HYPERBOLIC SUMMATION INVOLVING CERTAIN ARITHMETIC FUNCTIONS AND THE INTEGER PART FUNCTION

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Abstract

Let f be an arithmetic function satisfying certain growth conditions. This paper establishes an asymptotic formula for the hyperbolic sum

$$\sum_{n_1 n_2 \cdots n_r \le x} f\left(\left\lfloor \frac{x}{n_1 n_2 \cdots n_r} \right\rfloor\right),$$

where $r \ge 2$ is an integer, $\lfloor \cdot \rfloor$ denotes the integer part function, and $\tau_r(n)$ counts the number of ways to write n as a product of r positive integers. Under the assumptions $f(n) \ll n^{\alpha}$ for some $\alpha \in [0,1)$ and $\sum_{n \le x} f(n) \ll x(\log\log x)^{\beta}$ with $\beta \ge 0$, we obtain a main term expressed as x times a polynomial in $\log x$ of degree $r-1$ with explicitly described coefficients, together with an error term $O\big(x(\log x)^{r-2}(\log\log x)^{\beta}\big)$. Several applications to additive arithmetic functions are provided.

1. Introduction

Let $\lfloor t \rfloor$ denote the integer part of a real number t, and let f be an arithmetic function satisfying certain conditions. Define the sum

$$S_f(x) := \sum_{n \le x} f\left(\left\lfloor \frac{x}{n} \right\rfloor\right). \tag{1}$$

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This sum has attracted considerable attention in recent literature. For instance, Bordellès and others [1] investigated various properties of (1). Independently, Wu [12] and Zhai [13] established that for an arithmetic function f satisfying $f(n) \ll n^{\alpha}(\log n)^{\theta}$ with $\alpha \in [0,1)$ and $\theta \geq 0$, one has

$$S_f(x) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O\big(x^{(1+\alpha)/2}(\log x)^{\theta}\big).$$

For the divisor function $f = \tau$, Ma and Sun [9] proved the refined estimate

$$S_\tau(x) = \sum_{n \leq x} \tau\Big(\Big\lfloor \frac{x}{n} \Big\rfloor\Big) = x \sum_{n=1}^{\infty} \frac{\tau(n)}{n(n+1)} + O\big(x^{11/23+\varepsilon}\big).$$

For additive functions, such as $\omega(n)$ denoting the number of distinct prime divisors of n, different techniques are required. Bouderbala and Karras [3] showed that

$$\sum_{n \leq x} \omega\Big(\Big\lfloor \frac{x}{n} \Big\rfloor\Big) = Cx + O\big(x^{1/2} \log x\big),$$

where $C = \sum_{n=1}^{\infty} \omega(n)/\big(n(n+1)\big) \approx 0.5918$. Bordellès [2] subsequently improved the error term to $O\big(x^{455/914} \log x\big)$ with $455/914 \approx 0.4978$.

In the general hyperbolic setting with $r \geq 2$ variables, we consider sums of the form

$$S_{f,r}(x) := \sum_{n_1 n_2 \cdots n_r \leq x} f\Big(\Big\lfloor \frac{x}{n_1 n_2 \cdots n_r} \Big\rfloor\Big).$$

This can be rewritten as

$$S_{f,r}(x) = \sum_{n \leq x} f\Big(\Big\lfloor \frac{x}{n} \Big\rfloor\Big) \tau_r(n),$$

where $\tau_r(n)$ counts the number of ways to express n as a product of r positive integers, with $\tau_2(n) = \tau(n)$ being the usual divisor function. For $r = 2$, Karras, Li, and Stucky [8] established the asymptotic formula

$$\sum_{n_1 n_2 \leq x} f\Big(\Big\lfloor \frac{x}{n_1 n_2} \Big\rfloor\Big) = C_1(f) x \log x + C_2(f) x + O\big(x^{(4+3\alpha)/7+\varepsilon}\big),$$

where $C_1(f)$ and $C_2(f)$ are explicit constants depending on f.

We now present our main results concerning the general case $r \geq 2$.

2. Main Results

Theorem 1. *Let $r \geq 2$ be an integer, and let f be an arithmetic function. Assume there exists $\alpha \in [0,1)$ such that $f(n) \ll n^\alpha$ and*

$$\sum_{n \leq x} f(n) \ll x(\log\log x)^\beta$$

for some $\beta \geq 0$. Then

$$S_{f,r}(x) = x \sum_{j=0}^{r-1} \sum_{i=0}^{j} \binom{j}{i} (-1)^i a_j C_i (\log x)^{j-i} + O\big(x(\log x)^{r-2}(\log\log x)^\beta\big),$$

where a_j $(0 \leq j \leq r-1)$ are the coefficients of the polynomial $P_r(t) = \sum_{j=0}^{r-1} a_j t^j$ appearing in the asymptotic formula for the r-fold divisor sum, and C_i $(0 \leq i \leq r-1)$ are real constants defined by the convergent series

$$C_i = \sum_{n=1}^{\infty} f(n) \left(\frac{(\log n)^i}{n} - \frac{(\log(n+1))^i}{n+1} \right).$$

To establish Theorem 1, we require the following auxiliary result.

Lemma 1. *Let f be an arithmetic function such that $f(n) \ll n^\alpha$ for some constant $\alpha \in [0,1)$. Then for any fixed integer $i \geq 0$, the series*

$$\sum_{n=1}^{\infty} f(n) \left(\frac{(\log n)^i}{n} - \frac{(\log(n+1))^i}{n+1} \right)$$

converges absolutely.

Proof. As $n \to \infty$, we have the expansion

$$\log(n+1) = \log n + \log\left(1 + \frac{1}{n}\right) = \log n + \frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right).$$

Consequently,

$$\log(n+1) = \log n \left(1 + \frac{1}{n \log n} - \frac{1}{2n^2 \log n} + O\left(\frac{1}{n^3 \log n}\right)\right).$$

For any fixed integer $i \geq 0$, applying the binomial theorem yields

$$(\log(n+1))^i = (\log n)^i + i\frac{(\log n)^{i-1}}{n} + O\left(\frac{(\log n)^{i-2}}{n^2}\right).$$

Therefore,

$$\begin{aligned}\frac{(\log n)^i}{n} - \frac{(\log(n+1))^i}{n+1} &= \frac{(\log n)^i}{n} - \frac{(\log n)^i}{n+1} - i\frac{(\log n)^{i-1}}{n(n+1)} + O\left(\frac{(\log n)^{i-2}}{n^3}\right) \\ &= (\log n)^i \left(\frac{1}{n(n+1)}\right) + O\left(\frac{(\log n)^{i-1}}{n^2}\right) \\ &\ll \frac{(\log n)^i}{n^2}.\end{aligned}$$

Hence,

$$f(n)\left(\frac{(\log n)^i}{n} - \frac{(\log(n+1))^i}{n+1}\right) \ll \frac{(\log n)^i}{n^{2-\alpha}}.$$

Since $\alpha \in [0,1)$, we have $2-\alpha > 1$, and the series $\sum_{n=1}^{\infty} (\log n)^i / n^{2-\alpha}$ converges for any integer $i \geq 0$. This establishes absolute convergence of the series in question. $\square$

Proof of Theorem 1. We begin by rewriting the sum $S_{f,r}(x)$ as

$$S_{f,r}(x) = \sum_{n \le x} f\left(\left\lfloor \frac{x}{n} \right\rfloor\right) \tau_r(n) = \sum_{n\le x} \sum_{d=\lfloor x/n \rfloor} f(d)\tau_r(n) = \sum_{d \le x} f(d) \sum_{\frac{x}{d+1} < n \le \frac{x}{d}} \tau_r(n).$$

Using the identity

$$\sum_{\frac{x}{d+1} < n \le \frac{x}{d}} \tau_r(n) = \sum_{n \le x/d} \tau_r(n) - \sum_{n \le x/(d+1)} \tau_r(n),$$

and the classical asymptotic formula for the r-fold divisor sum (see Titchmarsh [11])

$$\sum_{n \le y} \tau_r(n) = yP_r(\log y) + O\big(y^{1-1/r}(\log y)^{r-2}\big), \tag{2}$$

where $P_r(t) = \sum_{j=0}^{r-1} a_j t^j$ is a polynomial of degree $r-1$, we obtain

$$\begin{aligned}S_{f,r}(x) = \sum_{d \le x} f(d) &\left[\frac{x}{d} P_r\left(\log \frac{x}{d}\right) - \frac{x}{d+1} P_r\left(\log \frac{x}{d+1}\right)\right] \\ &+ O\left(\sum_{d\le x} f(d)\left(\left(\frac{x}{d}\right)^{1-1/r} (\log x)^{r-2} + \left(\frac{x}{d+1}\right)^{1-1/r} (\log x)^{r-2}\right)\right).\end{aligned}$$

Expanding $P_r(\log(x/d)) = \sum_{j=0}^{r-1} a_j (\log x - \log d)^j$ and applying the binomial theorem gives

$$P_r\left(\log \frac{x}{d}\right) = \sum_{j=0}^{r-1} a_j \sum_{i=0}^{j} \binom{j}{i} (\log x)^{j-i} (-\log d)^i.$$

Consequently,

$$S_{f,r}(x) = x\sum_{j=0}^{r-1}\sum_{i=0}^{j}\binom{j}{i}(-1)^i a_j(\log x)^{j-i}\sum_{d\le x} f(d)\left(\frac{(\log d)^i}{d} - \frac{(\log(d+1))^i}{d+1}\right)$$
$$+ O\left(x^{1-1/r}(\log x)^{r-2}\sum_{d\le x}\frac{f(d)}{d^{1-1/r}}\right).$$

By Lemma 1, the inner sum converges to C_i with error $O(x^{\alpha-1})$. Thus,

$$\sum_{d\le x} f(d)\left(\frac{(\log d)^i}{d} - \frac{(\log(d+1))^i}{d+1}\right) = C_i + O(x^{\alpha-1}).$$

For the error term, we apply Abel's summation formula to estimate

$$\sum_{d\le x}\frac{f(d)}{d^{1-1/r}}.$$

Let $F(t) = \sum_{d\le t} f(d) \ll t(\log\log t)^\beta$. Then

$$\begin{aligned}\sum_{d\le x}\frac{f(d)}{d^{1-1/r}} &= \frac{F(x)}{x^{1-1/r}} + (1-1/r)\int_1^x \frac{F(t)}{t^{2-1/r}}\,dt\\ &\ll (\log\log x)^\beta + (1-1/r)\int_1^x \frac{(\log\log t)^\beta}{t^{1-1/r}}\,dt\\ &\ll (\log x)^{1/r}(\log\log x)^\beta.\end{aligned}$$

Therefore,

$$x^{1-1/r}(\log x)^{r-2}\sum_{d\le x}\frac{f(d)}{d^{1-1/r}} \ll x(\log x)^{r-2}(\log\log x)^\beta.$$

Combining all estimates yields the desired asymptotic formula. □

Proposition 1. *Let f be an arithmetic function defined by*

$$f(n) = \sum_{p^\alpha \| n} g(\alpha),$$

where g is an arithmetic function satisfying $g(0) = 0$, $g(1) \neq 0$, and $g(n) = O(2^{n/2})$. Then

$$S_{f,r}(x) = x\sum_{j=0}^{r-1}\sum_{i=0}^{j}\binom{j}{i}(-1)^i a_j C_i(\log x)^{j-i} + O\big(x(\log x)^{r-2}\log\log x\big).$$

Proof. For $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, we have $\alpha_i \le \log n / \log 2$. Hence,

$$f(n) = \sum_{p^\alpha \| n} g(\alpha) \ll \sum_{p^\alpha \| n} 2^{\alpha/2} \ll n^{1/2} \sum_{p^\alpha \| n} 1 \ll n^{1/2} \log\log n = O\big(n^{1/2+\varepsilon}\big)$$

for any $\varepsilon > 0$. Thus the first condition of Theorem 1 holds with $\alpha = 1/2 + \varepsilon < 1$. The second condition follows from Duncan [5, Theorem 1], which gives $\sum_{n \le x} f(n) \ll x \log\log x$, corresponding to $\beta = 1$. Substituting $\beta = 1$ into the error term of Theorem 1 completes the proof. □

3. Examples

We now present several applications of our main theorem.

Example 1. For a fixed integer $k \ge 0$, let $g(n) = n^k$ and define

$$f(n) = \Omega_k(n) = \sum_{p^\alpha \| n} \alpha^k.$$

Notably, $\Omega_0(n) = \omega(n)$ and $\Omega_1(n) = \Omega(n)$. Using results from Duncan [4] or Hassani [6], we have

$$\sum_{n \le x} \Omega_k(n) \ll x \log\log x.$$

Therefore, by Proposition 1,

$$S_{\Omega_k, r}(x) = x \sum_{j=0}^{r-1} \sum_{i=0}^{j} \binom{j}{i} (-1)^i a_j C_i (\log x)^{j-i} + O\big(x (\log x)^{r-2} \log\log x\big).$$

Example 2. The same asymptotic formula holds when $f(n)$ is replaced by $\tau(\tau(n))$ or $\Omega(\tau(n))$, as shown by Heppner [7]. Similarly, the formula applies to the functions $\omega(n)/p(n)$ and $\Omega(n)/p(n)$, where $p(n)$ denotes the smallest prime divisor of n (see Zhang [14]).

Example 3. For the function $f(n) = \omega^2(n)$, Shapiro [10, p. 347] established that

$$\sum_{n \le x} \omega^2(n) \ll x (\log\log x)^2.$$

Hence, by Theorem 1 with $\beta = 2$,

$$S_{\omega^2, r}(x) = x \sum_{j=0}^{r-1} \sum_{i=0}^{j} \binom{j}{i} (-1)^i a_j C_i (\log x)^{j-i} + O\big(x (\log x)^{r-2} (\log\log x)^2\big).$$

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