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A TOPOLOGICAL PROOF OF THE INFINITUDE OF PRIME NUMBERS IN CERTAIN ARITHMETIC PROGRESSIONS

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Abstract

In this note, we present a topological proof of the infinitude of prime numbers in the arithmetic progressions $4\mathbb{N}_0 + 3$, $3\mathbb{N}_0 + 2$, and $6\mathbb{N}_0 + 5$.

- To my wife and our first child

1. Introduction

Let $\mathbb{N}_0$, $\mathbb{N}$, and $\mathbb{P}$ denote the sets of non-negative integers, positive integers, and prime numbers, respectively. Let $a, b \in \mathbb{N}$. Dirichlet's theorem on arithmetic progressions states that the set

$$a\mathbb{N}_0 + b := \{b, a + b, 2a + b, 3a + b, \dots, an + b, a(n + 1) + b, \dots\} \qquad (1)$$

contains infinitely many prime numbers if and only if $\gcd(a, b) = 1$. Here, $\gcd(a, b)$ denotes the greatest common divisor of a and b.

The proof of Dirichlet's theorem on arithmetic progressions is not elementary (see [1, Chapter 7]). However, there are elementary proofs for special cases (see [8, Chapter 5, Theorem 10 and Theorem 11]). Moreover, a Euclidean-style proof of the infinitude of primes in arithmetic progressions of the form (1) with $\gcd(a, b) = 1$ exists if and only if $b^2 \equiv 1 \pmod{a}$ (see [6, Theorem 1]). In general, as far as the author knows, there is no topological proof of Dirichlet's theorem or any of its special cases. In this note, we present a topological proof of the infinitude of prime numbers in the arithmetic progressions $4\mathbb{N}_0 + 3$, $3\mathbb{N}_0 + 2$, and $6\mathbb{N}_0 + 5$.

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2. The Comaximal Space $CM(\mathbb{N})$

The *comaximal space* $CM(\mathbb{N})$ is the topological space $(\mathbb{N}, \tau)$ where τ is generated by the collection of sets $\sigma_n := \{m \in \mathbb{N} : \gcd(n,m) = 1\}$ for $n \in \mathbb{N}$. Note that

$$\sigma_n = \bigcup_{\substack{1 \le m \le n \\ \gcd(m,n)=1}} n\mathbb{N}_0 + m, \tag{2}$$

since for every integer x we have that $\gcd(n,m) = \gcd(n, nx+m)$; see [7, Theorem 1.9]. It is easily deduced from here that σ_n is infinite for every positive integer n. The comaximal space does not satisfy the T_0 separation axiom (see [4, Corollary 3.12]), is connected, locally connected, and path-connected (see [4, Corollary 2.6 and Corollary 3.10]). On the other hand, for every positive integer n, m and prime number p, it holds that $\sigma_{nm} = \sigma_n \cap \sigma_m$ (see the proof of [5, Theorem 1]), and $\mathbf{cl}_{CM(\mathbb{N})}(\{p\}) = p\mathbb{N}$ (see [5, Lemma 1]), where $\mathbf{cl}_{CM(\mathbb{N})}(\{n\})$ denotes the closure of the singleton set $\{n\}$ in the topological space $CM(\mathbb{N})$. Furthermore, in [5] (using the topology τ), a topological proof of the infinitude of prime numbers is presented (different from the proofs of Furstenberg [2] and Golomb [3]). In the same work, the infinitude of any non-empty subset of prime numbers is characterized in the following sense:

Lemma 1 ([5])**.** *If A is a non-empty subset of prime numbers, then A is infinite if and only if A is dense in $CM(\mathbb{N})$.*

Proof. Suppose that A is infinite. Then, for any positive integer $n > 1$, we can choose a prime $p \in A$ such that $p > n$, and consequently, $p \in \sigma_n$ since $\gcd(n,p) = 1$. Therefore, A is dense in $CM(\mathbb{N})$. On the other hand, assume that A is dense in $CM(\mathbb{N})$. Let $\{p_1, p_2, \ldots, p_k\} \subset A$ be a finite collection of prime numbers and consider the non-empty infinite basic element[1] σ_x where $x = p_1 \cdot p_2 \cdots p_k$. Note that none of the p_i belong to σ_x, but since A is dense in $CM(\mathbb{N})$, there must be another prime number q, different from each p_i, such that $q \in \sigma_x$. Consequently, A is infinite. $\square$

3. The Topological Proof

We will topologically prove that there are infinitely many prime numbers in the arithmetic progression $4\mathbb{N}_0 + 3$. The argument we use also works for the arithmetic progressions $3\mathbb{N}_0 + 2$ and $6\mathbb{N}_0 + 5$. First, consider the following lemma.

Lemma 2. *Let $a\mathbb{N}_0 + b$ be an arithmetic progression of the form (1) with* $\gcd(a,b) = 1$. *Then $a\mathbb{N}_0 + b$ is dense in $CM(\mathbb{N})$.*

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[1] See Equation (2)

Proof. We first show that for every prime number p, we have $(a\mathbb{N}_0 + b) \cap \sigma_p \neq \emptyset$. Suppose the contrary, that is, assume there exists a prime number p such that $(a\mathbb{N}_0 + b) \cap \sigma_p = \emptyset$. In this case, $\gcd(am + b, p) = p$ for all non-negative integers m. If $m = 0$, then $p \mid b$. This implies that $p \mid am$ for all non-negative integers m. If $\gcd(p, m) = 1$, then $p \mid a$, which contradicts the fact that $\gcd(a, b) = 1$. Therefore, $(a\mathbb{N}_0 + b) \cap \sigma_p \neq \emptyset$ for every prime number p.

Now we show that for every positive composite integer n, we have $(a\mathbb{N}_0 + b) \cap \sigma_n \neq \emptyset$. Let n be a positive composite integer. Write its prime factorization as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ (the p_i's are distinct primes). Let $c_i \in (a\mathbb{N}_0 + b) \cap \sigma_{p_i}$ for $i = 1, 2, \ldots, k$ (the existence of each c_i is guaranteed by the first part of the proof). Consider the following system of congruences:

$$\begin{cases} x \equiv c_1 \pmod{ap_1} \\ x \equiv c_2 \pmod{ap_2} \\ \quad\vdots \\ x \equiv c_k \pmod{ap_k} \end{cases} \tag{3}$$

Since each c_i is of the form $am + b$, we have that $a = a\gcd(p_i, p_j) = \gcd(ap_i, ap_j) \mid c_i - c_j$ for each $i, j = 1, 2, 3, \ldots, k$ (with $i \neq j$). By the Chinese Remainder Theorem, the system of congruences in Equation (3) has infinitely many solutions; let x_0 be one such solution. Note that $x_0 = ap_it_i + c_i \in (a\mathbb{N}_0 + b) \cap \sigma_{p_i}$ for each $i = 1, 2, 3, \ldots, k$ (for some integer t_i). Thus, $x_0 \in \bigcap_{i=1}^{k} \sigma_{p_i} \cap (a\mathbb{N}_0 + b)$. Since $\sigma_n = \bigcap_{i=1}^{k} \sigma_{p_i}$, it follows that $x_0 \in \sigma_n \cap (a\mathbb{N}_0 + b)$. Consequently, for every positive composite integer n, we have $\sigma_n \cap a\mathbb{N}_0 + b \neq \emptyset$.

Finally, since $\sigma_1 = \mathbb{N}$, we have $\sigma_n \cap (a\mathbb{N}_0 + b) \neq \emptyset$ for all positive integers n. Therefore, $a\mathbb{N}_0 + b$ is dense in $CM(\mathbb{N})$. $\square$

Remark 1. Lemma 2 is trivial if one invokes Dirichlet's theorem on arithmetic progressions. However, we avoid invoking this theorem in order to prevent circular reasoning.

We are now ready for the topological proof.

Proposition 1. *There are infinitely many prime numbers in the arithmetic progression* $4\mathbb{N}_0 + 3$.

Proof. Since every positive integer of the form $4k + 3$ is divisible by a prime number of the same form, we have

$$4\mathbb{N}_0 + 3 \subset \bigcup_{p \in \mathbb{P} \cap (4\mathbb{N}_0 + 3)} p\mathbb{N} \cap (4\mathbb{N}_0 + 3). \tag{4}$$

On the other hand, using properties of the closure operator, we get

$$\begin{aligned}\bigcup_{p\in\mathbb{P}\cap(4\mathbb{N}_0+3)} p\mathbb{N}\cap(4\mathbb{N}_0+3) &= \bigcup_{p\in\mathbb{P}\cap(4\mathbb{N}_0+3)} \mathbf{cl}_{CM(\mathbb{N})}(\{p\})\cap(4\mathbb{N}_0+3)\\ &= \bigcup_{p\in\mathbb{P}\cap(4\mathbb{N}_0+3)} \mathbf{cl}_{CM(4\mathbb{N}_0+3)}(\{p\})\\ &\subset \mathbf{cl}_{CM(4\mathbb{N}_0+3)}\left(\bigcup_{p\in\mathbb{P}\cap(4\mathbb{N}_0+3)}\{p\}\right)\\ &= \mathbf{cl}_{CM(4\mathbb{N}_0+3)}(\mathbb{P}\cap(4\mathbb{N}_0+3))\\ &\subset 4\mathbb{N}_0+3.\end{aligned}\tag{5}$$

Thus, from Equation (4) and Equation (5), it follows that

$$\mathbf{cl}_{CM(4\mathbb{N}_0+3)}(\mathbb{P}\cap(4\mathbb{N}_0+3)) = 4\mathbb{N}_0+3.$$

This implies that $\mathbb{P}\cap(4\mathbb{N}_0+3)$ is dense in $4\mathbb{N}_0+3$ with the subspace topology $M(4\mathbb{N}_0+3)$ induced by τ. By Lemma 2, the set $4\mathbb{N}_0+3$ is dense in $CM(\mathbb{N})$. Hence, by transitivity of density, $\mathbb{P}\cap(4\mathbb{N}_0+3)$ is dense in $CM(\mathbb{N})$. Finally, by Lemma 1, the set $\mathbb{P}\cap(4\mathbb{N}_0+3)$ is infinite. $\square$

To obtain topological proofs of the infinitude of prime numbers in the arithmetic progressions $3\mathbb{N}_0+2$ and $6\mathbb{N}_0+5$, it suffices to replace $4\mathbb{N}_0+3$ with $3\mathbb{N}_0+2$ and $6\mathbb{N}_0+5$, respectively, in the proof of the last proposition (Proposition 1). Note that the key point of the proof is Equation (4), which is also satisfied for the arithmetic progressions $3\mathbb{N}_0+2$ and $6\mathbb{N}_0+5$. In general, this argument, as presented, only works for arithmetic progressions of the form (1) with $\gcd(a,b)=1$ such that each of their elements has at least one prime factor of the same form.

4. Final Comment

In the second paragraph of the Introduction, we mentioned that a Euclidean-style proof of the infinitude of primes in arithmetic progressions of the form (1) with $\gcd(a,b)=1$ exists if and only if $b^2\equiv 1 \pmod a$. Thus, one may ask: does a topological-style proof of the infinitude of primes in arithmetic progressions of the form (1) with $\gcd(a,b)=1$ exist if and only if $b^2\equiv 1 \pmod a$? We do not have a general answer to this question; however, we can assert that the answer is negative if one requires a topological proof of the type presented in this note. Indeed, it suffices to consider the arithmetic progression $4\mathbb{N}_0+1$, for which the condition $b^2\equiv 1 \pmod a$ holds, but Equation (4) fails. Having said this, we leave the reader with the following question: for which arithmetic progressions of the form (1) with $\gcd(a,b)=1$ can a topological proof be obtained?

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