EXTENDING RECENT WORK OF NATH, SAIKIA, AND SARMA ON k-TUPLE ℓ-REGULAR PARTITIONS

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Abstract

Let $T_{\ell,k}(n)$ denote the number of ℓ-regular k-tuple partitions of n. In a recent work, Nath, Saikia, and Sarma derived several families of congruences for $T_{\ell,k}(n)$, with particular emphasis on the cases $T_{2,3}(n)$ and $T_{4,3}(n)$. In the concluding remarks of their paper, they conjectured that $T_{2,3}(n)$ satisfies an infinite set of congruences modulo 6. In this paper, we confirm their conjecture by proving a much more general result using elementary q-series techniques. We also present new families of congruences satisfied by $T_{\ell,k}(n)$.

1. Introduction

A *partition* of a non-negative integer n is a non-increasing sequence of positive integers, called parts, whose sum is n. By convention, zero has only one partition, namely, the empty sequence. Let $p(n)$ denote the number of partitions of n.

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Ramanujan [15] famously established the celebrated congruences

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Since Ramanujan's pioneering work, mathematicians have investigated further congruences for $p(n)$ and the arithmetic properties of its various generalizations. In this paper, we consider the following generalization. If

$$n_1, n_2, \ldots, n_k \geq 0 \quad \text{with} \quad n_1 + n_2 + \cdots + n_k = n,$$

and if λ_i is a partition of n_i for each $i = 1, \ldots, k$, then the sequence

$$(\lambda_1, \lambda_2, \ldots, \lambda_k)$$

is called a *k-tuple* partition of n. For instance, if $\lambda_1 = (3, 2, 1)$ and $\lambda_2 = (7, 6, 2)$, then (λ_1, λ_2) forms a 2-tuple partition of 21.

A partition is called *ℓ-regular* if none of its parts is divisible by ℓ. Correspondingly, a k-tuple partition is said to be *ℓ-regular* if each λ_i is an ℓ-regular partition of n_i for $1 \leq i \leq k$. We denote the number of k-tuple ℓ-regular partitions of n by $T_{\ell,k}(n)$ and, in particular, define

$$T_\ell(n) := T_{\ell,3}(n).$$

The theory of ℓ-regular partitions has been extensively developed in the literature (see, for example, [6], [8], and [10]). More recently, ℓ-regular k-tuple partitions have attracted significant interest, and various divisibility properties of $T_{\ell,k}(n)$ have been explored. For instance, the case $(\ell, k) = (3, 3)$ was studied by Adiga and Dasappa [1], da Silva and Sellers [5], and Gireesh and Mahadeva Naika [7]; the cases $(\ell, k) = (3, 9)$ and $(3, 27)$ by Baruah and Das [2]; the case $(\ell, k) = (3, 6)$ by Murugan and Fathima [11]; and both $(\ell, k) = (2, 3)$ and $(3, 3)$ by Nadji and Ahmia [12]. Additionally, Rahman and Saikia [14] examined the cases $(\ell, k) = (5, 3)$ and $(5, 5)$, while Vidya [16] considered cases with $k = 3$ and $\ell \in \{2, 4, 10, 20\}$.

In a recent work [13], Nath, Saikia, and Sarma analyzed the cases $(\ell, k) = (2, 3)$ and $(4, 3)$, and in some instances for general (ℓ, k). In particular, they proved [13, Theorem 1.3] that for $n \geq 0$ and $\alpha \geq 0$, we have

$$T_2\left(3^{4\alpha+2}n + \sum_{i=0}^{2\alpha} 3^{2i} + 3^{4\alpha+1}\right) \equiv 0 \pmod{24}, \tag{1}$$

$$T_2\left(3^{4\alpha+2}n + \sum_{i=0}^{2\alpha} 3^{2i} + 2 \cdot 3^{4\alpha+1}\right) \equiv 0 \pmod{24}. \tag{2}$$

They also proved [13, Theorem 1.6] that if $p \equiv 5$ or $7 \pmod 8$ and $n, \alpha \geq 0$ with $p \nmid n$, then

$$T_2\left(9p^{2\alpha+1}n + \frac{9p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod 6. \tag{3}$$

Moreover, their analysis led them to propose the following conjecture:

Conjecture 1 (Nath, Saikia, Sarma [13])**.** Let $p \geq 5$ be a prime with $\left(\frac{-2}{p}\right)_L = -1$ and, let t be a positive integer with $(t, 6) = 1$ and $p \mid t$. Then for all $n \geq 0$ and $1 \leq j \leq p-1$, we have

$$T_2\left(9 \cdot t^2 n + \frac{9 \cdot t^2 j}{p} + \frac{57 \cdot t^2 - 1}{8}\right) \equiv 0 \pmod 6.$$

Motivated by this conjecture, we establish the following theorem as a stronger version of it.

Theorem 1. *Let t be a positive integer with $\gcd(t, 6) = 1$. Then, for $n \geq 0$ and $N = 33$ or 57, we have*

$$T_2\left(9n + \frac{Nt^2-1}{8}\right) \equiv 0 \pmod{24}. \tag{4}$$

Corollary 1. *Conjecture 1 is true.*

Proof. Using $N = 57$ and replacing n by $t^2 n + \frac{t^2 j}{p}$ in (4) completes the proof. □

With the goal of extending such congruences even further, we prove the following infinite family of congruences modulo 8 in extremely elementary fashion.

Theorem 2. *Let $p \equiv 3, 5$ or $7 \pmod 8$ be a prime. Then, for $n, \alpha \geq 0$ with $p \nmid n$, we have*

$$T_2\left(p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod 8.$$

In addition, if $3 \nmid n$, then

$$T_2\left(p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod{24}. \tag{5}$$

Lastly, we note that (3) holds modulo 24, and it also holds for primes $p \equiv 3 \pmod 8$.

Corollary 2. *Let $p \equiv 3, 5$ or $7 \pmod 8$ be a prime and $p \neq 3$. Then, for $n, \alpha \geq 0$ with $p \nmid n$, we have*

$$T_2\left(9p^{2\alpha+1}n + \frac{9p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod{24}.$$

Proof. Replacing n by $9n+p$ in (5) gives the result. □

After collecting a number of necessary mathematical tools in Section 2, we provide elementary proofs of Theorem 1 and Theorem 2 in Section 3, and we share several closing comments in Section 4.

2. Preliminaries

Our proofs of the aforementioned theorems are entirely elementary, relying solely on generating function manipulations and classical q-series results. In this section, we present several elementary facts, obtained from elementary q-series analysis, that will be utilized in the course of our proofs below.

We recall the q-Pochhammer symbol defined by

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$$

and denote

$$f_k := (q^k;q^k)_\infty.$$

With this notation, it is clear that the generating function for $T_{\ell,k}(n)$ is given by

$$\sum_{n\geq 0} T_{\ell,k}(n)q^n = \frac{f_\ell^k}{f_1^k},$$

and, in particular, the generating function for $T_\ell(n)$ is

$$\sum_{n\geq 0} T_\ell(n)q^n = \frac{f_\ell^3}{f_1^3}.$$

We now collect the results which will be necessary in our work below.

Lemma 1. *We have*

$$f_1 = \sum_{m\in\mathbb{Z}} (-1)^m q^{m(3m-1)/2}. \tag{6}$$

Proof. A proof of this identity can be found in [9, Section 1.6]. □

Lemma 2. *We have*

$$f_1^3 = \sum_{m\geq 0} (-1)^m (2m+1) q^{m(m+1)/2}. \tag{7}$$

Proof. This identity can be found in [9, (1.7.1)]. □

Lemma 3. *We have*

$$\frac{f_1^5}{f_2^2} = \sum_{m \in \mathbb{Z}} (6m+1) q^{m(3m+1)/2}. \tag{8}$$

Proof. This identity appears in [9, (10.7.3)]. □

Lemma 4. *We have*

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}. \tag{9}$$

Proof. Note that

$$\begin{aligned}
\frac{f_2^3}{f_1 f_4} &= \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty (q^4;q^4)_\infty} \\
&= \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty (q^4;q^4)_\infty} \\
&= \frac{(q^2;q^2)_\infty (q^2;q^4)_\infty}{(q;q^2)_\infty} \\
&= (q^2;q^2)_\infty (-q;q^2)_\infty \\
&= (-q;-q)_\infty.
\end{aligned}$$

□

Lemma 5. *We have*

$$\frac{f_1^2}{f_2} = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} = 1 + 2 \sum_{n \geq 1} (-1)^n q^{n^2}. \tag{10}$$

Proof. See [9, (1.5.8)] for a proof of this result. □

Corollary 3. *We have*

$$\frac{f_2^2}{f_4} = 1 + 2 \sum_{n \geq 1} (-1)^n q^{2n^2}. \tag{11}$$

Proof. This follows from Lemma 5 by replacing q by q^2 everywhere in (10). □

We now provide the congruence–related tools necessary to complete our proofs in the next section. We begin with an extremely well–known result which, in essence, follows from the Binomial Theorem.

Lemma 6. *For a prime p and positive integers k and l,*

$$f_l^{p^k} \equiv f_{lp}^{p^{k-1}} \pmod{p^k}. \tag{12}$$

Proof. See [4, Lemma 3] for a proof. □

In [13, Theorem 1.1], Nath, Saikia, and Sarma use Lemma 6 to prove that, if p is a prime and $1 \le r \le p-1$, then

$$T_{\ell,p}(pn+r) \equiv 0 \pmod{p}. \tag{13}$$

Note that Lemma 6 can be generalized in a natural way in order to obtain the following:

Lemma 7. *For a prime p and positive integers $k, l,$ and s with $k-s \ge 0$,*

$$f_l^{p^k m} \equiv f_{lp^s}^{p^{k-s}m} \pmod{p^{k-s+1}}. \tag{14}$$

Proof. The congruence follows by applying (12) s times. □

With Lemma 7 in hand, we can introduce a generalized version of the congruence in (13), the proof of which is almost immediate.

Theorem 3. *Let p be a prime and $m, \ell > 0$ be integers. For $\alpha, s > 0$ satisfying $\alpha - s \ge 0$ and $1 \le r \le p^s - 1$,*

$$T_{\ell,p^\alpha m}(p^s n + r) \equiv 0 \pmod{p^{\alpha-s+1}}.$$

Proof. We use Lemma 7 to obtain

$$\sum_{n\ge 0} T_{\ell,p^\alpha m}(n)p^n = \frac{f_\ell^{p^\alpha m}}{f_1^{p^\alpha m}} \equiv \frac{f_{p^s\ell}^{p^{\alpha-s}m}}{f_{p^s}^{p^{\alpha-s}m}} \pmod{p^{\alpha-s+1}}.$$

Notice that the right-hand side is a function of q^{p^s}. Thus, the coefficients of $q^{p^s n+r}$ with $1 \le r \le p^s - 1$ equal zero. This completes the proof. □

We close this section by proving a parity characterization for the particular function $T_2(n)$.

Lemma 8. *For all $n \ge 0$,*

$$T_2(n) \equiv \begin{cases} 1 \pmod{2}, & \text{if } n = m(m+1)/2 \text{ for some } m \\ 0 \pmod{2}, & \text{otherwise.} \end{cases}$$

Proof. We see that

$$\begin{aligned} \sum_{n\ge 0} T_2(n)q^n &= \frac{f_2^3}{f_1^3} \\ &\equiv \frac{f_1^6}{f_1^3} \pmod{2} \quad \text{thanks to (12)} \\ &= f_1^3. \end{aligned}$$

Thanks to (7), the result follows. □

With the above tools in hand, we now proceed to prove Theorem 1 and Theorem 2.

3. Proofs of Our Results

We begin by sharing our proof of Theorem 1 which generalizes the original conjecture of Nath, Saikia, and Sarma [13].

Proof of Theorem 1. Using $\alpha = 0$ in (1) and (2) gives

$$T_2(9n+4) \equiv 0 \pmod{24},$$
$$T_2(9n+7) \equiv 0 \pmod{24},$$

respectively. Thus, writing

$$k = 9n + \frac{Nt^2 - 1}{8},$$

it suffices to show that $k \equiv 4$ or $7 \pmod 9$. Since $\gcd(t, 6) = 1$, t must have one of the forms $6m + 1$ or $6m + 5$. Therefore,

$$Nt^2 \equiv \begin{cases} 6 \pmod 9 & \text{if } N = 33, \\ 3 \pmod 9 & \text{if } N = 57. \end{cases}$$

Observing that $8^{-1} \equiv -1 \pmod 9$, we have

$$k \equiv \begin{cases} 4 \pmod 9 & \text{if } N = 33, \\ 7 \pmod 9 & \text{if } N = 57. \end{cases}$$

□

Next, we provide an elementary proof of Theorem 2.

Proof of Theorem 2. Thanks to (12), we have

$$\sum_{n \geq 0} T_2(n) q^n = \frac{f_2^3}{f_1^3} = \frac{f_2^4}{f_1^3 f_2} \equiv \frac{f_1^5}{f_2^2} f_2 \pmod 8. \tag{15}$$

Using (6) and (8) in (15), we obtain

$$\sum_{n \geq 0} T_2(n) q^n \equiv \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-)^k (6m+1) q^{m(3m+1)/2 + k(3k-1)} \pmod 8.$$

We now need to check whether $l := p^{2\alpha+1} n + \frac{p^{2\alpha+2} - 1}{8}$ can be represented as $\frac{m(3m+1)}{2} + k(3k-1)$. Equivalently, we check whether

$$24l + 3 = (6m+1)^2 + 2(6k-1)^2.$$

Let $\nu_p(N)$ be the highest power of p dividing N. We first consider primes $p \equiv 5$ or $7 \pmod 8$. Then, we have $\left(\frac{-2}{p}\right) = -1$. So, if $N = x^2 + 2y^2$, then $2|\nu_p(N)$. Also, for $p \nmid n$, we have $\nu_p(24l+3) = 2\alpha+1$. Thus, $24l+3$ cannot be of the form x^2+2y^2. Hence, for $p \equiv 5$ or $7 \pmod 8$,

$$T_2\left(p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod 8.$$

The above proof only deals with the cases $p \equiv 5$ or $7 \pmod 8$, which still leaves us with the case $p \equiv 3 \pmod 8$. This requires a different idea. In this case, we replace q by $-q$ and then, using (9) and (12), we obtain

$$\sum_{n\geq 0} T_2(n)(-q)^n = f_2^3\left(\frac{f_1 f_4}{f_2^3}\right)^3 = \frac{f_1^3 f_4^3 f_2^2}{f_2^8} \equiv \frac{f_1^3 f_4^3 f_2^2}{f_4^4} = f_1^3\frac{f_2^2}{f_4} \pmod 8. \tag{16}$$

Substituting (7) and (11) in (16) gives

$$\sum_{n\geq 0} T_2(n)(-q)^n \equiv \sum_{m\in\mathbb{Z}} (-1)^m(2m+1)q^{m(m+1)/2}\left(1 + 2\sum_{k\geq 1}(-1)^k q^{2k^2}\right) \pmod 8.$$

Now we need to check whether we have

$$l = \frac{m(m+1)}{2} + 2k^2,$$

that is,

$$8l+1 = (2m+1)^2 + (4k)^2.$$

For primes $p \equiv 3 \pmod 4$, we note that $\left(\frac{-1}{p}\right) = -1$. So, for any $N = x^2 + y^2$, we have $2|\nu_p(N)$. However, for $p \nmid n$, $\nu_p(8l+1) = 2\alpha+1$. This completes the proof of the congruences modulo 8.

For the second part, it suffices to show that $T_2(l) \equiv 0 \pmod 3$. For this, we check whether $3|l$ (in order to apply (13)). Since $8^{-1} \equiv -1 \pmod 3$, we have

$$l = p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{8} \equiv \begin{cases} n \pmod 3 & \text{if } p \equiv 1 \pmod 3, \\ -n \pmod 3 & \text{if } p \equiv -1 \pmod 3, \\ 1 \pmod 3 & \text{if } p = 3. \end{cases}$$

Since $3 \nmid n$, we have $3 \nmid l$ and, by (13), $T_2(l) \equiv 0 \pmod 3$.

□

4. Closing Thoughts

We conclude by mentioning some additional observations. In the case when $\alpha = 0$ and n is replaced by $pn + s$ for $1 \leq s \leq p - 1$, Theorem 2 shows that

$$T_2(p^2n + r) \equiv 0 \pmod{8},$$

when $p \equiv 3, 5,$ or $7 \pmod{8}$ is prime, and $r = ps + \frac{p^2-1}{8}$. Our computations further indicate that $T_2(n)$ is divisible by 32 in the following cases:

$$\begin{aligned}
T_2(25n + 8) &\equiv 0 \pmod{32},\\
T_2(25n + 13) &\equiv 0 \pmod{32},\\
T_2(25n + 18) &\equiv 0 \pmod{32},\\
T_2(25n + 23) &\equiv 0 \pmod{32},\\
T_2(49n + 13) &\equiv 0 \pmod{32},\\
T_2(49n + 20) &\equiv 0 \pmod{32},\\
T_2(49n + 27) &\equiv 0 \pmod{32},\\
T_2(49n + 34) &\equiv 0 \pmod{32},\\
T_2(49n + 41) &\equiv 0 \pmod{32},\\
T_2(49n + 48) &\equiv 0 \pmod{32}.
\end{aligned}$$

It would be interesting to determine whether these congruences hold for all $n \geq 0$, and whether they extend to $T_2(p^2n + r)$ for primes $p > 7$. Moreover, our computations suggest that, with only a few exceptions, the congruence

$$T_2(7n + 6) \equiv 0 \pmod{2^9} \tag{17}$$

appears to hold. We leave these questions for the interested reader. After our preprint was posted on arXiv, Shi-Chao Chen kindly informed us that the congruence (17) had been proved for $(n, 7) = 1$ in their paper [3, Theorem 1].

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