ON THE NAGELL–LJUNGGREN EQUATION $\frac{x^5-1}{x-1} = y^3$

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Abstract

In this note, we prove that the Diophantine equation $\frac{x^5-1}{x-1} = y^3$ admits no integer solutions (x, y) other than $(0, 1)$.

1. Introduction

In 1920, Nagell [5, 6] obtained the first results on the Diophantine equation

$$\frac{x^n-1}{x-1} = y^q, \quad \text{in integers} \quad x > 1, y > 1, n > 2, q \geq 2. \tag{1}$$

Two decades later, Ljunggren [3] resolved the remaining issues in Nagell's proof and established the following result, which is presented in [1].

Theorem 1 ([3]). *Except for the solutions*

$$\frac{3^5-1}{3-1} = 11^2, \ \frac{7^4-1}{7-1} = 20^2, \ \textit{and} \ \frac{18^3-1}{18-1} = 7^3, \tag{2}$$

Equation (1) admits no other solutions (x, y, n, q) if one of the following conditions holds: (i) $q = 2$, (ii) $3 \mid n$, (iii) $4 \mid n$, and (iv) $q = 3$ with $n \not\equiv 5 \pmod 6$.

In 2007, Bugeaud and Mihăilescu [1] proved the following remarkable theorem, in which $\Omega(n)$ denotes the total number of prime divisors of n, counted with multiplicities.

Theorem 2 ([1]). *Let (x, y, n, q) be a solution of Equation (1) but not of Equation (2). Then, the least prime divisor of n is at least equal to 29 and $\Omega(n) \leq 4$. Furthermore, n is prime if $q = 3$. Moreover, if q divides n, then $n = q$.*

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The first assertion of Theorem 2 follows from Théorème 2 of [2] and [4]. This result is established by a highly technical method involving linear forms in logarithms.

In this note, we use a more elementary approach to show that Equation (1) has no solutions when $n = 5$ and $q = 3$. The theorem is stated as follows.

Theorem 3. *The Diophantine equation* $\frac{x^5-1}{x-1} = y^3$ *admits no integer solutions other than* $(x, y) = (0, 1)$.

Remark 1. We observe that in Theorem 3, the variables x and y are only required to be integers, without the additional restrictions $x > 1$ and $y > 1$. Furthermore, this is also a special case ruled out by condition (iv) of Theorem 1.

Invoking the third assertion of Theorem 2 (with $q = 3$) along with Theorem 3, we readily obtain the following corollary.

Corollary 1. *Equation (1) admits no solutions when* $q = 3$ *and* $5 \mid n$.

Let $\rho = \frac{1+\sqrt{5}}{2}$, $\overline{\rho} = \frac{1-\sqrt{5}}{2}$. Let $\mathcal{O}_{\mathbb{K}}$ denote the ring of integers of the number field K. Since $5 \equiv 1 \pmod 4$, we have $\mathcal{O}_{\mathbb{Q}(\sqrt{5})} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}] = \mathbb{Z}[\rho] = a + b\rho$, where $a, b \in \mathbb{Z}$. Denote by $h(d)$ the class number of the real quadratic field $K = \mathbb{Q}(\sqrt{d})$. Since $h(5) = 1$, it follows that $\mathbb{Z}[\rho]$ is a unique factorization domain. This means that in $\mathbb{Z}[\rho]$, irreducible and prime elements coincide.

For an arbitrary element $\alpha \in \mathbb{Z}[\rho]$, we denote its conjugate by $\overline{\alpha} = a + b\overline{\rho}$. One can easily check that $\overline{\rho} + \rho = 1$; $\rho\overline{\rho} = -1$; $\rho^2 = \rho + 1$; $\rho^3 = 2\rho + 1$. The units in $\mathbb{Z}[\rho]$ are $\pm\rho^k$, where $k \in \mathbb{Z}$.

2. Proof of Theorem 3

Proof. We denote $v_p(m)$ as the p-adic valuation of an integer m, that is, the highest exponent of p that divides m. The following lifting the exponent lemma is elementary. The proof is omitted.

Lemma 1. *If* $p > 2$ *is a prime such that* $p \mid x - y, p \nmid x, p \nmid y, x, y, n \in \mathbb{Z}, n > 0$ *then*

$$v_p(x^n - y^n) = v_p(x - y) + v_p(n).$$

We now begin the proof of Theorem 3. We can rewrite the equation as follows

$$\frac{x^5-1}{x-1} = x^4 + x^3 + x^2 + x + 1 = \left(x^2 + \rho x + 1\right)\left(x^2 + \overline{\rho} x + 1\right) = y^3. \tag{3}$$

We observe that, if $y \equiv 0 \pmod 5$ then $x \equiv 1 \pmod 5$. Hence,

$$v_5\left(\frac{x^5-1}{x-1}\right) = v_5(x-1) + 1 - v_5(x-1) = 1 = 3v_5(y).$$

This is impossible. Therefore, $y \not\equiv 0 \pmod 5$.

Let $\alpha = \gcd\left(x^2+\rho x+1, x^2+\overline{\rho}x+1\right) = \gcd\left(x^2+\rho x+1, \sqrt{5}x\right)$. Since $\sqrt{5}$ is irreducible in $\mathbb{Z}[\rho]$ and $y \not\equiv 0 \pmod 5$, it follows that $\alpha = 1$. From Equation (3) we deduce that $\left(x^2+\rho x+1\right) = \varepsilon(a+b\rho)^3$, where ε is a unit and $a, b \in \mathbb{Z}$. It is easy to see that $\varepsilon = \pm\rho^k$, where $k \in \mathbb{Z}$. Since the exponent here is 3, which is odd, it suffices to consider the positive sign and $k = 0, 1, 2$, that is,

$$\begin{cases} x^2+\rho x+1 = \rho^k(a+b\rho)^3, \\ x^2+\overline{\rho}x+1 = \overline{\rho}^k(a+b\overline{\rho})^3, \\ \qquad\qquad y = (-1)^k(a^2+ab-b^2), \end{cases}$$

where $k = 0, 1, 2$ and $\gcd(a, b) = 1$.

We consider three separate cases for k.

`Case 1:` $k = 2$. We have

$$\begin{aligned} x^2+\rho x+1 = x^2+1+\rho x &= \rho^2(a+b\rho)^3 \\ &= a^3+3a^2b+6ab^2+3b^3+\rho(a^3+6a^2b+9ab^2+5b^3). \end{aligned}$$

We obtain

$$\begin{cases} x^2+1 = a^3+3a^2b+6ab^2+3b^3, \\ \qquad x = a^3+6a^2b+9ab^2+5b^3. \end{cases} \tag{4}$$

Hence,

$$(x+1)^2 = 3a^3+15a^2b+24ab^2+13b^3. \tag{5}$$

Taking Equation (5) modulo 3, we obtain $b \equiv 0, 1 \pmod 3$. If $b \equiv 0 \pmod 3$ then $3 \mid x+1$. Consequently $a \equiv 0 \pmod 3$. This contradicts the condition $\gcd(a, b) = 1$. Therefore, $b \equiv 1 \pmod 3$. From the first equation in (4), we obtain $a \equiv 1, 2 \pmod 3$.

If $a \equiv 2 \pmod 3$, it follows from the second equation in (4) that $x \equiv 1 \pmod 3$. We observe that

$$-(x-1)^2 = a^3+9a^2b+12ab^2+7b^3.$$

Hence,

$$9 \mid (a^3-8)+7(b^3-1)+15+3ab^2.$$

Since $2 \leq v_3(a^3-2^3)$ and $2 \leq v_3(b^3-1^3)$, whence 9 divides $3(5+ab^2)$. Which is impossible. We conclude that $a \equiv b \equiv 1 \pmod 3$.

We now have, from the second equation in (4), that $3 \mid x$. We deduce that $9 \mid x^2 = (a^3-1)+3b(a+b)^2$, that is, $3 \mid b(a+b)^2$. This is absurd.

`Case 2:` $k=1$. We have $\big(x^2+\rho x+1\big) = \rho(a+b\rho)^3$ and $\big(x^2+\overline{\rho}x+1\big) = \overline{\rho}(a+b\overline{\rho})^3$. Hence,

$$\begin{cases} x^2+1 = 3a^2b+3ab^2+2b^3, \\ x = a^3+6a^2b+9ab^2+5b^3, \\ -y = a^2+ab-b^2. \end{cases} \tag{6}$$

We observe that

$$(x+1)^2 = 2a^3+15a^2b+21ab^2+12b^3. \tag{7}$$

Taking Equation (7) modulo 3 yields $a \equiv 0,2 \pmod 3$. If $a \equiv 0 \pmod 3$ then $3 \mid x+1$ and we get $b \equiv 0 \pmod 3$. This contradicts $\gcd(a,b)=1$. Hence, $a \equiv 2 \pmod 3$. From the first equation in (6), we derive $b \equiv 1,2 \pmod 3$. If $b \equiv 1 \pmod 3$, it follows from the second equation in (6) that $x \equiv 1 \pmod 3$. Then, Equation (3) yields $y \equiv 2 \pmod 3$. Obviously, this contradicts the third equation in (6), since $-y \equiv 2 \pmod 3$. We conclude that $a, b \equiv 2 \pmod 3$. It follows from the second equation in (6) that $x \equiv 0 \pmod 3$. Then, Equation (3) yields $y \equiv 1 \pmod 3$. Similarly, this contradicts the third equation in (6), since $-y \equiv 1 \pmod 3$.

`Case 3:` $k=0$. We have $\big(x^2+\rho x+1\big) = (a+b\rho)^3$ and $\big(x^2+\overline{\rho}x+1\big) = (a+b\overline{\rho})^3$. Therefore,

$$\begin{cases} x^2+1 = a^3+3ab^2+b^3, \\ x = 2b^3+3a^2b+3ab^2, \\ y = a^2+ab-b^2. \end{cases} \tag{8}$$

• For $x>0$, it follows from the second equation in (8) that $b(2b^2+3ab+3a^2)>0$. Therefore, $b>0$. By Equation (3), we have $y>0$. Therefore, if $a<0$ then $y = a(a+b)-b^2>0$, that is, $-a>b$. From the first equation in (8), we obtain $b^3 > -a(a^2+3b^2) > 4b^3$. Which is impossible. Thus, $a, b > 0$. Since $x^2+1 \geq 2x$, we derive

$$a^3+3ab^2+b^3 \geq (2b^3+3a^2b+3ab^2)^2 > 4b^6+9a^4b^2+9a^2b^4.$$

This contradicts the inequalities $a^3 < 9a^4b^2$, $3ab^2 < 9a^4b^2$, and $b^3 < 4b^6$.

• For $x<0$, similarly we have $b<0$. Let $c=-b>0$. Since $x^2+1 = a^3+3ac^2-c^3>0$, we deduce that $a>0$. Since $y = a^2-ac-c^2>0$, we derive $a>c$. Moreover, $a-c/2 > \sqrt{5}c/2$. Hence, $2a>3c$. We have

$$-x = 2c^3+3a^2c-3ac^2 = c(2c^2+a^2)+ca(2a-3c) > c(2c^2+a^2) > 0.$$

Thus, $x^2 > (2c^3 + ca^2)^2 > a^3 + 3ac^2 - c^3 = x^2 + 1$. Which is impossible.

- For $x = 0$, we obtain $y = 1$.

The theorem is proved. □

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