#A41 INTEGERS 26 (2026)

NEW PROOFS OF CERTAIN THETA FUNCTION IDENTITIES VIA RAMANUJAN'S FUNCTION

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Received: 8/11/25, Accepted: 2/24/26, Published: 4/3/26

Abstract

We present new proofs of two theta function identities equivalent to Ramanujan's modular equations of degree five by employing some identities involving Ramanujan's function $k(q)$ due to Cooper, Chern, and Tang.

1. Introduction

Throughout this paper, we define $(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$ for complex numbers a and q with $|q|<1$ and

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty.$$

Recall that Ramanujan's general theta function is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a,-b,ab;ab)_\infty$$

for all complex numbers a and b with $|ab|<1$, where the last equality is the Jacobi triple product identity. Three special cases of $f(a,b)$ are given by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty},$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

DOI: 10.5281/zenodo.19403704

For $|z| < 1$, we also recall the Gaussian hypergeometric function ${}_2F_1$ given by

$$ {}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, $$

where $(a)_n := a(a+1)\cdots(a+n-1)$ for $n \geq 1$ and $(a)_0 = 1$. Following Ramanujan [7], we define a modular equation of degree $n \in \mathbb{N}$ by a relation involving α and β such that

$$ n \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; 1-\alpha)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; \alpha)} = \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; 1-\beta)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; \beta)}; $$

in this case, we say that β has degree n over α. In his second notebook [7, pp. 236–237, Entry 13(i), (ii)], Ramanujan recorded the following modular equations of degree five without proof:

$$ (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1, \tag{1} $$

$$ \left(\frac{\alpha^5}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8} = 1 + 2^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}. \tag{2} $$

In the third volume of his remarkable edition on Ramanujan's notebooks, Berndt [2] proved (1) and (2) by employing the method of parametrization. Applying the transformation formulas found in [2, pp. 122–124, Entry 10, Entry 11, Entry 12], Berndt showed that these modular equations are equivalent to the following respective theta function identities.

Theorem 1. *We have the identities*

$$ \frac{\varphi^5(q)}{\varphi(q^5)} + \frac{\varphi^5(-q^2)}{\varphi(-q^{10})} + 2\frac{f^5(q)}{f(q^5)} = 4\frac{\psi^5(q)}{\psi(q^5)}, \tag{3} $$

$$ \varphi^2(q)\varphi^2(q^5) - \varphi^2(-q)\varphi^2(-q^5) - 16q^3\psi^2(q^2)\psi^2(q^{10}) = 8qf^2(-q^2)f^2(-q^{10}). \tag{4} $$

Berndt [2, p. 285] remarked that no direct proofs of (3) and (4) have been constructed. Baruah, Bora, and Ojah [1] proved Equation (4) using certain theta function identities.

In this note, we present a new proof of Theorem 1 by applying certain identities involving the Ramanujan's function $k(q) := r(q)r^2(q^2)$ due to Cooper [4] and Chern and Tang [3], where $r(q)$ is the Rogers–Ramanujan continued fraction

$$ r(q) := \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}} = q^{1/5} \frac{(q, q^4; q^5)_\infty}{(q^2, q^3; q^5)_\infty}. $$

We then see that $k(q)$ has the infinite product representation given by

$$k(q) = q\frac{(q, q^2, q^8, q^9; q^{10})_\infty}{(q^3, q^4, q^6, q^7; q^{10})_\infty}.$$

The function $k(q)$ first appears in the second notebook [7, p. 326] of Ramanujan, who recorded the identities

$$r^5(q) = k(q)\left(\frac{1-k(q)}{1+k(q)}\right)^2, \qquad r^5(q^2) = k^2(q)\left(\frac{1+k(q)}{1-k(q)}\right).$$

In his lost notebook [8, p. 56], Ramanujan also recorded the identities

$$\begin{aligned} r(q^{1/2}) &= \frac{k^{1/10}(q)(1+k(q))^{4/5}(1-k(q))^{1/5}}{\sqrt{k(q)} + \sqrt{1+k(q)-k^2(q)}}, \\ r(q^4) &= \left(\frac{1-k(q)}{1+k(q)}\right)^{1/10} \frac{2k^{4/5}(q)}{\sqrt{1-k^2(q)} + \sqrt{1-4k(q)-k^2(q)}}, \end{aligned}$$

which were established by Kang [5] using theta function identities. We refer the reader to [4, 5, 9] for a list of identities involving $k(q)$. We remark that the aforementioned proof relies on a certain polynomial relation between $k(q)$ and $k(q^2)$ (see Theorem 4).

2. A New Proof of Theorem 1

We provide a new proof of Theorem 1 in this section. We start with listing the following important identities involving $k(q)$ found by Cooper [4, Theorem 3.5] and Chern and Tang [3, Theorem 3.3].

Lemma 2. *We have the identities*

$$\frac{f(-q^2)f^5(-q^5)}{qf(-q)f^5(-q^{10})} = \frac{1}{k(q)} - k(q), \tag{5}$$

$$\frac{f^4(-q^2)f^2(-q^5)}{qf^2(-q)f^4(-q^{10})} = \frac{1}{k(q)} + 1 - k(q), \tag{6}$$

$$\frac{f(-q)^3f(-q^5)}{qf(-q^2)f^3(-q^{10})} = \frac{1}{k(q)} - 4 - k(q). \tag{7}$$

Lemma 3. *We have the identity*

$$\frac{k(q)}{k(q^2)} - \frac{k(q^2)}{k(q)} = \frac{f(-q)f^3(-q^5)}{qf^4(-q^{10})}.$$

A consequence of Lemma 3 is the following polynomial relation between $k(q)$ and $k(q^2)$, which will play a key role in the proof of Theorem 1. We remark that Lee and Park [6] recently obtained this relation using the theory of modular functions.

Theorem 4. *We have the identity*

$$X^2 - Y + 2XY + X^2Y + Y^2 = 0,$$

where $X := k(q)$ *and* $Y := k(q^2)$.

Proof. We square the identity in Lemma 3 and use (5) and (7), so that

$$\begin{aligned}\left(\frac{X}{Y} - \frac{Y}{X}\right)^2 = \frac{f^2(-q)f^6(-q^5)}{q^2 f^8(-q^{10})} &= \frac{f(-q^2)f^5(-q^5)}{qf(-q)f^5(-q^{10})} \cdot \frac{f(-q)^3 f(-q^5)}{qf(-q^2)f^3(-q^{10})}\\ &= \left(\frac{1}{X} - X\right)\left(\frac{1}{X} - 4 - X\right).\end{aligned}$$

Expanding the above equation leads to

$$\frac{(X^2 + Y - 2XY - X^2Y + Y^2)(X^2 - Y + 2XY + X^2Y + Y^2)}{X^2Y^2} = 0. \tag{8}$$

Using the q-expansion

$$k(q) = q - q^2 - q^3 + 2q^4 - 2q^6 + 2q^7 + O(q^8)$$

of $k(q)$, we note that the second factor of the numerator of Equation (8) vanishes while the first factor does not. Hence, the desired identity follows. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. We begin with proving Equation (3). Replacing q with $-q$ and noting that $\psi(-q) = f(-q)f(-q^4)/f(-q^2)$ and $\varphi(-q) = f^2(-q)/f(-q^2)$, we rewrite Equation (3) in terms of $f(-q)$ and multiply both sides by $f(-q^5)/f^5(-q)$ so that

$$\frac{f^5(-q)f(-q^{10})}{f^5(-q^2)f(-q^5)} + \frac{f^{10}(-q^2)f(-q^5)f(-q^{20})}{f^5(-q)f^5(-q^4)f^2(-q^{10})} + 2 = 4\frac{f^5(-q^4)f(-q^{10})}{f^5(-q^2)f(-q^{20})}. \tag{9}$$

From Lemma 2, we have

$$\begin{aligned}A &:= \frac{f^5(-q)f(-q^{10})}{f^5(-q^2)f(-q^5)} = \frac{1-4X-X^2}{1+X-X^2},\\ B &:= \frac{f^5(-q^2)f(-q^{20})}{f^5(-q^4)f(-q^{10})} = \frac{1-4Y-Y^2}{1+Y-Y^2},\end{aligned}$$

where $X := k(q)$ and $Y := k(q^2)$. Then

$$\frac{5}{1-A} = \frac{1}{X} - X + 1, \tag{10}$$

$$C := \frac{5}{1-B} = \frac{1}{Y} - Y + 1. \tag{11}$$

In view of Theorem 4 and Equation (11), we deduce that

$$1 - 2X - X^2 = \frac{X^2}{Y} + Y = X^2(C + Y - 1) + Y = X^2(C-1) + Y(X^2+1),$$

and solving this equation for Y, we get

$$Y = \frac{1 - 2X - X^2C}{X^2+1}. \tag{12}$$

Substituting Equation (12) into Theorem 4 and clearing denominators, we obtain

$$X^2C^2 - (1 - 2X + 2X^3 + X^4)C + (1 + X - X^2)^2 = 0,$$

or, in view of Equation (10),

$$C^2 - \left(\frac{1}{X^2} - \frac{2}{X} + 2X + X^2\right)C + \frac{25}{(1-A)^2} = 0. \tag{13}$$

Observe from Equation (10) that

$$\frac{1}{X^2} - \frac{2}{X} + 2X + X^2 = \left(\frac{1}{X} - X\right)^2 - 2\left(\frac{1}{X} - X\right) + 2 = \frac{5(A^2+2A+2)}{(1-A)^2}. \tag{14}$$

Thus, combining (11), (13), and (14), we obtain

$$\frac{25}{(1-B)^2} - \frac{5(A^2+2A+2)}{(1-A)^2} \cdot \frac{5}{1-B} + \frac{25}{(1-A)^2} = 0,$$

which, after clearing denominators, simplifies to

$$B^2 + A^2B + 2AB - 4A = 0. \tag{15}$$

Thus, we infer from Equation (15) that

$$\begin{aligned}\frac{f^5(-q)f(-q^{10})}{f^5(-q^2)f(-q^5)} + \frac{f^{10}(-q^2)f(-q^5)f(-q^{20})}{f^5(-q)f^5(-q^4)f^2(-q^{10})} + 2 = A + \frac{B}{A} + 2 = \frac{A^2+2A+B}{A} \\ = \frac{4}{B} = 4\frac{f^5(-q^4)f(-q^{10})}{f^5(-q^2)f(-q^{20})},\end{aligned}$$

which is exactly Equation (9). This completes the proof of Equation (3).

We next establish Equation (4). We rewrite this in terms of $f(-q)$ and divide both sides by $qf^2(-q^2)f^2(-q^{10})$ so that

$$\frac{f^8(-q^2)f^8(-q^{10})}{qf^4(-q)f^4(-q^4)f^4(-q^5)f^4(-q^{20})} - \frac{f^4(-q)f^4(-q^5)}{qf^4(-q^2)f^4(-q^{10})} - 16q^2\frac{f^4(-q^4)f^4(-q^{20})}{f^4(-q^2)f^4(-q^{10})} = 8. \tag{16}$$

Using Lemma 2, we set

$$P := \frac{f^4(-q)f^4(-q^5)}{qf^4(-q^2)f^4(-q^{10})} = \left(\frac{1}{X} - 4 - X\right)\frac{1-X^2}{1+X-X^2},$$
$$Q := \frac{f^4(-q^2)f^4(-q^{10})}{q^2f^4(-q^4)f^4(-q^{20})} = \left(\frac{1}{Y} - 4 - Y\right)\frac{1-Y^2}{1+Y-Y^2}.$$

We see from the definitions of A and B that

$$P = \frac{A(A+4)}{1-A}, \tag{17}$$

$$Q = \frac{B(B+4)}{1-B}. \tag{18}$$

We rewrite Equation (17) as

$$P\left(\frac{1}{A} - 1\right) = A + 4.$$

Multiplying both sides of the above equation by B and invoking Equation (15), we infer that

$$AB + 4B = P\left(\frac{B}{A} - B\right) = P\left(\frac{4}{B} - A - B - 2\right),$$

which yields

$$A = \frac{4P - 2BP - B^2(P+4)}{B(B+P)}. \tag{19}$$

Plugging in Equation (19) into Equation (15) and clearing denominators, we deduce that

$$B(B+4)(B-1)P^2 + 8(B-1)(B^2+2B+2)P + B^2(B+4)^2 = 0. \tag{20}$$

We note from Equation (18) that

$$\frac{B^2+2B+2}{1-B} = \frac{B^2+4B}{1-B} + 2 = Q + 2. \tag{21}$$

Thus, dividing both sides of Equation (20) by $(1-B)^2$ and applying (18) and (21), we arrive at

$$-P^2Q - 8PQ - 16P + Q^2 = 0,$$

which implies that

$$\frac{f^8(-q^2)f^8(-q^{10})}{qf^4(-q)f^4(-q^4)f^4(-q^5)f^4(-q^{20})} - \frac{f^4(-q)f^4(-q^5)}{qf^4(-q^2)f^4(-q^{10})} - 16q^2\frac{f^4(-q^4)f^4(-q^{20})}{f^4(-q^2)f^4(-q^{10})}$$
$$= \frac{Q}{P} - P - \frac{16}{Q} = \frac{Q^2 - P^2Q - 16P}{PQ} = 8$$

as desired. This completes the proofs of (16) and (4). $\square$

Acknowledgement. The author would like to thank the anonymous referee and the managing editor Bruce Landman for helpful comments that improved the clarity and presentation of this paper.

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