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ASYMPTOTICS OF PARTIAL SUMS OF DIRICHLET CONVOLUTIONS OF DIGITAL SUM FUNCTIONS IN ARBITRARY BASES

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Abstract

The sum of digital sum functions has been studied extensively, asymptotically and in terms of error terms. Some variations of the sum of digital sum functions were also studied previously—for example, the moments (the sum of powers) of the binary digital sum functions. In 2018, Srichan studied another variation: the partial sums of the Dirichlet convolution of the binary digital sum function. He found an asymptotic approximation and its error term. Srichan further extended his results to arbitrary k-th convolutions. In this paper, we generalize his results to digital sum functions in arbitrary bases. In contrast to previous work in the binary case, our analysis corrects a reliance on integer-only versions of the Trollope–Delange formula and establishes uniform asymptotic results for Dirichlet convolutions of digital sum functions in arbitrary bases.

1. Introduction

For integers $q > 1$ and $n > 0$, we define the digital sum function in base q, denoted $s_q(n)$, by

$$s_q(n) = \sum_{i=0}^{m} a_i$$

where $n = \sum_{i=0}^{m} a_i q^i$ is the base-q expansion of n with integers $m \geq 0$, $0 \leq a_i < q$, and $a_m > 0$. We also set $s_q(0) = 0$.

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Bush [3] first showed in 1940 that

$$\sum_{n\le x} s_q(n) \sim \frac{q-1}{2\log q} x\log x.$$

In 1948, Bellman and Shapiro [2] proved that when $q=2$,

$$\sum_{n\le x} s_2(n) = \frac{1}{2} x\log_2 x + \mathcal{O}(x\log\log x).$$

Their method was also revisited in an undergraduate project by Gadd and the second author [9], who obtained the same error term for arbitrary bases, although the optimal bound had already been established. In 1949, Mirsky [11] proved that

$$\sum_{n\le x} s_q(n) = \frac{q-1}{2} x\log_q x + \mathcal{O}(x),$$

where $\mathcal{O}(x)$ is the best possible error term. This result has since been reproved by various authors using different methods (see [1], [4], and [12]). In 1968, Trollope [14] derived an explicit formula for the error term in the case $q=2$. In 1975, Delange [7] extended Trollope's results to all bases, using a simpler method. His theorem can be stated as follows:

Theorem 1. *Let $F:\mathbb{R}\to\mathbb{R}$ be the continuous, 1-periodic, nowhere differentiable function defined by*

$$F(x) = \frac{q-1}{2}(1+\lfloor x\rfloor - x) + q^{1+\lfloor x\rfloor - x} h(q^{x-\lfloor x\rfloor -1}),$$

where

$$h(x) = \sum_{r=0}^{\infty} \frac{g(q^r x)}{q^r},$$

and

$$g(x) = \int_0^x \left(\lfloor qt\rfloor - q\lfloor t\rfloor - \frac{q-1}{2}\right) dt.$$

Then

$$\sum_{n\le x} s_q(n) = \frac{q-1}{2\log q} x\log x + xF\left(\frac{\log x}{\log q}\right) - h(x) + (1+\lfloor x\rfloor - x)s_q(\lfloor x\rfloor). \qquad (1)$$

There is another version of the formula due to Cooper and Kennedy [5], who extended Trollope's method from base 2 to arbitrary bases. Their expression for the error term, however, is valid only for natural numbers x. In a recent paper [8], the present authors proved directly that the error-term expressions obtained by Trollope's method, as extended by Cooper and Kennedy, and by Delange's analytic approach coincide for all natural numbers x. This confirms the consistency of the two approaches in the integer case.

As noted by Srichan [13], the Trollope–Delange formula is a powerful tool for studying problems involving digital sum functions. For example, the moments of the binary digital sum function were studied by Coquet [6] and later refined by Grabner, Kirschenhofer, Prodinger and Tichy [10]:

$$\sum_{n<x} s_2(n)^s = \left(\frac{\log_2 x}{2}\right)^2 x + x\sum_{j=1}^{s-1}(\log_2 x)^j \nu_j(\log_2 x),$$

where $\nu_j(x)$ are continuous nowhere differentiable functions of period 1.

As a variation, Srichan [13] studied the sum of the Dirichlet convolution of the binary digital sum function and derived asymptotics for both the second convolution and its higher-order analogues. In this paper, we generalize his results to arbitrary bases. Section 2 states our main theorems, Section 3 develops the necessary lemmas, and Section 4 contains the proofs of the main theorems. As a note, the function F in our paper when $q = 2$ is equal to Srichan's $\frac{F}{2}$ in his paper. Thus, you may notice that the coefficient $\frac{1}{2}$ is gone in all the results here.

While Srichan's analysis of Dirichlet convolutions of the binary digital sum function yields correct leading-order asymptotics, it relies on a version of the Trollope–Delange formula that applies only to integer arguments. In this paper, we address this issue by working systematically with the full real-variable version of the formula and by extending the analysis to digital sum functions in arbitrary bases. This allows us to derive uniform asymptotic formulas for k-fold Dirichlet convolutions and to clarify the role of error terms in the general setting.

2. Main Results

Define

$$s_q^{(2)}(n) := s_q * s_q(n) = \sum_{d|n} s_q(d)s_q(n/d).$$

To derive a formula for $\sum_{n\leq x} s_2^{(2)}(n)$, for $x > 1$, Srichan first derived an upper bound and a lower bound for the sum. However, he mistakenly used the natural number version of the Trollope–Delange formula instead of the more general version that holds for real numbers. Although the missing component in the error term did not affect the final result, several intermediate formulas should be corrected. However, the upper and lower bounds for the sum are too long to present if we add back the missing terms. Instead, we will prove the asymptotic formula for arbitrary bases directly by estimating the growth of each term in this paper.

Theorem 2. *As $x \to \infty$, we have*

$$\sum_{n \le x} s_q^{(2)}(n) = \frac{(q-1)^2 x \log^3 x}{24 \log^2 q} + \mathcal{O}(x \log^2 x).$$

Remark 1. When $q = 2$, Theorem 2 recovers Srichan's main asymptotic result. The proof given here avoids reliance on intermediate bounds that depend on integer-only formulas and extends without modification to arbitrary bases.

Srichan extended his result to the k-th convolution of the binary digital sum for any integer $k \ge 2$. Here we will extend our result to the k-convolution as well. For $k \ge 2$, we define the k-convolution for an arbitrary base digital sum as

$$s_q^{(k)}(n) := s_q * s_q^{(k-1)}(n) = \sum_{d|n} s_q(d) s_q^{(k-1)}(n/d).$$

Now, we have the following result.

Theorem 3. *For any integer $k \ge 2$,*

$$\sum_{n \le x} s_q^{(k)}(n) = \frac{a(k)}{24 \log^2 q} x \log^{2k-1} x + O(x \log^{2k-2} x), \tag{2}$$

where

$$a(2) = (q-1)^2, a(k) = \frac{a(k-1)}{2 \log q} \sum_{j=0}^{2k-3} \binom{2k-3}{j} \frac{(-1)^j}{j+2} (q-1).$$

3. Lemmas

Throughout this section, we repeatedly use Abel's summation formula (partial summation). For an arithmetic function $a(n)$ with $A(x) = \sum_{n \le x} a(n)$ and a continuously differentiable function $\phi(x)$ on $[1, x]$, Abel's summation formula states that

$$\sum_{n \le x} a(n)\phi(n) = A(x)\phi(x) - \int_1^x A(u)\phi'(u)\, du. \tag{3}$$

We apply (3) with suitable choices of $\phi(x)$ to derive asymptotic formulas for weighted sums involving $s_q(n)$.

Lemma 1. *As $x \to \infty$,*

$$\sum_{n \le x} \frac{s_q(n)}{n} = \frac{(q-1)\log^2 x}{4 \log q} + \frac{(q-1)\log x}{2 \log q} + F\left(\frac{\log x}{\log q}\right) + G(x) + \frac{J(x)}{x} + K(x), \tag{4}$$

$$\begin{aligned}\sum_{n\le x}\frac{s_q(n)\log n}{n} &= \frac{(q-1)\log^3 x}{6\log q}+\frac{(q-1)\log^2 x}{4\log q}+\log x\cdot F\left(\frac{\log x}{\log q}\right)\\ &\quad +(\log x-1)G(x)-H(x)+\frac{\log x}{x}J(x)\\ &\quad +(\log x-1)K(x)-L(x), \end{aligned}\tag{5}$$

where

$$J(x)=(1+\lfloor x\rfloor-x)s_q(\lfloor x\rfloor)-h(x),\qquad G(x)=\int_1^x F\left(\frac{\log u}{\log q}\right)\frac{du}{u},$$

$$K(x)=\int_1^x J(u)\,\frac{du}{u^2},\qquad H(x)=\int_1^x\int_1^u F\left(\frac{\log v}{\log q}\right)\frac{dvdu}{vu},$$

$$\textit{and}\quad L(x)=\int_1^x\int_1^u J(v)\,\frac{dvdu}{v^2u}.$$

Proof. Applying (3) with $\phi(x)=\frac{1}{x}$ and $A(x)=\sum\limits_{n\le x}s_q(n)$, together with (1), we get

$$\begin{aligned}\sum_{n\le x}\frac{s_q(n)}{n} &= A(x)\phi(x)-\int_1^x A(u)\phi'(u)\,du\\ &= \frac{1}{x}\left(\frac{(q-1)x\log x}{2\log q}+xF\left(\frac{\log x}{\log q}\right)+J(x)\right)\\ &\qquad +\int_1^x\left(\frac{(q-1)u\log u}{2\log q}+uF\left(\frac{\log u}{\log q}\right)+J(u)\right)\frac{du}{u^2}\\ &= \frac{(q-1)\log x}{2\log q}+F\left(\frac{\log x}{\log q}\right)+\frac{J(x)}{x}+\frac{(q-1)}{2\log q}\int_1^x\frac{\log u}{u}\,du\\ &\qquad +\int_1^x F\left(\frac{\log u}{\log q}\right)\frac{du}{u}+\int_1^x J(u)\frac{du}{u^2}\\ &= \frac{(q-1)\log x}{2\log q}+F\left(\frac{\log x}{\log q}\right)+\frac{J(x)}{x}+\frac{(q-1)\log^2 x}{4\log q}+G(x)+K(x)\\ &= \frac{(q-1)\log^2 x}{4\log q}+\frac{(q-1)\log x}{2\log q}+F\left(\frac{\log x}{\log q}\right)+G(x)+\frac{J(x)}{x}+K(x).\end{aligned}$$

Applying (3) again with $\phi(x)=\log x$ and $A(x)=\sum\limits_{n\le x}\frac{s_q(n)}{n}$, together with (4), we get

$$\begin{aligned}
\sum_{1\le n\le x} \frac{s_q(n)\log n}{n} &= \log x\Bigg(\frac{(q-1)\log^2 x}{4\log q} + \frac{(q-1)\log x}{2\log q} + F\left(\frac{\log x}{\log q}\right) + G(x)\\
&\quad + \frac{J(x)}{x} + K(x)\Bigg)\\
&\quad - \int_1^x \Bigg(\frac{(q-1)\log^2 u}{4\log q} + \frac{(q-1)\log u}{2\log q} + F\left(\frac{\log u}{\log q}\right) + G(u)\\
&\quad + \frac{J(u)}{u} + K(u)\Bigg)\frac{du}{u}\\
&= \frac{(q-1)\log^3 x}{4\log q} + \frac{(q-1)\log^2 x}{2\log q} + \log x\cdot F\left(\frac{\log x}{\log q}\right)\\
&\quad + \log x\cdot G(x) + \frac{\log x}{x}J(x) + \log x\cdot K(x)\\
&\quad - \frac{(q-1)}{4\log q}\int_1^x \frac{\log^2 u}{u}\,du - \frac{(q-1)}{2\log q}\int_1^x \frac{\log u}{u}\,du\\
&\quad - \int_1^x F\left(\frac{\log u}{\log q}\right)\frac{du}{u} - \int_1^x G(u)\,\frac{du}{u} - \int_1^x J(u)\,\frac{du}{u^2}\\
&\quad - \int_1^x K(u)\,\frac{du}{u}\\
&= \frac{(q-1)\log^3 x}{4\log q} + \frac{(q-1)\log^2 x}{2\log q} + \log x\cdot F\left(\frac{\log x}{\log q}\right)\\
&\quad + \log x\cdot G(x) + \frac{\log x}{x}J(x) + \log x\cdot K(x) - \frac{(q-1)\log^3 x}{12\log q}\\
&\quad - \frac{(q-1)\log^2 x}{4\log q} - G(x) - H(x) - K(x) - L(x)\\
&= \frac{(q-1)\log^3 x}{6\log q} + \frac{(q-1)\log^2 x}{4\log q} + \log x\cdot F\left(\frac{\log x}{\log q}\right)\\
&\quad + (\log x - 1)G(x) - H(x) + \frac{\log x}{x}J(x) + (\log x - 1)K(x)\\
&\quad - L(x).
\end{aligned}$$

□

To find the asymptotic formula of the sum of Dirichlet convolutions of digital sum functions, we need to evaluate the growth of the functions g, h, F, G, H, J, K, and L.

Lemma 2. *Let* $\{t\} = t - \lfloor t \rfloor$ *be the fractional part of* t. *Let*

$$g(x) = \int_0^x \left(\lfloor qt \rfloor - q\lfloor t \rfloor - \frac{q-1}{2} \right) dt = \int_0^x \left(\lfloor q\{t\} \rfloor - \frac{q-1}{2} \right) dt.$$

Then, we have $g(x) = O(1)$.

Proof. Let $b = \left\lfloor \frac{q}{2} \right\rfloor$. In the authors' previous paper [8], the integrand function in $g(x)$ was illustrated as a periodic step function. Thus, we can find the minimal value of $g(x)$ by calculating the net area of the integrand function from 0 to $\frac{b}{q}$ by the sum

$$\sum_{i=1}^{b} \left(-\frac{q-1}{2} + i - 1 \right) \frac{1}{q} = \sum_{i=1}^{b} \left(\frac{-q-1}{2} + i \right) \frac{1}{q} = \frac{-q-1}{2q} b + \frac{b(b+1)}{2q} = \frac{b^2 - qb}{2q}.$$

Thus, $0 \geq g(x) \geq \dfrac{b^2 - qb}{2q}$ and $g(x) = O(1)$. □

Lemma 3. *We also have the following:*

$$h(x) = O(1), F(x) = O(1), J(x) = O(\log x), G(x) = O(\log x),$$
$$H(x) = O(\log^2 x), K(x) = O(1), \text{ and } L(x) = O(\log x).$$

Proof. Clearly, $b(q-b) \leq 2q^2$ as $b \leq q$. Thus, $b^2 - bq \geq -2q^2$ and $\frac{b^2 - bq}{2q} \geq -q$. Therefore, $g(x) \geq -q$. We then have

$$0 \geq h(x) = \sum_{r=0}^{\infty} \frac{g(q^r x)}{q^r} \geq -q \sum_{r=0}^{\infty} \frac{1}{q^r} = -\frac{q^2}{q-1}.$$

Thus, $h(x) = O(1)$. Also, since $0 \leq 1 + \lfloor x \rfloor - x \leq 1$ and h is a bounded function, $F(x) = \frac{q-1}{2}(1 + \lfloor x \rfloor - x) + q^{1 + \lfloor x \rfloor - x} h(q^{x - \lfloor x \rfloor - 1})$ is a bounded function as well, i.e., $F(x) = O(1)$. Now, we know

$$\begin{aligned} 0 \leq J(x) &= (1 + \lfloor x \rfloor - x) s_q(\lfloor x \rfloor) - h(x) \\ &\leq s_q(\lfloor x \rfloor) + \frac{q^2}{q-1} \\ &\leq (q-1) \left(\left\lfloor \frac{\log x}{\log q} \right\rfloor + 1 \right) + \frac{q^2}{q-1}. \end{aligned}$$

Thus, $J(x) = O(\log x)$.

Since $F(x)$ is a bounded function,

$$G(x) = \int_1^x F\left(\frac{\log u}{\log q} \right) \frac{du}{u} = O(\log x).$$

Since $H(x) = \int_1^x G(u)\frac{du}{u}$, we have $H(x) = O(\log^2 x)$. We can also see that since $K(x) = \int_1^x J(u)\,\frac{du}{u^2}$, we have $K(x) = O(1)$. And finally, since $L(x) = \int_1^x K(u)\,\frac{du}{u}$, we have $L(x) = O(\log x)$. ☐

The following inequality will be needed to bound a sum with the function F:

Lemma 4. *We have*

$$\left| \sum_{n \le y} \frac{s_q(n)}{n} F\left(\frac{\log(x/n)}{\log q}\right) \right| \le F_M \sum_{n \le y} \frac{s_q(n)}{n},$$

where $F_M := \max\{|F(t)|, t \in \mathbb{R}\}$.

Proof. Since $|F|$ is continuous and periodic with period 1 by Theorem 1, it attains a maximum on $[0, 1]$ by the Extreme Value Theorem. ☐

For the k-th convolution, we will need the following formula which is the more general version of (5).

Lemma 5. *For any positive integer* j, *we have*

$$\sum_{n \le x} \frac{s_q(n)}{n} \log^j n = \frac{(q-1)\log^{j+2} x}{2(j+2)\log q} + O(\log^{j+1} x).$$

Proof. This follows from an application of (3) together with (5). ☐

4. Proofs

Proof of Theorem 2. Let $x > 1$. Following Srichan's idea, we will use Dirichlet's hyperbolic method on the sum of the Dirichlet convolution:

$$\sum_{n \le x} s_q^{(2)}(n) = \sum_{n \le x} \sum_{d \mid n} s_q(d) s_q(n/d) = 2 \sum_{n \le \sqrt{x}} s_q(n) \sum_{m \le x/n} s_q(m) - \left(\sum_{n \le \sqrt{x}} s_q(n) \right)^2.$$

Inserting (1) in the above formula, we have

$$\sum_{n \le x} s_q^{(2)}(n) = 2 \sum_{n \le \sqrt{x}} s_q(n) \left(\frac{(q-1)x \log(x/n)}{2n \log q} + \frac{x}{n} F\left(\frac{\log(x/n)}{\log q}\right) + J\left(\frac{x}{n}\right) \right)$$
$$- \left(\frac{(q-1)\sqrt{x}\log x}{4 \log q} + \sqrt{x} F\left(\frac{\log x}{2 \log q}\right) + J(\sqrt{x}) \right)^2$$

$$\begin{aligned}
&= \frac{(q-1)x\log x}{\log q}\sum_{n\le\sqrt{x}}\frac{s_q(n)}{n} - \frac{(q-1)x}{\log q}\sum_{n\le\sqrt{x}}\frac{s_q(n)\log n}{n} \\
&\quad + 2x\sum_{n\le\sqrt{x}}\frac{s_q(n)}{n}F\left(\frac{\log(x/n)}{\log q}\right) + 2\sum_{n\le\sqrt{x}} s_q(n)J\left(\frac{x}{n}\right) - \frac{(q-1)^2x\log^2 x}{16\log^2 q} \\
&\quad - \frac{(q-1)x\log x}{2\log q}F\left(\frac{\log x}{2\log q}\right) - xF^2\left(\frac{\log x}{2\log q}\right) - \frac{(q-1)\sqrt{x}\log x}{2\log q}J(\sqrt{x}) \\
&\quad - 2\sqrt{x}F\left(\frac{\log x}{2\log q}\right)J(\sqrt{x}) - J^2(\sqrt{x}).
\end{aligned}$$

The expression is lengthy, so we analyze each sum separately. We begin with the first term and use (4):

$$\begin{aligned}
\frac{(q-1)x\log x}{\log q}\sum_{n\le\sqrt{x}}\frac{s_q(n)}{n} &= \frac{(q-1)x\log x}{\log q}\Bigg(\frac{(q-1)\log^2\sqrt{x}}{4\log q} + \frac{(q-1)\log\sqrt{x}}{2\log q} \\
&\qquad + F\left(\frac{\log\sqrt{x}}{\log q}\right) + G(\sqrt{x}) + \frac{J(\sqrt{x})}{\sqrt{x}} + K(\sqrt{x})\Bigg) \\
&= \frac{(q-1)x\log x}{\log q}\Bigg(\frac{(q-1)\log^2 x}{16\log q} + \frac{(q-1)\log x}{4\log q} \\
&\qquad + O(\log x)\Bigg) \\
&= \frac{(q-1)^2x\log^3 x}{16\log^2 q} + \frac{(q-1)^2x\log^2 x}{4\log^2 q} \\
&\qquad + \frac{(q-1)x\log x}{\log q}\cdot O(\log x) \\
&= \frac{(q-1)^2x\log^3 x}{16\log^2 q} + O(x\log^2 x).
\end{aligned}$$

Consider the second term and use (5):

$$\begin{aligned}
\frac{(q-1)x}{\log q}\sum_{n\le\sqrt{x}}\frac{s_q(n)\log n}{n} &= \frac{(q-1)x}{\log q}\Bigg(\frac{(q-1)\log^3\sqrt{x}}{6\log q} + \frac{(q-1)\log^2\sqrt{x}}{4\log q} \\
&\qquad + \log\sqrt{x}\cdot F\left(\frac{\log\sqrt{x}}{\log q}\right) + (\log\sqrt{x}-1)G(\sqrt{x}) \\
&\qquad - H(\sqrt{x}) + \frac{\log\sqrt{x}}{\sqrt{x}}J(\sqrt{x}) + (\log\sqrt{x}-1)K(\sqrt{x}) \\
&\qquad - L(\sqrt{x})\Bigg)
\end{aligned}$$

$$\begin{aligned}&= \frac{(q-1)x}{\log q}\left(\frac{(q-1)\log^3 x}{48\log q} + \frac{(q-1)\log^2 x}{16\log q} + O(\log^2 x)\right)\\ &= \frac{(q-1)^2 x\log^3 x}{48\log^2 q} + O(x\log^2 x).\end{aligned}$$

Consider the absolute value of the third term and use Lemma 4:

$$\left|2x\sum_{n\le\sqrt{x}}\frac{s_q(n)}{n}F\left(\frac{\log(x/n)}{\log q}\right)\right| \le 2xF_M\sum_{n\le\sqrt{x}}\frac{s_q(n)}{n} = O(x\log^2 x).$$

The fourth term is

$$2\sum_{n\le\sqrt{x}} s_q(n)J\left(\frac{x}{n}\right).$$

Recall that $J(y)=O(\log y)$ as $y\to\infty$. Then there exist numbers C and y_0 such that $|J(y)|\le C\log y$ for $y\ge y_0$. For $1\le n\le\sqrt{x}$, we have $x/n\ge\sqrt{x}$. Hence, for x large enough ($\sqrt{x}\ge y_0$),

$$|J(x/n)|\le C\log(x/n)\le C\log x \quad \text{uniformly in } n\le\sqrt{x}.$$

Since $s_q(n)\ge 0$, we have the following bound:

$$\left|2\sum_{n\le\sqrt{x}} s_q(n)J\left(\frac{x}{n}\right)\right| \le 2C\log x\sum_{n\le\sqrt{x}} s_q(n) = O(\sqrt{x}\log^2 x).$$

The remainder equals

$$\begin{aligned}&\frac{x(q-1)^2\log^2 x}{16\log^2 q} + xF^2\left(\frac{\log x}{2\log q}\right) + \frac{(q-1)x\log x}{2\log q}F\left(\frac{\log x}{2\log q}\right)\\ &\qquad + J(\sqrt{x})\left(J(\sqrt{x}) + \frac{(q-1)\sqrt{x}\log x}{2\log q} + 2\sqrt{x}F\left(\frac{\log x}{2\log q}\right)\right)\\ &= \frac{x(q-1)^2\log^2 x}{16\log^2 q} + O(\sqrt{x}\log^2 x)\\ &= O(x\log^2 x).\end{aligned}$$

We now combine the above estimates to obtain

$$\begin{aligned}\sum_{n\le x} s_q^{(2)}(n) &= \frac{(q-1)^2 x\log^3 x}{16\log^2 q} - \frac{(q-1)x}{\log q}\left(\frac{(q-1)\log^3 x}{48\log q}\right) + O(x\log^2 x)\\ &= \frac{(q-1)^2 x\log^3 x}{24\log^2 q} + O(x\log^2 x).\end{aligned}$$

□

We now prove Theorem 3 for a general base.

Proof of Theorem 3. We use induction on k. Equation (2) is true for $k = 2$ as we just proved it above. Assume Equation (2) is true for all $k < l$. By definition,

$$\begin{aligned}\sum_{n\le x} s_q^{(l)}(n) &= \sum_{0\le n\le x}\sum_{d\mid n} s_q(d)s_q^{(l-1)}(n/d)\\ &= \sum_{n\le x} s_q(n)\sum_{m\le x/n} s_q^{(l-1)}(m).\end{aligned}$$

From our hypothesis, we get

$$\begin{aligned}\sum_{n\le x} s_q^{(l)}(n) &= \sum_{n\le x} s_q(n)\left(\frac{a(l-1)}{24\log^2 q}\cdot\frac{x}{n}\log^{2l-3}\left(\frac{x}{n}\right)+O\left(\frac{x}{n}\log^{2l-4}\left(\frac{x}{n}\right)\right)\right)\\ &= \frac{a(l-1)x}{24\log^2 q}\sum_{n\le x}\frac{s_q(n)}{n}(\log x-\log n)^{2l-3}\\ &\quad + O\left(x\sum_{n\le x}\frac{s_q(n)}{n}(\log x-\log n)^{2l-4}\right).\end{aligned} \tag{6}$$

Let, for an integer $r \ge 0$,

$$S_r(x) := \sum_{n\le x}\frac{s_q(n)}{n}(\log x-\log n)^r.$$

The main term and the error term in (6) involve $S_{2l-3}(x)$ and $S_{2l-4}(x)$, respectively, so it suffices to estimate $S_r(x)$ for $r = 2l-3$ and $r = 2l-4$ in the same way. By the Binomial Theorem, we have

$$\begin{aligned}\sum_{n\le x}\frac{s_q(n)}{n}(\log x-\log n)^{2l-4} &= \sum_{n\le x}\frac{s_q(n)}{n}\sum_{j=0}^{2l-4}\binom{2l-4}{j}(-1)^j\log^{2l-4-j}x\log^j n\\ &= \log^{2l-4}x\sum_{j=0}^{2l-4}\binom{2l-4}{j}(-1)^j\log^{-j}x\sum_{n\le x}\frac{s_q(n)}{n}\log^j n.\end{aligned}$$

By Lemma 5, we have

$$\begin{aligned}&\sum_{n\le x}\frac{s_q(n)}{n}(\log x-\log n)^{2l-4}\\ &= \log^{2l-4}x\sum_{j=0}^{2l-4}\binom{2l-4}{j}(-1)^j\log^{-j}x\left(\frac{(q-1)\log^{j+2}x}{2(j+2)\log q}+O(\log^{j+1}x)\right)\\ &= \frac{(q-1)}{2\log q}\log^{2l-2}x\sum_{j=0}^{2l-4}\binom{2l-4}{j}\frac{(-1)^j}{j+2}+O(\log^{2l-3}x).\end{aligned}$$

Similarly,

$$\sum_{n\le x}\frac{s_q(n)}{n}(\log x-\log n)^{2l-3}=\frac{(q-1)}{2\log q}\log^{2l-1}x\sum_{j=0}^{2l-3}\binom{2l-3}{j}\frac{(-1)^j}{j+2}+O(\log^{2l-2}x).$$

Substituting these into (6), we obtain

$$\begin{aligned}\sum_{n\le x}s_q^{(l)}(n)&=\frac{a(l-1)x}{24\log^2 q}\left(\frac{(q-1)}{2\log q}\log^{2l-1}x\sum_{j=0}^{2l-3}\binom{2l-3}{j}\frac{(-1)^j}{j+2}+O(\log^{2l-2}x)\right)\\&\quad+O\left(x\log^{2l-2}x\right)\\&=\frac{a(l-1)x}{24\log^2 q}\left(\frac{(q-1)}{2\log q}\log^{2l-1}x\sum_{j=0}^{2l-3}\binom{2l-3}{j}\frac{(-1)^j}{j+2}\right)+O\left(x\log^{2l-2}x\right).\end{aligned}$$

Let $a(l)=\frac{a(l-1)(q-1)}{2\log q}\sum_{j=0}^{2l-3}\binom{2l-3}{j}\frac{(-1)^j}{j+2}$. Thus, we have

$$\sum_{n\le x}s_q^{(l)}(n)=\frac{a(l)}{24\log^2 q}x\log^{2l-1}x+O\left(x\log^{2l-2}x\right).$$

□

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