\prod_{i=1}^{j} \big(x+1-i\big) = \prod_{i=0}^{j-1} \big(x-i\big) = \sum_{k=0}^{j} (-1)^{j-k} \left[{j \atop k}\right] x^k$. Setting $x = z-(n+1)$ in this equation yields

$$\prod_{i=1}^{j} \big(z-(n+i)\big) = \sum_{k=0}^{j} (-1)^{j-k} \left[{j \atop k}\right] \big(z-(n+1)\big)^k.$$

By (2.4) and the binomial theorem, we have

$$\begin{aligned}\sum_{k=0}^{j}(-1)^k e_{j,k}(n) z^{j-k} &= \sum_{k=0}^{j}(-1)^{j-k}\begin{bmatrix} j \\ k \end{bmatrix}\Big(\sum_{\ell=0}^{k}\binom{k}{\ell} z^{\ell}(-1)^{k-\ell}(n+1)^{k-\ell}\Big) \\ &= \sum_{k=0}^{j}\begin{bmatrix} j \\ k \end{bmatrix}\Big(\sum_{\ell=0}^{k}(-1)^{j-\ell}\binom{k}{\ell}(n+1)^{k-\ell} z^{\ell}\Big).\end{aligned} \tag{3.5}$$

Let $S(z)$ be the right-hand side of this identity. Then

$$\begin{aligned}S(z) = {} & \begin{bmatrix} j \\ 0 \end{bmatrix}(-1)^j\binom{0}{0}(n+1)^0 z^0 \\ & + \begin{bmatrix} j \\ 1 \end{bmatrix}\Big((-1)^j\binom{1}{0}(n+1)^1 z^0 + (-1)^{j-1}\binom{1}{1}(n+1)^0 z^1\Big) \\ & + \begin{bmatrix} j \\ 2 \end{bmatrix}\Big((-1)^j\binom{2}{0}(n+1)^2 z^0 + (-1)^{j-1}\binom{2}{1}(n+1)^1 z^1 \\ & \quad + (-1)^{j-2}\binom{2}{2}(n+1)^0 z^2\Big) + \cdots \\ & + \begin{bmatrix} j \\ j \end{bmatrix}\Big((-1)^j\binom{j}{0}(n+1)^j z^0 + (-1)^{j-1}\binom{j}{1}(n+1)^{j-1} z^1 + \cdots \\ & + (-1)^{j-j}\binom{j}{j}(n+1)^0 z^j\Big).\end{aligned}$$

By collecting like powers of z, we find that

$$\begin{aligned}S(z) = {} & (-1)^j\Big(\sum_{\ell=0}^{j}\binom{\ell}{0}\begin{bmatrix} j \\ \ell \end{bmatrix}(n+1)^{\ell}\Big)z^0 + (-1)^{j-1}\Big(\sum_{\ell=1}^{j}\binom{\ell}{1}\begin{bmatrix} j \\ \ell \end{bmatrix}(n+1)^{\ell-1}\Big)z^1 \\ & + (-1)^{j-2}\Big(\sum_{\ell=2}^{j}\binom{\ell}{2}\begin{bmatrix} j \\ \ell \end{bmatrix}(n+1)^{\ell-2}\Big)z^2 + \cdots \\ & + (-1)^{j-j}\Big(\sum_{\ell=j}^{j}\binom{\ell}{j}\begin{bmatrix} j \\ \ell \end{bmatrix}(n+1)^{\ell-j}\Big)z^j.\end{aligned}$$

Clearly, the coefficient of z^{j-k} in $S(z)$, denoted by $[z^{j-k}]\,S(z)$, is given by (see [4, Equation (5.53)])

$$\begin{aligned}[z^{j-k}]\,S(z) &= (-1)^{j-(j-k)}\sum_{\ell=j-k}^{j}\binom{\ell}{j-k}\begin{bmatrix} j \\ \ell \end{bmatrix}(n+1)^{\ell-(j-k)} \\ &= (-1)^k\sum_{\ell=j-k}^{j}\binom{\ell}{\ell-(j-k)}\begin{bmatrix} j \\ \ell \end{bmatrix}(n+1)^{\ell-(j-k)} \\ &= (-1)^k\sum_{\ell=0}^{k}\binom{j-k+\ell}{\ell}\begin{bmatrix} j \\ j-k+\ell \end{bmatrix}(n+1)^{\ell},\end{aligned}$$

since $\ell \in \{j-k, j-k+1, \dots, j\}$ if and only if $\ell \in \{0, 1, \dots, k\}$. Hence, from (3.5) it follows that

$$\sum_{k=0}^{j}(-1)^k e_{j,k}(n) z^{j-k} = \sum_{k=0}^{j}\Big((-1)^k \sum_{\ell=0}^{k}\binom{j-k+\ell}{\ell}\begin{bmatrix} j \\ j-k+\ell \end{bmatrix}(n+1)^{\ell}\Big) z^{j-k}.$$

Equating the coefficients of z^{j-k} of the above polynomials gives Equation (3.2). Note that, by definition, $e_{j,k}(n) = 0$ for $j < 0$ or $k < 0$. Setting $z = -1$ in (2.4) we obtain

$$\prod_{i=1}^{j}\big(-1-(n+i)\big) = (-1)^j \prod_{i=1}^{j}\big(n+i+1\big) = \sum_{k=0}^{j}(-1)^k e_{j,k}(n)(-1)^{j-k}.$$

Dividing this equation by $(-1)^j$ yields $\prod_{i=1}^{j}\big(n+i+1\big) = \sum_{k=0}^{j} e_{j,k}(n)$, which is formula (3.3), since $\prod_{i=1}^{j}\big(n+i+1\big) = (n+2)^{\overline{j}}$. Finally, setting $z = 1$ in (2.4) yields

$$\prod_{i=1}^{j}\big(1-(n+i)\big) = (-1)^j \prod_{i=1}^{j}\big(n+i-1\big) = \sum_{k=0}^{j}(-1)^k e_{j,k}(n),$$

which is Equation (3.4), since $\prod_{i=1}^{j}\big(n+i-1\big) = n^{\overline{j}}$. This completes the proof of the proposition. □

For $n = 0$, from (3.1) and (3.2) we obtain the following corollary (see [3, Proposition 3.15, Equation (3.17)]).

Corollary 3. *For all $j, k \in \mathbb{N}_0$, we have*

$$e_{0,k}(0) = \delta_{0,k}, \quad e_{j,k}(0) = e_{j-1,k}(0) + j e_{j-1,k-1}(0), \tag{3.6}$$

with the solution

$$e_{j,k}(0) = \sum_{\ell=0}^{k}\binom{j-k+\ell}{\ell}\begin{bmatrix} j \\ j-k+\ell \end{bmatrix} = \begin{bmatrix} j+1 \\ j+1-k \end{bmatrix}. \tag{3.7}$$

We will now demonstrate the analogous properties of the entries $b_{j,k}(n)$ of the matrix $B(n)$. Note that (3.10) gives the row sums, and (3.11) the alternating row sums, of the matrix $B(n)$.

Proposition 2. *Let $j, k \in \mathbb{N}_0$. Then for all $n \in \mathbb{N}_0$, we have*

$$b_{0,k}(n) = \delta_{0,k}, \quad b_{j,k}(n) = (n+j) b_{j-1,k}(n) - b_{j-1,k-1}(n), \tag{3.8}$$

with the solution

$$b_{j,k}(n) = (-1)^k \sum_{\ell=k}^{j}\binom{\ell}{k}\begin{bmatrix} j \\ \ell \end{bmatrix}(n+1)^{\ell-k}. \tag{3.9}$$

Furthermore, for all $n \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{j} b_{j,k}(n) = n^{\overline{j}} \tag{3.10}$$

and

$$\sum_{k=0}^{j} (-1)^k b_{j,k}(n) = (n+2)^{\overline{j}}. \tag{3.11}$$

Proof. By (2.5), for $j = 0$, we have $b_{0,k}(n) = (-1)^k e_{0,-k}(n) = \delta_{0,k}$. Furthermore, by (3.1), we have $b_{0,k}(n) = (-1)^k e_{0,k}(n) = \delta_{0,k}$, and

$$\begin{aligned} b_{j,k}(n) &= (-1)^k e_{j,j-k}(n) = (-1)^k \big(e_{j-1,j-k}(n) + (n+j) e_{j-1,j-k-1}(n)\big) \\ &= -(-1)^{k-1} e_{j-1,(j-1)-(k-1)}(n) + (n+j)(-1)^k e_{j-1,j-1-k}(n) \\ &= -b_{j-1,k-1}(n) + (n+j) b_{j-1,k}(n). \end{aligned}$$

This proves (3.8). The solution (3.9) now follows from (3.2) by taking $j-k$ instead of k and multiplying the sum by $(-1)^k$. We obtain

$$b_{j,k}(n) = (-1)^k \sum_{\ell=0}^{j-k} \binom{k+\ell}{\ell} \begin{bmatrix} j \\ k+\ell \end{bmatrix} (n+1)^{\ell},$$

which is Equation (3.9), noting that $\binom{\ell}{\ell-k} = \binom{\ell}{k}$. This is because the sum from the first term to the last is the same as the sum from the last term to the first. Note that, by definition, $b_{j,k}(n) = 0$ for $j < 0$ or $k < 0$. Furthemore, using (2.4) with $j-k$ instead of k, we obtain

$$(-1)^j \prod_{i=1}^{j} \big(n+i-z\big) = \prod_{i=1}^{j} \big(z-(n+i)\big) = \sum_{k=0}^{j} (-1)^{j-k} e_{j,j-k}(n) z^{j-(j-k)}.$$

After dividing by $(-1)^j$, it follows that

$$\prod_{i=1}^{j} \big(n+i-z\big) = \sum_{k=0}^{j} (-1)^k e_{j,j-k}(n) z^k = \sum_{k=0}^{j} b_{j,k}(n) z^k. \tag{3.12}$$

Setting $z = 1$ in (3.12) immediately yields the formula (3.10) by noting that $\prod_{i=1}^{j} \big(n+i-1\big) = n^{\overline{j}}$. Finally, the formula (3.11) now follows by setting $z = -1$ in (3.12) and noting that $\prod_{i=1}^{j} \big(n+i+1\big) = (n+2)^{\overline{j}}$. This completes the proof of the proposition. □

For $n = 0$, from (3.8) and (3.9) we obtain the following corollary.

Corollary 4. *For all $k, j \in \mathbb{N}_0$, we have*

$$b_{0,k}(0) = \delta_{0,k}, \quad b_{j,k}(0) = j b_{j-1,k}(0) - b_{j-1,k-1}(0), \tag{3.13}$$

with the solution

$$b_{j,k}(0) = (-1)^k \sum_{\ell=k}^{j} \binom{\ell}{k} \begin{bmatrix} j \\ \ell \end{bmatrix} = (-1)^k \begin{bmatrix} j+1 \\ k+1 \end{bmatrix}. \tag{3.14}$$

Note that the second identity in (3.14) follows from (3.7) since, by definition, $b_{j,k}(0) = (-1)^k e_{j,j-k}(0) = (-1)^k \left[\begin{smallmatrix} j+1 \\ k+1 \end{smallmatrix}\right]$.

Remark 1. In [9, Equation (3.11), Corollary 4, and Remark 6], we proved the recurrence

$$c_{0,m}(n) = \delta_{0,m}, \quad c_{\ell,m}(n) = \big(m+n+1\big) c_{\ell-1,m}(n) - m c_{\ell-1,m-1}(n), \tag{3.15}$$

for all $\ell, m, n \in \mathbb{N}_0$, with the solution

$$c_{\ell,m}(n) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (k+n+1)^\ell = (-1)^m m! \sum_{j=m}^{\ell} \binom{\ell}{j} \begin{Bmatrix} j \\ m \end{Bmatrix} (n+1)^{\ell-j} \tag{3.16}$$

(see [9, Identity (3.10)] for $x = 1$ and $y = n+1$). Note that, by definition, $c_{\ell,m}(n) = 0$ for $\ell < 0$ or $m < 0$.

Notice that the formulas (3.2), (3.9), and (3.16) are also valid for all $n \in \mathbb{Z}$. In particular, for $n = -1$, we have

$$e_{j,k}(-1) = \begin{bmatrix} j \\ j-k \end{bmatrix}, \tag{3.17}$$

$$b_{j,k}(-1) = (-1)^k \begin{bmatrix} j \\ k \end{bmatrix}, \tag{3.18}$$

and (see [4, Identity (6.19)])

$$c_{\ell,m}(-1) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} k^\ell = (-1)^m m! \begin{Bmatrix} \ell \\ m \end{Bmatrix}. \tag{3.19}$$

These special cases are very important because they can be used to determine all cases for any $n \in \mathbb{Z}$. Indeed, the following proposition holds true.

Proposition 3. *Let $j, k, \ell, m \in \mathbb{N}_0$. Then, for all $n \in \mathbb{Z}$, we have*

$$e_{j,k}(n) = \sum_{\ell=0}^{k} \binom{j-k+\ell}{\ell} (n+1)^\ell e_{j,k-\ell}(-1), \tag{3.20}$$

$$b_{j,k}(n) = \sum_{\ell=k}^{j} (-1)^{\ell-k} \binom{\ell}{k} (n+1)^{\ell-k} b_{j,\ell}(-1), \tag{3.21}$$

$$c_{\ell,m}(n) = \sum_{j=m}^{\ell} \binom{\ell}{j} (n+1)^{\ell-j} c_{j,m}(-1). \tag{3.22}$$

Proof. Equation (3.20) follows from (3.2) and (3.17) because $\left[{j \atop j-k+\ell}\right] = e_{j,k-\ell}(-1)$. Equation (3.21) follows from (3.9) and (3.18) because $\left[{j \atop \ell}\right] = (-1)^{\ell} b_{j,\ell}(-1)$. Finally, Equation (3.22) follows from (3.16) and (3.19) because $(-1)^m m! \left\{{j \atop m}\right\} = c_{j,m}(-1)$. □

4. Connection with Problem 6.94 of Graham, Knuth, and Patashnik

The three recurrences (3.1), (3.8), and (3.15) are special cases of the general two-parameter recurrence for the double sequence $\left|{j \atop k}\right|$ defined by

$$\left|{0 \atop k}\right| = \delta_{0,k}, \quad \left|{j \atop k}\right| = (\alpha j + \beta k + \gamma) \left|{j-1 \atop k}\right| + (\alpha' j + \beta' k + \gamma') \left|{j-1 \atop k-1}\right|, \quad j, k \in \mathbb{Z}, \tag{4.1}$$

assuming that $\left|{j \atop k}\right| = 0$ when $j < 0$ or $k < 0$. In fact,

- $e_{j,k}(n) : (\alpha, \beta, \gamma; \alpha', \beta', \gamma') = (0, 0, 1; 1, 0, n)$
- $b_{j,k}(n) : (\alpha, \beta, \gamma; \alpha', \beta', \gamma') = (1, 0, n; 0, 0, -1)$
- $c_{\ell,m}(n) : (\alpha, \beta, \gamma; \alpha', \beta', \gamma') = (0, 1, n+1; 0, -1, 0)$.

The recurrence (4.1) is known as "Problem 6.94", since it appears as a "research problem" in [4, Problem 6.94, pp. 319 and 564]. In this problem, the authors propose to develop a general theory of the solutions of (4.1) and ask what special values $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ yield "fundamental solutions" in terms of which the general solution can be expressed. Note that (4.1) includes the recurrences for the binomial coefficients, the Stirling numbers of both kinds, the Eulerian numbers, the (signed and unsigned) Lah numbers, and many other combinatorial sequences.

The topic has been studied by a number of researchers either by a combinatorial approach or by solving the partial differential equation, that is, given by the bivariate exponential generating function of $\left|{j \atop k}\right|$. In particular, we mention the work of Théorêt [11], Wilf [12], Maier [5], Mansour and Shattuck [6], Spivey [10], and Barbero, Salas, and Villaseñor [1]. In [1], the authors classified the recurrence (4.1) into four different types, namely: Type I: $\beta\beta' \neq 0$, Type II: $\beta \neq 0$, $\beta' = 0$, Type

III: $\beta = 0$, $\beta' \neq 0$, and Type IV: $\beta = 0$, $\beta' = 0$. In our cases, $e_{j,k}(n)$ and $b_{j,k}(n)$ are of Type IV, while $c_{\ell,m}(n)$ is of Type I.

In this context, we mention a class of numbers $R_n^r(a,b)$, where $a, b \in \mathbb{C}$, $b \neq 0$, $r, n \in \mathbb{Z}$, related to the (signed) Stirling numbers of the first kind, which were studied by D. S. Mitrinović [7]. Assuming that $R_n^r(a,b) = 0$ when $n < 0$ or $r < 0$, the numbers $R_n^r(a,b)$ satisfy the following recurrence relation (see [7, Equation (1.2)]):

$$R_{n+1}^r(a,b) = R_n^{r-1}(a,b) - (a+bn)R_n^r(a,b)$$

or, with $j-1$ instead of n, and k instead of r,

$$R_j^k(a,b) = R_{j-1}^{k-1}(a,b) - (a-b+bj)R_{j-1}^k(a,b). \tag{4.2}$$

This recurrence relation is of Type IV, since $(\alpha, \beta, \gamma; \alpha', \beta', \gamma') = (-b, 0, b-a; 0, 0, 1)$. Mitrinović [7, Equation (1.4)] has given the following solution to this recurrence:

$$R_j^k(a,b) = \sum_{\ell=0}^{j-k} (-1)^\ell \binom{k+\ell}{\ell} a^\ell b^{j-k-\ell} s(j, k+\ell), \tag{4.3}$$

where $s(j,k+\ell) = (-1)^{j-(k+\ell)} \left[{j \atop k+\ell}\right]$ is the signed Stirling number of the first kind. Using the unsigned Stirling numbers of the first kind, Equation (4.3) can also be written as

$$R_j^k(a,b) = (-1)^{j-k} \sum_{\ell=0}^{j-k} \binom{k+\ell}{\ell} \left[{j \atop k+\ell}\right] a^\ell b^{j-k-\ell}. \tag{4.4}$$

For $a = 0$, from (4.2) and (4.4) we obtain the following corollary.

Corollary 5. *Let $b \in \mathbb{C} \setminus \{0\}$. Then for all $k, j \in \mathbb{Z}$, we have the recurrence*

$$R_0^k(0,b) = \delta_{0,k}, \quad R_j^k(0,b) = R_{j-1}^{k-1}(0,b) + (b-bj)R_{j-1}^k(a,b),$$

where $R_j^k(0,b) = 0$ when $j < 0$ or $k < 0$. Its solution is given by

$$R_j^k(0,b) = (-b)^{j-k} \left[{j \atop k}\right]. \tag{4.5}$$

This special case will enable us to determine all other cases for any $a \in \mathbb{C}$.

Proposition 4. *Let $a, b \in \mathbb{C}$, $b \neq 0$. Then, for all $j, k \in \mathbb{N}_0$, we have*

$$R_j^k(a,b) = \sum_{\ell=0}^{j-k} (-1)^\ell \binom{k+\ell}{\ell} a^\ell R_j^{k+\ell}(0,b) \tag{4.6}$$

Proof. Equation (4.6) follows immediately from (4.4) and (4.5) for $k+\ell$ instead of k. ☐

Note that (4.6) is a generalization of the special case $a = 1$, $b = 2$ given by Mitrinović [7, p. 2355].

Remark 2. The numbers $R_j^k(a,b)$, $a, b \in \mathbb{C}$, $b \neq 0$, $j, k \in \mathbb{Z}$, are closely related to $e_{j,k}(n)$ and $b_{j,k}(n)$. In fact, in a second paper by D. S. Mitrinović and R. S. Mitrinović [8, pp. 2–4], the authors studied the special case $a = p$, $p \in \mathbb{N}$, $b = 1$, and $j-k$ instead of k, namely $R_j^{j-k}(p,1)$. They calculated extensive tables for the cases $p = 2, 3, 4, 5$. For $p = n+1$, it follows from (4.4) and (3.2) that

$$(-1)^k R_j^{j-k}(n+1,1) = (-1)^k(-1)^k \sum_{\ell=0}^{k} \binom{j-k+\ell}{\ell} \left[{j \atop j-k+\ell} \right] (n+1)^\ell = e_{j,k}(n).$$

Consequently, by (2.5), we have

$$b_{j,k}(n) = (-1)^k e_{j,j-k}(n) = (-1)^k(-1)^{j-k} R_j^k(n+1,1) = (-1)^j R_j^k(n+1,1).$$

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