



THE ERDŐS-STRAUS CONJECTURE AND THE STRUCTURE OF PRIMES

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Abstract

The famous Erdős-Straus Conjecture states that every fraction $4/n$, where $n \geq 2$, can be written as the sum of three unit fractions. This paper shows that if p is a prime, then $4/p$ can be written as the sum of three unit fractions, where exactly two of the denominators are multiples of p , if and only if $p = qr - 4s_1s_2$, where $q \equiv 3 \pmod{4}$ and both s_1 and s_2 are divisors of $(q+1)/4$. It is conjectured that all primes can be written in this form.

1. Introduction

The following unsolved number theory problem has attracted attention for decades.

Conjecture 1. (Erdős-Straus) For every integer $n \geq 2$, we can write

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \tag{1}$$

for some $a, b, c \in \mathbb{Z}^+$.

Dating back to 1948, the conjecture has been verified [6] for all $n \leq 10^{17}$. Obviously we can write $4/n$ as the sum of four unit fractions (a greedy algorithm usually produces four distinct unit fractions), but the conjecture asserts that three unit fractions are always sufficient.

A well-studied approach has been to consider various residue classes where the desired decomposition is possible. For example, the equation

$$\frac{4}{n} = \frac{1}{n} + \frac{1}{(n+1)/3} + \frac{1}{n(n+1)/3}$$

yields a solution to (1) whenever $n \equiv 2 \pmod 3$. This can be made explicit by writing $n = 3k + 2$ to produce

$$\frac{4}{3k + 2} = \frac{1}{3k + 2} + \frac{1}{k + 1} + \frac{1}{(k + 1)(3k + 2)}.$$

Formulas of this type can be constructed to cover many residue classes, leaving one to wonder whether covering all $n \geq 2$ is possible. Indeed, Webb [8] used sieving methods to show that $4/n$ can be written in the form (1) for almost all n . Covering all the positive integers with polynomial identities, however, is impossible, as argued by Mordell [4, pp.287–290]. A result of Schinzel [7] makes this explicit: the equation

$$\frac{4}{ax + b} = \frac{1}{F_1(x)} + \frac{1}{F_2(x)} + \frac{1}{F_3(x)}$$

has no solution in polynomials F_1, F_2, F_3 with integer coefficients and positive leading coefficients if b is a quadratic residue modulo a .

The work of Elsholtz and Tao [3] also points to the limitations of the covering approach. For a given n , we say (a, b, c) is a *Type I* solution of Equation (1) if exactly one of the values is divisible by n , and a *Type II* solution if exactly two of the values are divisible by n . They showed that if n is odd, then n^2 has neither Type I nor Type II solutions. This does not mean that the Erdős-Straus Conjecture is false; if $4/n$ has a solution, then multiplying each denominator by n gives a solution for $4/n^2$. These limitations simply point to the need for a different approach.

Since

$$\frac{4}{mn} = \frac{1}{m} \cdot \frac{4}{n},$$

it is well-known that it suffices to restrict our attention to $4/p$ for all prime numbers p . It has been shown [1, 3] that if p is prime and Equation (1) is satisfied for $4/p$ with $a \leq b \leq c$, then $p \nmid a$ and $p|c$. Numerical evidence suggests that for all odd primes p , $4/p$ has both Type I and Type II solutions. The main result of this paper asserts that $4/p$ has a Type II solution if and only if p can be expressed in a particular, simple form.

Theorem 1. *Let p be a prime number. Then there exist $a, b, c \in \mathbb{Z}^+$ such that $a \leq b \leq c$, $p \nmid a$, p is a divisor of both b and c , and*

$$\frac{4}{p} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

if and only if

$$p = qr - 4s_1s_2 \tag{2}$$

for some $q, r, s_1, s_2 \in \mathbb{Z}^+$ such that $q \equiv 3 \pmod 4$ and both s_1 and s_2 are divisors of $(q + 1)/4$.

Other results connecting the structure of primes to the Erdős-Straus Conjecture can be found in Bradford [2]. Section 2 offers a proof of the Theorem. Section 3 explores the structure of primes in light of the Theorem.

2. Proof of the Theorem

To prove one direction of the theorem, we will use a non-obvious result due to Bradford [1]: if

$$\frac{4}{p} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

where p is prime and $a \leq b \leq c$, then

$$c = \frac{abp}{\gcd(b, p) \gcd(ab, a + b)}.$$

For Type II solutions, this simplifies to

$$c = \frac{ab}{\gcd(ab, a + b)}. \tag{3}$$

Proof of Theorem 1. To settle the forward direction of the proof, assume that we are given $a, b, c \in \mathbb{Z}^+$ such that $a \leq b \leq c$, $p \nmid a$, p divides both b and c , and

$$\frac{4}{p} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Define

$$q = \frac{4b}{p} - 1, \quad r = 4a - p,$$

$$s_1 = \gcd\left(\frac{ab}{\gcd(a, b)}, \frac{a + b}{\gcd(a, b)}\right), \quad s_2 = \gcd(a, b).$$

The conditions on a, b, c guarantee that $q, r, s_1, s_2 \in \mathbb{Z}^+$ and $q \equiv 3 \pmod{4}$. We also have $(q + 1)/4 = b/p$, and since $\gcd(a, b) = \gcd(a, b/p)$ is a divisor of b/p , this yields $s_2 | (q + 1)/4$.

Showing $s_1 | (q + 1)/4$ is more involved. First note that we can write $a = s_2x$ and $b = s_2y$ for some $x, y \in \mathbb{Z}^+$ with $\gcd(x, y) = 1$, thus implying $s_1 = \gcd(s_2xy, x + y)$. If t is a prime such that $t^k | s_1$ for some $k \in \mathbb{Z}^+$, then $t | (x + y)$. This forces $t \nmid x$ and $t \nmid y$, hence $t^k | s_2$. This argument applies to all primes t , hence $s_1 | s_2$. Since $s_2 | (q + 1)/4$, we have $s_1 | (q + 1)/4$.

Lastly, assembling these components together and using (3) yields

$$\begin{aligned} qr - 4s_1s_2 &= \left(\frac{4b}{p} - 1\right) \cdot (4a - p) - 4 \gcd\left(\frac{ab}{\gcd(a, b)}, \frac{a + b}{\gcd(a, b)}\right) \cdot \gcd(a, b) \\ &= \frac{16ab}{p} - 4a - 4b + p - 4 \gcd(ab, a + b) \\ &= p + 4ab \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 4a - 4b - 4 \gcd(ab, a + b) \\ &= p + 4 \left(\frac{ab}{c} - \gcd(ab, a + b)\right) \\ &= p. \end{aligned}$$

To settle the reverse direction of the proof, assume that $p = qr - 4s_1s_2$ for some $q, r, s_1, s_2 \in \mathbb{Z}^+$ such that $q \equiv 3 \pmod{4}$ and both s_1 and s_2 are divisors of $(q + 1)/4$. The following equation holds for all scalars q, r, s_1, s_2 when the denominators are non-zero:

$$\frac{4}{qr - 4s_1s_2} = \frac{1}{r\left(\frac{q+1}{4}\right) - s_1s_2} + \frac{1}{\left(\frac{q+1}{4}\right) \cdot (qr - 4s_1s_2)} + \frac{s_1s_2}{\left(\frac{q+1}{4}\right) \cdot \left(r\left(\frac{q+1}{4}\right) - s_1s_2\right) \cdot (qr - 4s_1s_2)}. \tag{4}$$

For this direction of the proof, the conditions imply $(q + 1)/4 \in \mathbb{Z}^+$ and the three denominators of Equation (4) are positive integers. Since both s_1 and s_2 divide $(q + 1)/4$, dividing the top and bottom of the last fraction by s_1s_2 and simplifying implies that the right side of (4) is the sum of three unit fractions. The last two unit fractions have $p = qr - 4s_1s_2$ in their denominators, thus implying

$$a = r\left(\frac{q+1}{4}\right) - s_1s_2$$

and b and c take the values

$$\left(\frac{q+1}{4}\right) \cdot (qr - 4s_1s_2) \tag{5}$$

and

$$\frac{1}{s_1}\left(\frac{q+1}{4}\right) \cdot \left(\frac{r}{s_2}\left(\frac{q+1}{4}\right) - s_1\right) \cdot (qr - 4s_1s_2), \tag{6}$$

arranged so that $b \leq c$. □

Note that it is unclear which of the expressions (5) or (6) is larger. For example, the possible Type II representations of $4/41$ using Equation (4), omitting the swapping of s_1 and s_2 , are listed in Table 1.

3. The Structure of Primes

The theorem behooves us to determine when a prime p takes the form $p = qr - 4s_1s_2$. Indeed, the extensive testing on the Erdős-Straus Conjecture suggests the following claim.

Conjecture 2. (Prime Representation Conjecture) Every prime p can be written in the form $p = qr - 4s_1s_2$ for some $q, r, s_1, s_2 \in \mathbb{Z}^+$ such that $q \equiv 3 \pmod{4}$ and both s_1 and s_2 are divisors of $(q + 1)/4$.

q	r	s_1	s_2	representation of $4/41$
3	15	1	1	$\frac{1}{14} + \frac{1}{41} + \frac{1}{574}$
7	7	1	2	$\frac{1}{12} + \frac{1}{82} + \frac{1}{492}$
11	7	3	3	$\frac{1}{12} + \frac{1}{123} + \frac{1}{164}$
15	3	1	1	$\frac{1}{11} + \frac{1}{164} + \frac{1}{1804}$
15	7	4	4	$\frac{1}{12} + \frac{1}{164} + \frac{1}{123}$
47	7	6	12	$\frac{1}{12} + \frac{1}{492} + \frac{1}{82}$

Table 1: Type II representations of $4/41$.

The trivial case $p = 2$ is settled with $2 = 3 \cdot 2 - 4 \cdot 1 \cdot 1$, corresponding, via Equation (4), to $4/2 = 1/1 + 1/2 + 1/2$. More interesting is when $p \equiv 3 \pmod 4$, which can be settled in at least two ways:

$$p = (2p + 1) \cdot 1 - 4 \cdot 1 \cdot \left(\frac{p + 1}{4}\right) \Leftrightarrow \frac{4}{p} = \frac{1}{\left(\frac{p+1}{4}\right)} + \frac{1}{\left(\frac{p+1}{2}\right)p} + \frac{1}{\left(\frac{p+1}{2}\right)p}$$

and

$$p = (p + 4) \cdot 1 - 4 \cdot 1 \cdot 1 \Leftrightarrow \frac{4}{p} = \frac{1}{\left(\frac{p+1}{4}\right)} + \frac{1}{\left(\frac{p+5}{4}\right)p} + \frac{1}{\left(\frac{p+1}{4}\right)\left(\frac{p+5}{4}\right)p}.$$

For any prime p , the divisibility conditions required for Equation (2) are automatically met when s_1 and s_2 are restricted to the values 1 and $(q + 1)/4$. Since r is unrestricted, it can be shifted, if necessary, to r' :

$$q \cdot r - 4 \cdot 1 \cdot 1 = q \cdot r - 4, \tag{7}$$

$$q \cdot r - 4 \cdot 1 \cdot \left(\frac{q + 1}{4}\right) = q \cdot r' - 1, \tag{8}$$

$$q \cdot r - 4 \cdot \left(\frac{q + 1}{4}\right) \cdot \left(\frac{q + 1}{4}\right) = q \cdot r' - \left(\frac{q + 1}{4}\right). \tag{9}$$

Numerical exploration suggests that most primes can be represented with one of these three forms. The corresponding forms for Equation (4) (without ordering b and c) follow:

$$\begin{aligned} \frac{4}{qr - 4} &= \frac{1}{r \left(\frac{q+1}{4}\right) - 1} + \frac{1}{\left(\frac{q+1}{4}\right) \cdot (qr - 4)} + \frac{1}{\left(\frac{q+1}{4}\right) \cdot \left(r \left(\frac{q+1}{4}\right) - 1\right) \cdot (qr - 4)}, \\ \frac{4}{qr - 1} &= \frac{1}{r \left(\frac{q+1}{4}\right)} + \frac{1}{\left(\frac{q+1}{4}\right) \cdot (qr - 1)} + \frac{1}{r \left(\frac{q+1}{4}\right) \cdot (qr - 1)}, \\ \frac{4}{qr - \frac{q+1}{4}} &= \frac{1}{r \left(\frac{q+1}{4}\right)} + \frac{1}{\left(\frac{q+1}{4}\right) \cdot \left(qr - \frac{q+1}{4}\right)} + \frac{1}{r \left(qr - \frac{q+1}{4}\right)}. \end{aligned}$$

$q = 3$	$3r - 1$	$q = 19$	$19r - 1$
$q = 7$	$7r - 1$		$19r - 4$
	$7r - 2$		$19r - 5$
	$7r - 4$		
$q = 11$	$11r - 1$	$q = 23$	$23r - 1$
	$11r - 3$		$23r - 2$
	$11r - 4$		$23r - 3$
$q = 15$			$23r - 4$
	$15r - 1$		$23r - 6$
	$15r - 2$		$23r - 8$
	$15r - 4$		$23r - 12$
			$23r - 13$
			$23r - 16$
	$15r - 8$		

Table 2: Residue classes covered for $q \leq 23$.

These forms are essentially those listed in [5, p.207] and [9].

Not all primes, however, fit one of the forms (7)-(9). The prime $p = 193$ is such an example, but it can be written in four other ways (not counting swapping s_1 and s_2), each with $1 < s_1 < (q + 1)/4$ and $s_2 = (q + 1)/4$:

$$193 = 15 \cdot 15 - 4 \cdot 2 \cdot 4 = 39 \cdot 7 - 4 \cdot 2 \cdot 10 = 99 \cdot 7 - 4 \cdot 5 \cdot 25 = 103 \cdot 15 - 4 \cdot 13 \cdot 26.$$

For most values of $q \equiv 3 \pmod{4}$, the number $(q + 1)/4$ is not prime, so other choices of s_1 and s_2 may produce other congruence classes besides Equations (7)–(9). Table 2 lists these congruence classes covered for $q \leq 23$, where any shifted values of r are simply written as r . The prime $p = 211$ cannot be written with $q \leq 23$, but one has $211 = 43 \cdot 5 - 4 \cdot 1 \cdot 1$. Further computation finds that the prime $p = 1009$ is the smallest prime where $s_1 \geq 3$ and $s_2 \geq 3$ are necessary; one solution is $1009 = 23 \cdot 47 - 4 \cdot 3 \cdot 6$.

References

- [1] K. Bradford, *A note on the Erdős-Straus conjecture*, *Integers* **21** (2021), # A24, 10pp.
- [2] K. Bradford, *Elementary patterns from the Erdős-Straus conjecture*, *Integers* **25** (2025), # A54, 10pp.
- [3] C. Elsholtz and T. Tao, *Counting the number of solutions to the Erdős-Straus equation on unit fractions*, *J. Aust. Math. Soc.*, **94** (1) (2013), 50–105.
- [4] L.J. Mordell. *Diophantine Equations*, Academic Press, New York, 1967.
- [5] P. Pollack. *Not Always Buried Deep. A Second Course in Elementary Number Theory*, AMS, Providence, 2009.
- [6] S. Salez, *The Erdős-Straus conjecture New modular equations and checking up to $N = 10^{17}$* , preprint, [arXiv:1406.6307](https://arxiv.org/abs/1406.6307).

- [7] A. Schinzel, *On sums of three unit fractions with polynomial denominators*, *Funct. Approx. Comment. Math.*, **28** (2000), 187–194.
- [8] W.A. Webb, *On $4/n = 1/x_1 + 1/x_2 + 1/x_3$* , *Proc. Amer. Math. Soc.*, **25** (3) (1970), 578–584.
- [9] X.Q. Yang, *A Note on $4/n = 1/x + 1/y + 1/z$* , *Proc. Amer. Math. Soc.*, **85** (4) (1982), 496–498.