#A45 INTEGERS 26 (2026)

LÜROTH EXPANSIONS IN DIOPHANTINE APPROXIMATION: METRIC PROPERTIES AND CONJECTURES

Ying Wai Lee
Department of Mathematics, University of York, York, United Kingdom

Received: 8/6/25, Accepted: 3/6/26, Published: 4/3/26

Abstract

The metric theory of Diophantine approximation for Lüroth convergents is studied. A Khintchine-type theorem in the Lüroth setting is established, yielding a sharp zero–one law governed by the usual summation criterion with a logarithmic correction, and thereby settling a measure-theoretic statement attributed to Tan and Zhou. By applying the Beresnevich and Velani Mass Transference Principle, the Hausdorff dimension is determined for the full non-negative range of approximation exponents and the critical Hausdorff measure is shown to be infinite. This strengthens a theorem of Cao, Wu, and Zhang and provides a simpler proof. Finally, a conjecture of Tan and Zhou is refuted, a corrected monotone formulation is proposed, and verification is given in a natural partial case.

1. Introduction

The metric theory of Diophantine approximation studies the Lebesgue measure and Hausdorff dimension of subsets of real numbers satisfying specific approximation properties. Let $\psi : \mathbb{N} \to [0, 1/2]$. Define $W(\psi)$, *the set of all ψ-well approximable numbers*, as

$$W(\psi) := \limsup_{q\to+\infty} \bigcup_{p\in\mathbb{Z}} \left\{ x \in [0,1) : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \right\}. \tag{1}$$

Equivalently, a number $x \in [0, 1)$ is ψ-well approximable if and only if there exist infinitely many $q \in \mathbb{N}$ such that for some $p \in \mathbb{Z}$,

$$\left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}. \tag{2}$$

The Khintchine Theorem is one of the fundamental results in metric Diophantine approximation, and serves as a starting point for much subsequent research. The

---

DOI: 10.5281/zenodo.19403786

theorem states that if ψ is non-increasing, then the Lebesgue measure of the set $W(\psi)$ satisfies the zero–one law, and is determined by the convergence or divergence of the series $\sum_{q=1}^{\infty}\psi(q)$. Specifically, if the series converges, then almost every $x\in[0,1)$ is not ψ-well approximable. Conversely, if the series diverges, then almost every $x\in[0,1)$ is ψ-well approximable.

Theorem 1 (Khintchine [6])**.** *Let $\psi:\mathbb{N}\to[0,1/2]$. Suppose ψ is non-increasing. Then*

$$\mathcal{L}(W(\psi))=\begin{cases}0, & \text{if } \sum_{q=1}^{\infty}\psi(q)<\infty;\\ 1, & \text{if } \sum_{q=1}^{\infty}\psi(q)=\infty,\end{cases}$$

where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}$.

Let $\tau\geq 0$. Define $W(\tau)$, *the set of all τ-well approximable numbers*, as

$$W(\tau):=\limsup_{q\to+\infty}\bigcup_{p\in\mathbb{Z}}\left\{x\in[0,1):\left|x-\frac{p}{q}\right|<\frac{1}{q^{1+\tau}}\right\}. \tag{3}$$

The set $W(\tau)$ coincides precisely with $W(\psi)$ when ψ is chosen as $\psi(q):=1/q^{\tau}$ for any $q\in\mathbb{N}$. A fundamental result concerning the Hausdorff dimension of $W(\tau)$ is given by the Jarník–Besicovitch Theorem, established independently by Jarník and Besicovitch.

Theorem 2 (Jarník [5]; Besicovitch [2])**.** *For any $\tau\geq 1$,*

$$\dim W(\tau)=\frac{2}{1+\tau}.$$

The Jarník–Besicovitch Theorem is generalized by the result of Dodson [4, Theorem 2], extending the theorem from $W(\tau)$ to $W(\psi)$ for general non-increasing ψ. The Hausdorff dimension of $W(\psi)$ is expressed in terms of the lower order at infinity of $1/\psi$.

Theorem 3 (Dodson [4])**.** *Let $\psi:\mathbb{N}\to[0,1/2]$. Suppose ψ is non-increasing. Then*

$$\dim W(\psi)=\frac{2}{1+\underline{\tau}_{\psi}}, \tag{4}$$

where $\underline{\tau}_{\psi}:=\liminf_{q\to+\infty}-\log\psi(q)/\log q\geq 1$.

In summary, the results of Khintchine, Jarník–Besicovitch, and Dodson primarily focus on the metric properties of well approximable numbers, specifically their Lebesgue measure and Hausdorff dimension. It is worth noting that the fractions p/q that approximate $x\in[0,1)$ in (2) are not required to be in lowest terms, nor to satisfy any additional constraints. However, imposing additional constraints on the approximating fractions may lead to a reduction in both measure and dimension.

Similar results are presented in the setting where the approximating fractions p/q in (2) are required to be Lüroth convergents of x.

2. Preliminaries

For any $x \in (0,1]$, there exists a unique sequence $(d_n)_{n\in\mathbb{N}}$ of positive integers that are not equal to 1, referred to as "*digits*", such that

$$x = [d_1, d_2, d_3, \dots, d_n, \dots] \tag{5}$$

$$:= \sum_{k=1}^{\infty} \frac{1}{d_k \prod_{j=1}^{k-1} d_j(d_j - 1)}, \tag{6}$$

where, for any $n \in \mathbb{N}$, we have $d_n = d_n(x) \in \mathbb{N} \setminus \{1\}$. Equation (5) is referred to as the *Lüroth representation* of x. Equation (6) is referred to as the *Lüroth expansion* of x. The sequence $(d_n)_{n\in\mathbb{N}}$ can be determined by an iterative process induced by the *Lüroth map* $T : (0,1] \to (0,1]$, defined by, for any $x \in (0,1]$,

$$T(x) := \left\lfloor \frac{1}{x} \right\rfloor \left(\left\lfloor \frac{1}{x} + 1 \right\rfloor x - 1 \right). \tag{7}$$

As illustrated in Figure 1, the Lüroth map consists of countably many linear pieces, each defined by a distinct linear equation. For any $x \in (0,1]$, the first digit of x in the Lüroth representation is given by $d_1(x) = \lfloor 1/x \rfloor + 1$, and for any $n \in \mathbb{N} \setminus \{1\}$ we have $d_n(x) = d_1(T^{n-1}(x))$.

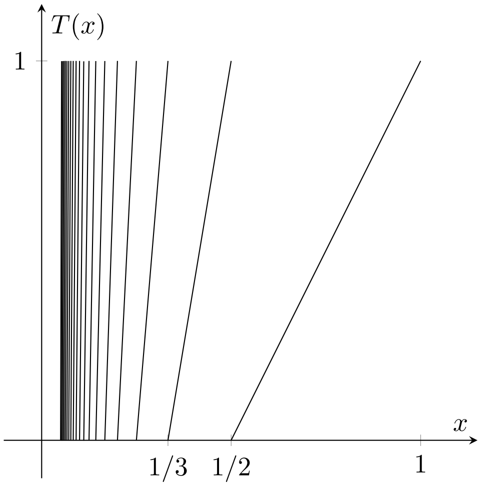

Figure 1: Lüroth map $T : (0,1] \to (0,1]$

Let $x \in (0,1]$ and $n \in \mathbb{N}$. Define the *n-th Lüroth convergent* x_n of x by the n-th partial sum in (6); that is:

$$\begin{aligned} x_n &:= [d_1, d_2, d_3, \ldots, d_n] := \sum_{k=1}^{n} \frac{1}{d_k \prod_{j=1}^{k-1} d_j(d_j-1)} \\ &= \frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \frac{1}{d_1(d_1-1)d_2(d_2-1)d_3} + \cdots \\ &\quad + \frac{1}{d_1(d_1-1)\cdots d_{n-1}(d_{n-1}-1)d_n}. \end{aligned}$$

The unsimplified numerator, $P_n(x)$, and denominator, $Q_n(x)$, of the n-th Lüroth convergent of x are defined in (8) and (9), respectively:

$$P_n(x) := [d_1, d_2, \ldots, d_n]Q_n(x), \tag{8}$$

$$Q_n(x) := d_n \prod_{j=1}^{n-1} d_j(d_j-1) = d_1(d_1-1)d_2(d_2-1)\cdots d_n. \tag{9}$$

Note that the fraction $P_n(x)/Q_n(x)$ may not be in its simplest form. For example, for $x = 27/71 = [3, 4, 3, 4, 3, 4, \ldots]$ and any $n \in \mathbb{N}\backslash\{1\}$ we have $\gcd(P_n(x), Q_n(x)) = 2 > 1$.

Let $\psi : \mathbb{N} \to (0, 1]$. Define $L(\psi)$, *the set of all Lüroth ψ-well approximable numbers*, as

$$L(\psi) := \limsup_{n\to+\infty} \left\{ x \in (0,1] : \left| x - \frac{P_n(x)}{Q_n(x)} \right| < \frac{\psi(Q_n(x))}{Q_n(x)} \right\}.$$

Let $\tau \geq 0$. Define $L(\tau)$, *the set of all Lüroth τ-well approximable numbers*, as

$$L(\tau) := \limsup_{n\to+\infty} \left\{ x \in (0,1] : \left| x - \frac{P_n(x)}{Q_n(x)} \right| < \frac{1}{Q_n(x)^{1+\tau}} \right\}. \tag{10}$$

The set $L(\tau)$ coincides precisely with $L(\psi)$ when ψ is chosen as $\psi(q) := 1/q^{\tau}$ for any $q \in \mathbb{N}$.

3. Main Results

Theorem 4 is an analog of the Khintchine Theorem, as stated in earlier work of Tan and Zhou [7]. Theorem 6 is an analog of the Jarník–Besicovitch Theorem, which improves the previous result of Cao, Wu, and Zhang [3, Theorem 1.2]. Theorems 7 and 8 are generalizations of Theorem 6 for general functions. Theorem 9 is a counterexample to the conjecture stated by Tan and Zhou [7, Conjecture 2], an analog of the result of Dodson. Theorem 10 is a partial result toward the revised conjecture.

Theorem 4 (Tan and Zhou [7, 8])**.** *Let* $\psi : \mathbb{N} \to (0,1]$*. Suppose* ψ *is non-increasing. Then*

$$\mathcal{L}\left(L(\psi)\right) = \begin{cases} 0, & \textit{if} \quad \sum_{q=1}^{\infty} \dfrac{-\psi(q)\log\psi(q)}{q} < \infty; \\ 1, & \textit{if} \quad \sum_{q=1}^{\infty} \dfrac{-\psi(q)\log\psi(q)}{q} = \infty. \end{cases}$$

Theorem 4 was originally claimed by Tan and Zhou [7], and can be seen as an analog of the Khintchine Theorem. The statement appears in [7], and is claimed to be studied in [8]. However, only an equivalent statement of the theorem is presented in [8], without explicitly stating the equivalence or providing a proof. A supplementary proof of Theorem 4 is therefore provided.

The following dimensional result is proved by Cao, Wu, and Zhang [3, Theorem 1.2], and can be seen as an analog of the Jarník–Besicovitch Theorem.

Theorem 5 (Cao, Wu, and Zhang [3])**.** *For any* $\tau \geq 1$*,*

$$\dim L(\tau) = \frac{1}{1+\tau}.$$

This theorem can be compared with the Jarník–Besicovitch Theorem. For any $\tau \geq 1$, we have $\dim W(\tau) = 2/(1+\tau)$, which is twice the Hausdorff dimension of $L(\tau)$.

In some works, for instance [7, 8], the result of Cao, Wu, and Zhang is claimed to hold for all non-negative real numbers $\tau \geq 0$, though no explicit explanation is provided. A proof of Theorem 6 is provided, thereby verifying the claim.

Theorem 6. *For any* $\tau \geq 0$*,*

$$\dim L(\tau) = \frac{1}{1+\tau},$$

and the $1/(1+\tau)$*-Hausdorff measure of* $L(\tau)$ *is infinite, that is:*

$$\mathcal{H}^{1/(1+\tau)}(L(\tau)) = +\infty.$$

By applying the Beresnevich and Velani Mass Transference Principle [1, Theorem 2], the range of exponents is extended from $\tau \geq 1$ to $\tau \geq 0$, the argument in [3] is simplified, and the Hausdorff measure at the critical exponent is obtained.

Before discussing the analog of Dodson's theorem (Theorem 3) in the Lüroth setting, a few generalizations to Theorem 6 are given. Let $\psi : \mathbb{N} \to (0,1]$. Define the *lower order at infinity* and *upper order at infinity* of $1/\psi$ as follows:

$$\underline{\tau}_{\psi} \coloneqq \liminf_{q\to+\infty} -\frac{\log\psi(q)}{\log q}; \qquad \overline{\tau}_{\psi} \coloneqq \limsup_{q\to+\infty} -\frac{\log\psi(q)}{\log q}. \tag{11}$$

Theorem 7. *Let* $\psi : \mathbb{N} \to (0,1]$*. We have*

$$\frac{1}{1+\overline{\tau}_\psi} \leq \dim L(\psi) \leq \frac{1}{1+\underline{\tau}_\psi}. \tag{12}$$

Theorem 7 provides both lower and upper bounds on the Hausdorff dimension of $L(\psi)$, and serves as a generalization of Theorem 6. For $\tau \geq 0$ and $\psi : \mathbb{N} \to (0,1]$ taken as $\psi(q) := 1/q^\tau$ for any $q \in \mathbb{N}$, Theorem 6 is recovered as $\underline{\tau}_\psi = \overline{\tau}_\psi = \tau$.

From the definition of $L(\psi)$ given above, the set $L(\psi)$ depends only on the values of ψ evaluated at a specific subset of $\mathbb{N}$ with arbitrarily small density. Indeed, for any $x \in (0,1]$ and $n \in \mathbb{N}$, the denominator $Q_n(x)$ of the n-th Lüroth convergent of x, is divisible by 2^{n-1}. Thus, the contribution of ψ evaluated at non-highly composite numbers does not affect the set $L(\psi)$, nor does it affect the Lebesgue measure and Hausdorff dimension of $L(\psi)$. Theorem 8 refines Theorem 7 by incorporating a precise range for the limit inferior and limit superior in (11).

Theorem 8. *Let* $\psi : \mathbb{N} \to (0,1]$*. Then,*

$$\frac{1}{1+\overline{\lambda}_\psi} \leq \dim L(\psi) \leq \frac{1}{1+\underline{\lambda}_\psi},$$

where $\underline{\lambda}_\psi$ *and* $\overline{\lambda}_\psi$ *are defined by*

$$\underline{\lambda}_\psi := \liminf_{k\to+\infty} \liminf_{q\in S_k} \frac{-\log\psi(q)}{\log q} \geq \underline{\tau}_\psi, \quad \overline{\lambda}_\psi := \liminf_{k\to+\infty} \limsup_{q\in S_k} \frac{-\log\psi(q)}{\log q} \leq \overline{\tau}_\psi,$$

and for any $k \in \mathbb{N}$ *we define the set* S_k *by*

$$S_k := \left\{ d_k \prod_{j=1}^{k-1} d_j(d_j-1) : \textit{for any } n \in \mathbb{N},\ d_n \in \mathbb{N}\setminus\{1\} \right\} \subset \mathbb{N}.$$

The following conjecture is stated by Tan and Zhou [7, Conjecture 2], regarding the Hausdorff dimension of $L(\psi)$, and can be seen as an analog of Dodson's theorem (Theorem 3).

Conjecture 1 (Tan and Zhou [7]). Let $\psi : \mathbb{N} \to (0,1]$. We have

$$\dim L(\psi) = \frac{1}{1+\underline{\tau}_\psi}.$$

Theorem 9 demonstrates that the conjecture can fail without the monotonicity assumption on ψ. Consequently, the conjecture should be reformulated to include that assumption.

Theorem 9. *For any* $\tau \geq 0$*, there exists* $\psi : \mathbb{N} \to (0,1]$ *such that* ψ *is not eventually non-increasing,* $\underline{\tau}_\psi = \tau$*, and*

$$\dim L(\psi) = 0 < \frac{1}{1+\underline{\tau}_\psi}.$$

Conjecture 2 (Revised Tan and Zhou Conjecture)**.** Let $\psi : \mathbb{N} \to (0,1]$. Suppose ψ is non-increasing. Then

$$\dim L(\psi) = \frac{1}{1+\underline{\tau}_\psi}.$$

Theorem 10 provides a partial result toward the revised conjecture with three extra assumptions: a sufficiently large lower order, a different monotonicity condition, and the divergence of a specific series. Let $\psi : \mathbb{N} \to (0,1]$. Let $t := 1/(1+\underline{\tau}_\psi)$, and define $\theta_t : \mathbb{N} \to (0,1]$ by setting, for each $q \in \mathbb{N}$,

$$\theta_t(q) := \min\left\{1, 2^{-t} q^{1-t} \psi^t(q)\right\}. \tag{13}$$

Theorem 10. *Let* $\psi : \mathbb{N} \to (0,1]$. *Suppose that* $\underline{\tau}_\psi > 1$, *that* θ_t *is non-increasing, and that*

$$\sum_{q=1}^{\infty} -\left(\frac{\psi(q)}{q}\right)^t \left((1-t)\log q + t \log \frac{\psi(q)}{2}\right) = \infty,$$

where $\underline{\tau}_\psi$ *and* θ_t *are defined in* (11) *and* (13). *Then*

$$\dim L(\psi) = \frac{1}{1+\underline{\tau}_\psi}.$$

4. Proofs

4.1. Proof of Theorem 4

Theorem 4 presents a measure-theoretic result originally stated by Tan and Zhou [7], which forms an analog of the Khintchine Theorem in the Lüroth setting. While the original statement is claimed in [7], a direct proof is not fully provided. This section offers a supplementary proof.

Theorem 4 is logically equivalent to [8, Corollary 1.3]. The following demonstrates how the former is implied by the latter.

Proof of Theorem 4. By [8, (4.1)], for any $x \in (0,1]$ and $n \in \mathbb{N}$,

$$\frac{1}{(d_n(x)-1)d_{n+1}(x)} < |xQ_n(x) - P_n(x)| < \frac{1}{(d_n(x)-1)(d_{n+1}(x)-1)}.$$

Since for any $x \in (0,1]$ and $n \in \mathbb{N}$, it holds that $d_n(x) \geq 2$, which implies

$$\frac{1}{2d_n(x)d_{n+1}(x)} < |xQ_n(x) - P_n(x)| < \frac{4}{d_n(x)d_{n+1}(x)}.$$

For any $n \in \mathbb{N}$, define

$$\begin{aligned} A_n &\coloneqq \left\{ x \in (0,1] : d_n(x) d_{n+1}(x) \geq \frac{4}{\psi(Q_n(x))} \right\}, \\ B_n &\coloneqq \left\{ x \in (0,1] : |xQ_n(x) - P_n(x)| < \psi(Q_n(x)) \right\}, \\ C_n &\coloneqq \left\{ x \in (0,1] : d_n(x) d_{n+1}(x) \geq \frac{1}{2\psi(Q_n(x))} \right\}. \end{aligned}$$

Then the following inclusions hold:

$$\limsup_{n\to\infty} A_n \subset L(\psi) = \limsup_{n\to\infty} B_n \subset \limsup_{n\to\infty} C_n.$$

In the convergent case, assume that

$$\sum_{q=1}^{\infty} -\frac{\psi(q)\log\psi(q)}{q} < +\infty.$$

In particular, $\lim_{q\to\infty}\psi(q) = 0$. Define the non-decreasing function $\varphi_0 : \mathbb{N} \to [2, +\infty)$, for any $q \in \mathbb{N}$, by

$$\varphi_0(q) \coloneqq \max\left(2, \frac{1}{2\psi(q)}\right).$$

Since $\lim_{q\to\infty}\psi(q) = 0$, there exists a minimal $q_0 \in \mathbb{N}$ such that for any $q \in \mathbb{N}$, if $q > q_0$ then $\varphi_0(q) = 1/(2\psi(q))$. By the convergent part of [8, Corollary 1.3] with

$$\begin{aligned} \sum_{q=1}^{\infty} \frac{\log\varphi_0(q)}{q\varphi_0(q)} &= \sum_{q=1}^{q_0} \frac{\log 2}{2q} + \sum_{q=q_0+1}^{\infty} -\frac{2\psi(q)\log(2\psi(q))}{q} \\ &< \sum_{q=1}^{q_0} \frac{\log 2}{2q} + 2\sum_{q=1}^{\infty} -\frac{\psi(q)\log\psi(q)}{q} < +\infty, \end{aligned}$$

and these set inclusions imply that $\mathcal{L}(L(\psi)) = 0$.

In the divergent case, assume that

$$\sum_{q=1}^{\infty} -\frac{\psi(q)\log\psi(q)}{q} = \infty.$$

Define the non-decreasing function $\varphi_1 : \mathbb{N} \to [2, +\infty)$, for any $q \in \mathbb{N}$, by

$$\varphi_1(q) \coloneqq \max\left(2, \frac{4}{\psi(q)}\right).$$

By the divergent part of [8, Corollary 1.3] with

$$\begin{aligned}\sum_{q=1}^{\infty}\frac{\log\varphi_1(q)}{q\varphi_1(q)} &\geq \sum_{q=1}^{\infty}\frac{-\psi(q)\log\left(\psi(q)/4\right)}{4q}\\ &\geq \frac{1}{4}\sum_{q=1}^{\infty}\frac{-\psi(q)\log\psi(q)}{q}.\end{aligned}$$

The right-hand side diverges to $+\infty$, and these set inclusions imply that $\mathcal{L}(L(\psi)) = 1$. This completes the proof. □

4.2. Proof of Theorem 6

Theorem 6 establishes the Hausdorff dimension of the set of Lüroth τ-well approximable numbers, improving the previous result of Cao, Wu, and Zhang [3, Theorem 1.2], which forms an analog of the Jarník–Besicovitch Theorem in the Lüroth setting.

Proposition 1 provides an inequality that every Lüroth convergent satisfies.

Proposition 1. *For any $x \in (0, 1]$ and $n \in \mathbb{N}$,*

$$0 < x - \frac{P_n(x)}{Q_n(x)} \leq \frac{1}{d_n(x)-1}\frac{1}{Q_n(x)}.$$

Proof. Pick any $x \in (0, 1]$ and $n \in \mathbb{N}$. By the definitions of $P_n(x)$ and $Q_n(x)$ in (8) and (9),

$$\begin{aligned}x - \frac{P_n(x)}{Q_n(x)} &= \sum_{k=n+1}^{\infty}\frac{1}{d_k\prod_{j=1}^{k-1}d_j(d_j-1)}\\ &= \frac{1}{d_n-1}\frac{1}{Q_n(x)}\sum_{k=n+1}^{\infty}\frac{1}{d_k\prod_{j=n+1}^{k-1}d_j(d_j-1)},\end{aligned}$$

where the value of the summation in the rightmost expression lies in the interval $(0, 1]$. □

Proposition 1 states that every real number in $(0, 1]$ is strictly greater than all of its Lüroth convergents. In other words, all Lüroth convergents approximate the real number from the left of the number line. Figure 2 is an illustration.

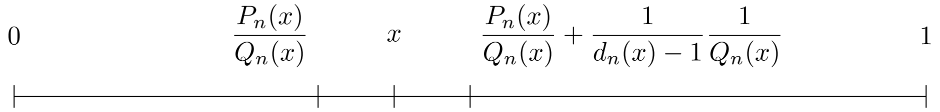

Figure 2: Lüroth convergents approximate from the left

By Proposition 1, for any $\tau \geq 0$, the set $L(\tau)$, defined in (10), can be expressed as:

$$L(\tau) = \limsup_{n\to+\infty} \left\{ x \in (0,1] : 0 < x - \frac{P_n(x)}{Q_n(x)} < \frac{1}{Q_n(x)^{1+\tau}} \right\}. \tag{14}$$

The proof of the upper bound for $\dim L(\tau)$ is established by a standard covering argument.

Proposition 2 (Upper Bound of $\dim L(\tau)$)**.** *For any $\tau \geq 0$,*

$$\dim L(\tau) \leq \frac{1}{1+\tau}.$$

Proof. Pick any $n \in \mathbb{N}$ and $d_1, d_2, \ldots, d_n \in \mathbb{N} \setminus \{1\}$. Define

$$\begin{aligned} Q_n &\coloneqq d_1(d_1-1)\cdots d_{n-1}(d_{n-1}-1)d_n \geq 2^n, \\ P_n &\coloneqq [d_1, d_2, \ldots, d_n]\, Q_n, \end{aligned}$$

and

$$I_{n,\tau}(d_1, d_2, \ldots, d_n) \coloneqq \left(\frac{P_n}{Q_n}, \frac{P_n}{Q_n} + \frac{1}{{Q_n}^{1+\tau}} \right).$$

Note that $\operatorname{diam} I_{n,\tau} = 1/{Q_n}^{1+\tau} \leq 1/2^n$. Pick any $\rho > 0$ and $N \in \mathbb{N}$. Suppose $N > -\log\rho/\log 2$. By (14), the following is a ρ-cover of $L(\tau)$.

$$\bigcup_{n=N}^{\infty} \bigcup_{(d_1,\ldots,d_n)\in(\mathbb{N}\setminus\{1\})^n} I_{n,\tau}(d_1, d_2, \ldots, d_n). \tag{15}$$

Pick any $\varepsilon > 0$ and $s > 1/(1+\tau)$. Without loss of generality, $N \in \mathbb{N}$ is large enough in terms of $\tau \geq 0$, $s > 0$, and $\varepsilon > 0$, so that

$$\sum_{n=N}^{\infty} {r_{\tau,s}}^{n-1} < \frac{\varepsilon}{\zeta((1+\tau)s)},$$

where ζ is the Riemann zeta function, $(1+\tau)s > 1$, and

$$r_{\tau,s} \coloneqq \sum_{d=2}^{\infty} \frac{1}{(d(d-1))^{(1+\tau)s}} \in (0,1).$$

Hence, the s-Hausdorff measure for the ρ-cover (15) is obtained as

$$\begin{aligned}
\mathcal{H}^s_\rho(L(\tau)) &\le \sum_{n=N}^{\infty} \sum_{(d_1,\dots,d_n)\in(\mathbb{N}\setminus\{1\})^n} \left(\operatorname{diam} I_{n,\tau}(d_1,\dots,d_n)\right)^s \\
&= \sum_{n=N}^{\infty} \sum_{(d_1,\dots,d_n)\in(\mathbb{N}\setminus\{1\})^n} \frac{1}{Q_n(d_1,\dots,d_n)^{(1+\tau)s}} \\
&= \sum_{n=N}^{\infty} \sum_{(d_1,\dots,d_n)\in(\mathbb{N}\setminus\{1\})^n} \frac{1}{\left(d_1(d_1-1)\cdots d_{n-1}(d_{n-1}-1)d_n\right)^{(1+\tau)s}} \\
&= \sum_{n=N}^{\infty} \left(\sum_{d=2}^{\infty} \frac{1}{\left(d(d-1)\right)^{(1+\tau)s}}\right)^{n-1} \left(\sum_{d=2}^{\infty} \frac{1}{d^{(1+\tau)s}}\right) \\
&\le \zeta((1+\tau)s) \sum_{n=N}^{\infty} {r_{\tau,s}}^{n-1} < \varepsilon.
\end{aligned}$$

Thus, for any $s > 1/(1+\tau)$ we have $\mathcal{H}^s(L(\tau)) = 0$. The result follows from the definition of Hausdorff dimension. □

The proof of the lower bound for $\dim L(\tau)$ follows from applying the Beresnevich and Velani Mass Transference Principle [1, Theorem 2], providing an alternative to the method presented in [3]. Proposition 3 is sufficient for the rest of the paper.

Proposition 3 (Applied Mass Transference Principle)**.** *Let* $(I_n)_{n\in\mathbb{N}}$ *be a sequence of intervals in* $[0,1]$ *and* $0 \le s \le 1$*. Suppose* $\lim_{n\to+\infty} \operatorname{diam} I_n = 0$ *and*

$$\mathcal{L}\left([0,1] \cap \limsup_{n\to+\infty} {I_n}^s\right) = 1,$$

where I^s *denotes the interval with the same center as* I *and with radius equal to the radius of* I *raised to the power of* s*. Then*

$$\mathcal{H}^s\left(\limsup_{n\to+\infty} I_n\right) = +\infty.$$

In particular, we have

$$\dim\left(\limsup_{n\to+\infty} I_n\right) \ge s.$$

Proof. The proof essentially applies [1, Theorem 2].

Suppose $0 < s < 1$. Define $f_s : \mathbb{R}^+ \to \mathbb{R}^+$ by $f_s(x) \coloneqq x^s$, for any $x \in \mathbb{R}^+$. Then f_s is a dimension function, that is, f_s is non-decreasing and $\lim_{x\to 0^+} f_s(x) = 0$. Define X_s and its 1-periodic extension Y_s by

$$X_s \coloneqq \limsup_{n\to+\infty} \left([0,1) \cap {I_n}^s\right), \qquad Y_s \coloneqq \bigcup_{z\in\mathbb{Z}} (X_s + z)$$

where $X_s + z$ is the translation of X_s by z, defined as $X_s + z \coloneqq \{x + z : x \in X_s\}$.

Pick any interval $I \subset \mathbb{R}$. Pick any $z \in \mathbb{Z}$. By assumption, $\mathcal{L}(X_s) = 1$, which implies $\mathcal{L}([z, z+1) \cap Y_s) = 1$ and $\mathcal{L}([z, z+1) \setminus Y_s) = 0$. By the additivity of Lebesgue measure,

$$\begin{aligned}\mathcal{L}\left(I \cap [z, z+1) \cap Y_s\right) &= \mathcal{L}(I \cap [z, z+1)) - \mathcal{L}(I \cap [z, z+1) \setminus Y_s) \\ &= \mathcal{L}(I \cap [z, z+1)).\end{aligned}$$

By the countable additivity of Lebesgue measure, the above implies that $\mathcal{L}(I \cap Y_s) = \mathcal{L}(I)$. By a re-enumeration, $Y_s = \limsup_{n\to\infty} {B_n}^s$, where for any $n \in \mathbb{N}$ the set B_n is an interval, and $\lim_{n\to+\infty} \operatorname{diam} B_n = 0$. Thus, the assumption of [1, Theorem 2] is satisfied and its conclusion implies that

$$\mathcal{H}^s\left([0,1] \cap Y_s\right) = \mathcal{H}^s([0,1]) = +\infty.$$

Since $X_s = [0, 1) \cap Y_s$, we have

$$\mathcal{H}^s\left(\limsup_{n\to+\infty} I_n\right) \geq \mathcal{H}^s\left(X_s\right) = +\infty.$$

The result follows from the definition of Hausdorff dimension. □

In order to apply Proposition 3 to establish a lower bound for $\dim L(\tau)$, the set $L(\tau)$ should be expressed as a lim sup set of intervals. Define S to be the set of all 3-tuples consisting of the Lüroth numerators, denominators, and last digits for all real numbers at all levels. Formally,

$$S \coloneqq \bigcup_{n\in\mathbb{N}} \bigcup_{x\in(0,1]} \left(P_n(x), Q_n(x), d_n(x)\right), \tag{16}$$

where $P_n(x)$ and $Q_n(x)$ are defined in (8) and (9), respectively, and $d_n(x) = d_n$ is given in (5). Note that S is countable. Let

$$S = ((P_k, Q_k, d_k))_{k\in\mathbb{N}} \tag{17}$$

be a re-enumeration so that $(Q_k)_{k\in\mathbb{N}}$ is non-decreasing. Proposition 4 can be compared with Proposition 1, which shows that every Lüroth convergent $P_n(x)/Q_n(x)$ satisfies the inequality in Proposition 1.

Proposition 4. *For any $x \in (0, 1]$ and $k \in \mathbb{N}$, if*

$$0 < x - \frac{P_k}{Q_k} \leq \frac{1}{(d_k - 1)Q_k},$$

then P_k/Q_k is a Lüroth convergent of x.

Proof. By the definition of S in (16), there exist $y \in (0,1]$ and $n \in \mathbb{N}$ such that $Q_k = Q_n(y)$. By the definition of $Q_n(y)$ in (9), there exist $d_1, d_2, \ldots, d_n \in \mathbb{N} \setminus \{1\}$ such that

$$Q_k = d_n \prod_{i=1}^{n-1} d_i(d_i - 1) = d_1(d_1-1)d_2(d_2-1)\cdots d_{n-1}(d_{n-1}-1)d_n.$$

The length of the cylinder of $[d_1, d_2, \ldots, d_n]$ is

$$\frac{1}{\prod_{i=1}^{n} d_i(d_i-1)} = \frac{1}{(d_n-1)Q_k}.$$

Therefore, P_k/Q_k is the n-th Lüroth convergent of x. □

Define, for any $\tau \geq 0$, a sequence of intervals $(B_{\tau,k})_{k\in\mathbb{N}}$ in $[0,1]$ by, for any $k \in \mathbb{N}$,

$$B_{\tau,k} \coloneqq \left(\frac{P_k}{Q_k}, \frac{P_k}{Q_k} + \frac{1}{{Q_k}^{1+\tau}}\right) \cap \left(\frac{P_k}{Q_k}, \frac{P_k}{Q_k} + \frac{1}{(d_k-1)Q_k}\right]. \tag{18}$$

Proposition 5. *For any $\tau \geq 0$,*

$$L(\tau) = \limsup_{k\to+\infty} B_{\tau,k}.$$

Proof. The inclusion $\subset$ follows directly from (14) with the definitions of S and $B_{\tau,k}$ in (16) and (18), respectively. The reverse inclusion $\supset$ follows from Proposition 4. □

Proposition 6 (Lower Bound of $\dim L(\tau)$)**.** *For any $\tau \geq 0$,*

$$\dim L(\tau) \geq \frac{1}{1+\tau}.$$

Proof. Pick any $x \in (0,1] \setminus \mathbb{Q}$. Since x is irrational, there exist infinitely many $n \in \mathbb{N}$ such that $d_n(x) \geq 3$. By Proposition 1, for any $n \in \mathbb{N}$, if $d_n(x) \geq 3$ then

$$0 < x - \frac{P_n(x)}{Q_n(x)} < \frac{1}{Q_n(x)}. \tag{19}$$

By (14), $(0,1] \setminus \mathbb{Q} \subset L(0)$. In particular, $\dim L(0) = 1$.

Suppose $\tau > 0$. By direct computation, for any $k \in \mathbb{N}$,

$$\left(\frac{P_k}{Q_k}, \frac{P_k}{Q_k} + \frac{1}{Q_k}\right) \subset {B_{\tau,k}}^{1/(1+\tau)},$$

where B^s denotes the interval with the same center as B and with radius equal to the radius of B raised to the power of s. This is because for any $x \in (0,1] \setminus \mathbb{Q}$, there exist infinitely many $n \in \mathbb{N}$ such that (19) holds. Consequently, we have

$$\begin{aligned}(0,1] \setminus \mathbb{Q} &\subset ((0,1] \setminus \mathbb{Q}) \cap \limsup_{k\to+\infty} \left(\frac{P_k}{Q_k}, \frac{P_k}{Q_k} + \frac{1}{Q_k}\right)\\ &\subset ((0,1] \setminus \mathbb{Q}) \cap \limsup_{k\to+\infty} {B_{\tau,k}}^{1/(1+\tau)}.\end{aligned}$$

By the re-enumeration on S in (17), we have

$$\lim_{k\to+\infty} Q_k = \infty \quad \text{and} \quad \lim_{k\to+\infty} \operatorname{diam} B_{\tau,k} = 0.$$

Therefore, the assumptions of Proposition 3 are satisfied. Indeed,

$$1 = \mathcal{H}^1\left((0,1] \setminus \mathbb{Q}\right) \leq \mathcal{H}^1\left((0,1] \setminus \mathbb{Q} \cap \limsup_{k\to+\infty} {B_{\tau,k}}^{1/(1+\tau)}\right) \leq 1.$$

By the conclusion of Propositions 3 and 5,

$$\mathcal{H}^{1/(1+\tau)}(L(\tau)) = \mathcal{H}^{1/(1+\tau)}\left(\limsup_{k\to+\infty} B_{\tau,k}\right) = +\infty. \tag{20}$$

In particular, we have

$$\dim L(\tau) = \dim\left(\limsup_{k\to+\infty} B_{\tau,k}\right) \geq \frac{1}{1+\tau}.$$

Thus, the lower bound of $\dim L(\tau)$ is established. □

Proof of Theorem 6. By combining Proposition 2 and 6, the Hausdorff dimension of $L(\tau)$ is established for any $\tau \geq 0$. By (20), the Hausdorff measure of $L(\tau)$ at the critical exponent is obtained. This completes the proof. □

4.3. Proof of Theorem 7

At first glance, Theorem 7 is stronger than Theorem 6. The proof shows that Theorem 7 follows by applying Theorem 6 twice.

Proof of Theorem 7. Suppose $\underline{\tau}_\psi > 0$. Pick any $0 < \varepsilon < \underline{\tau}_\psi$. By the definition of $\underline{\tau}_\psi$, there exists $N_\varepsilon \in \mathbb{N}$ such that for any $q \in \mathbb{N}$, if $q > N_\varepsilon$ then

$$\begin{aligned}0 < \underline{\tau}_\psi - \varepsilon &< -\frac{\log \psi(q)}{\log q}\\ \psi(q) &< \frac{1}{q^{\underline{\tau}_\psi - \varepsilon}}.\end{aligned}$$

Thus, $L(\psi) \subset L(\underline{\tau}_\psi - \varepsilon)$. By Theorem 6 and the monotonicity of Hausdorff dimension,

$$\dim L(\psi) \leq \dim L(\underline{\tau}_\psi - \varepsilon) = \frac{1}{1 + \underline{\tau}_\psi - \varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, the upper bound is proved.

Pick any $\varepsilon > 0$. By the definition of $\overline{\tau}_\psi$, there exists $N_\varepsilon \in \mathbb{N}$ such that for any $q \in \mathbb{N}$, if $q > N_\varepsilon$ then

$$\begin{aligned} -\frac{\log \psi(q)}{\log q} &< \overline{\tau}_\psi + \varepsilon \\ \frac{1}{q^{\overline{\tau}_\psi + \varepsilon}} &< \psi(q). \end{aligned}$$

Thus, $L(\overline{\tau}_\psi + \varepsilon) \subset L(\psi)$. By Theorem 6 and the monotonicity of Hausdorff dimension,

$$\frac{1}{1 + \overline{\tau}_\psi + \varepsilon} = \dim L(\overline{\tau}_\psi + \varepsilon) \leq \dim L(\psi).$$

Since $\varepsilon > 0$ is arbitrary, the lower bound is proved. This completes the proof. □

4.4. Proof of Theorem 8

Theorem 8 refines the general dimensional bounds established in Theorem 7. The proof for the upper bound follows a similar approach as in Theorem 7. For the lower bound, Proposition 3 is applied once again.

Proof of Theorem 8. Suppose $\underline{\lambda}_\psi > 0$. Pick any $0 < \varepsilon < \underline{\lambda}_\psi$. By the definition of $\underline{\lambda}_\psi$, there exists $K_\varepsilon \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, there exists $q_k \in \mathbb{N}$ such that for any $q \in S_k$, if $k > K_\varepsilon$ and $q > q_k$ then

$$\begin{aligned} \underline{\lambda}_\psi - \varepsilon &< -\frac{\log \psi(q)}{\log q} \\ \psi(q) &< \frac{1}{q^{\underline{\lambda}_\psi - \varepsilon}}. \end{aligned}$$

Thus, $L(\psi) \subset L(\underline{\lambda}_\psi - \varepsilon)$. By Theorem 6 and the monotonicity of Hausdorff dimension,

$$\dim L(\psi) \leq \dim L(\underline{\lambda}_\psi - \varepsilon) = \frac{1}{1 + \underline{\lambda}_\psi - \varepsilon}.$$

Pick any $\varepsilon > 0$. By the definition of $\overline{\lambda}_\psi$, there exists a strictly increasing sequence $(k_j)_{j \in \mathbb{N}}$ of positive integers such that for any $j \in \mathbb{N}$,

$$\alpha_j := \limsup_{q \in S_{k_j}} \frac{-\log \psi(q)}{\log q} < \overline{\lambda}_\psi + \frac{\varepsilon}{2}.$$

For any $j \in \mathbb{N}$, by the definition of the limit superior, there exists $q_{k_j} \in S_{k_j}$ such that for any $q \in S_{k_j}$, if $q > q_{k_j}$ then

$$\overline{\lambda}_\psi + \varepsilon > \alpha_j + \frac{\varepsilon}{2} > \frac{-\log \psi(q)}{\log q}.$$

Thus, for any $\varepsilon > 0$, there exists a strictly increasing sequence $(k_j)_{j\in\mathbb{N}}$ of positive integers such that for any $j \in \mathbb{N}$, there exists $q_{k_j} \in S_{k_j}$ such that for any $q \in S_{k_j}$, if $q > q_{k_j}$ then

$$\frac{-\log \psi(q)}{\log q} < \overline{\lambda}_\psi + \varepsilon.$$

Without loss of generality, for any $j \in \mathbb{N}$ we have $q_{k_{j+1}} > q_{k_j}$. For any $j \in \mathbb{N}$, define $a_j := \lceil \log q_{k_j} / \log 2 \rceil + 1 - k_j \geq 1$, where $\lceil\ \rceil$ is the ceiling function. Then, for any $j \in \mathbb{N}$ and $q \in S_{k_j+a_j} \subset S_{k_j}$, which implies $q > q_{k_j}$ and

$$\frac{1}{q^{\overline{\lambda}_\psi + \varepsilon}} < \psi(q). \tag{21}$$

Thus, for any $j \in \mathbb{N}$ and $d_1, \ldots, d_{k_j+a_j} \in \mathbb{N} \setminus \{1\}$,

$$\left(\frac{P}{Q}, \frac{P}{Q} + \frac{1}{Q^{1+\overline{\lambda}_\psi+\varepsilon}}\right) \subset \left(\frac{P}{Q}, \frac{P}{Q} + \frac{\psi(Q)}{Q}\right)$$

where $Q = d_1(d_1-1)\cdots d_{k_j+a_j} \in S_{k_j+a_j}$ and $P = [d_1, \ldots, d_{k_j+a_j}]Q$. Thus,

$$\left(\frac{P}{Q}, \frac{P}{Q} + \frac{1}{Q}\right] \subset \left(\frac{P}{Q}, \frac{P}{Q} + \frac{1}{Q^{1+\overline{\lambda}_\psi+\varepsilon}}\right)^{1/(1+\overline{\lambda}_\psi+\varepsilon)} \subset \left(\frac{P}{Q}, \frac{P}{Q} + \frac{\psi(Q)}{Q}\right)^{1/(1+\overline{\lambda}_\psi+\varepsilon)} \tag{22}$$

where B^s denotes the interval with the same center as B and with a radius equal to the radius of B raised to the power of s. By Proposition 1 and (22), for any $j \in \mathbb{N}$,

$$(0,1] \subset \bigcup_{d_1,\ldots,d_{k_j+a_j} \in \mathbb{N}\setminus\{1\}} \left(\frac{P}{Q}, \frac{P}{Q} + \frac{\psi(Q)}{Q}\right)^{1/(1+\overline{\lambda}_\psi+\varepsilon)}. \tag{23}$$

Let $((P_{k_i}, Q_{k_i}, d_{k_i}))_{i\in\mathbb{N}}$ be the sub-re-enumeration of $S = ((P_k, Q_k, d_k))_{k\in\mathbb{N}}$ in (17) so that for any $i \in \mathbb{N}$ and $q = Q_{k_i}$, (21) is satisfied. Define a sequence of intervals $(B_{\psi,\varepsilon,i})_{i\in\mathbb{N}}$ by, for any $i \in \mathbb{N}$,

$$B_{\psi,\varepsilon,i} := \left(\frac{P_{k_i}}{Q_{k_i}}, \frac{P_{k_i}}{Q_{k_i}} + \frac{1}{{Q_{k_i}}^{1+\overline{\lambda}_\psi+\varepsilon}}\right) \cap \left(\frac{P_{k_i}}{Q_{k_i}}, \frac{P_{k_i}}{Q_{k_i}} + \frac{1}{(d_{k_i}-1)Q_{k_i}}\right].$$

Note that for any $\varepsilon > 0$,

$$\limsup_{i\to+\infty} B_{\psi,\varepsilon,i} \subset L(\psi).$$

By direct computation, for any $i \in \mathbb{N}$,

$$\left(\frac{P_{k_i}}{Q_{k_i}}, \frac{P_{k_i}}{Q_{k_i}} + \frac{1}{Q_{k_i}}\right) \subset {B_{\psi,\varepsilon,i}}^{1/(1+\overline{\lambda}_\psi+\varepsilon)}.$$

By (23), for any $j \in \mathbb{N}$, the intervals at the $(k_j + a_j)$-th level cover $(0,1]$. Thus,

$$(0,1] \cap \limsup_{i\to+\infty} {B_{\psi,\varepsilon,i}}^{1/(1+\overline{\lambda}_\psi+\varepsilon)} = (0,1].$$

In particular, we have

$$\mathcal{L}\left([0,1] \cap \limsup_{i\to+\infty} {B_{\psi,\varepsilon,i}}^{1/(1+\overline{\lambda}_\psi+\varepsilon)}\right) = 1.$$

By the sub-re-enumeration on S, we have $\lim_{i\to+\infty} Q_{k_i} = \infty$ and consequently $\lim_{i\to+\infty} \operatorname{diam} B_{\psi,\varepsilon,i} = 0$. By Proposition 3 and the monotonicity of Hausdorff dimension,

$$\dim L(\psi) \geq \dim\left(\limsup_{i\to+\infty} B_{\psi,\varepsilon,i}\right) \geq \frac{1}{1+\overline{\lambda}_\psi+\varepsilon}.$$

This completes the proof. □

4.5. Proof of Theorem 9

Theorem 9 presents a counterexample and suggests a refinement to the conjecture stated by Tan and Zhou [7]. The proof constructs the function that takes small values at integers that are highly divisible by 2, and consequently does not satisfy the monotonicity condition.

Proof of Theorem 9. Let $\tau \geq 0$. Define $\psi \coloneqq \psi_\tau : \mathbb{N} \to (0,1]$ by, for any $q \in \mathbb{N}$,

$$\psi(q) = \frac{1}{q^{\tau+\nu_2(q)}},$$

where $\nu_2(q) \coloneqq \max\{k \in \mathbb{N}\cup\{0\} : 2^k \mid q\}$ is the 2-adic valuation of q. For any $n \in \mathbb{N}$, if $n > \tau + \sqrt{\tau}$ then

$$\psi(2^n+1) = \frac{1}{(2^n+1)^\tau} \geq \frac{1}{2^{(\tau+1)n}} > \frac{1}{2^{(\tau+n)n}} = \psi(2^n).$$

Thus, ψ is not eventually non-increasing. By direct computation, $\underline{\tau}_\psi = \tau$ can be verified. It remains to prove that $\dim L(\psi) = 0$.

For any $q, k \in \mathbb{N}$, if 2^k divides q then $\nu_2(q) \geq k$ and

$$0 < \psi(q) \leq \frac{1}{q^{\tau+k}}. \tag{24}$$

Pick any $j \in \mathbb{N}$, any $\rho > 0$, and any $N \in \mathbb{N}$. Suppose $N > -\log \rho / \log 2$. The following forms a ρ-cover of $L(\psi)$:

$$\bigcup_{n=N}^{\infty} \bigcup_{(d_1,\ldots,d_n)\in(\mathbb{N}\setminus\{1\})^n} \left(\frac{P_n}{Q_n}, \frac{P_n}{Q_n} + \frac{\psi(Q_n)}{Q_n} \right), \tag{25}$$

where for any $n \in \mathbb{N}$ we set $Q_n \coloneqq d_1(d_1-1)\cdots d_{n-1}(d_{n-1}-1)d_n \geq 2^n$ and $P_n \coloneqq [d_1, d_2, \ldots, d_n]Q_n$. Pick any $s > 1/(1+\tau+j)$. Define

$$\begin{aligned} r_{\tau,j,s} &\coloneqq \sum_{d=2}^{\infty} \left(\frac{1}{d(d-1)} \right)^{(1+\tau+j)s} \in (0,1); \\ C_{\tau,j,s} &\coloneqq \sum_{d=2}^{\infty} \frac{1}{d^{(1+\tau+j)s}} < +\infty. \end{aligned}$$

Pick any $\varepsilon > 0$. Without loss of generality, $N > j$ and

$$C_{\tau,j,s} \sum_{n=N}^{\infty} {r_{\tau,j,s}}^{n-1} < \varepsilon.$$

Note that for any $n \in \mathbb{N}$ and $d_1, d_2, \ldots, d_n \in \mathbb{N} \setminus \{1\}$, if $n \geq N$ then $n > j$ and $2^j \mid d_1(d_1-1)\cdots d_{n-1}(d_{n-1}-1)d_n$. By applying (24), the s-Hausdorff measure for the ρ-cover (25) is obtained as

$$\begin{aligned} \mathcal{H}^s_\rho(L(\psi)) &\leq \sum_{n=N}^{\infty} \sum_{(d_1,\ldots,d_n)\in(\mathbb{N}\setminus\{1\})^n} \left(\frac{\psi(d_1(d_1-1)\cdots d_n)}{d_1(d_1-1)\cdots d_n} \right)^s \\ &\leq \sum_{n=N}^{\infty} \sum_{(d_1,\ldots,d_n)\in(\mathbb{N}\setminus\{1\})^n} \left(\frac{1}{(d_1(d_1-1)\cdots d_n)^{1+\tau+j}} \right)^s \\ &= \sum_{n=N}^{\infty} \left(\sum_{d=2}^{\infty} \left(\frac{1}{d(d-1)} \right)^{(1+\tau+j)s} \right)^{n-1} \sum_{d=2}^{\infty} \frac{1}{d^{(1+\tau+j)s}} \\ &= C_{\tau,j,s} \sum_{n=N}^{\infty} {r_{\tau,j,s}}^{n-1} < \varepsilon. \end{aligned}$$

Hence, $\mathcal{H}^s(L(\psi)) = 0$ and consequently for any $j \in \mathbb{N}$,

$$\dim L(\psi) \leq \frac{1}{1+\tau+j}.$$

This completes the proof. □

4.6. Proof of Theorem 10

Theorem 10 provides a partial result toward confirming the revised conjecture. By Theorem 7, it suffices to prove the following lower bound:

$$\dim L(\psi) \geq \frac{1}{1+\underline{\tau}_\psi}.$$

The assumption $\underline{\tau}_\psi > 1$ guarantees that the set $L(\psi)$ can be expressed in a lim sup set of intervals, which allows an application of Proposition 3 to establish the lower bound. Proposition 7 is a similar result to Proposition 5.

Proposition 7. *Let $\psi : \mathbb{N} \to (0,1]$. Suppose $\underline{\tau}_\psi > 1$. Then*

$$L(\psi) = \limsup_{k\to+\infty} \left(\frac{P_k}{Q_k}, \frac{P_k}{Q_k} + \frac{\psi(Q_k)}{Q_k} \right),$$

where $((P_k, Q_k, d_k))_{k\in\mathbb{N}}$ is the re-enumeration of S in (17).

Proof. The inclusion $\subset$ follows directly from the definition of S and (14). It remains to prove the reverse inclusion $\supset$.

By the assumption that $\underline{\tau}_\psi > 1$, there exists $q_0 \in \mathbb{N}$ such that for any $q \in \mathbb{N}$, if $q > q_0$ then

$$0 < \psi(q) < \frac{1}{q}. \tag{26}$$

Pick any $x \in \limsup_{k\to+\infty}(P_k/Q_k, P_k/Q_k + \psi(Q_k)/Q_k)$. By the definition of the limit superior, there exists a strictly increasing sequence $(k_j)_{j\in\mathbb{N}}$ of positive integers such that for any $j \in \mathbb{N}$ we have $x \in (P_{k_j}/Q_{k_j}, P_{k_j}/Q_{k_j} + \psi(Q_{k_j})/Q_{k_j})$. Since $(Q_k)_{k\in\mathbb{N}}$ is non-decreasing, $\lim_{k\to\infty} Q_k = \infty$ and there exists $j_0 \in \mathbb{N}$ such that $Q_{k_{j_0}} > q_0$. For any $j \in \mathbb{N}$, if $j > j_0$ then $Q_{k_j} \geq Q_{k_{j_0}} > q_0$ and (26) implies that

$$0 < x - \frac{P_{k_j}}{Q_{k_j}} < \frac{\psi(Q_{k_j})}{Q_{k_j}} < \frac{1}{{Q_{k_j}}^2} < \frac{1}{(d_{k_j}-1)Q_{k_j}}.$$

By Proposition 4, P_{k_j}/Q_{k_j} is a Lüroth convergent of x. Hence $x \in L(\psi)$. □

Proof of Theorem 10. By the divergent assumption,

$$\sum_{q=1}^{\infty} -\frac{\theta_t(q)\log\theta_t(q)}{q} \geq \sum_{q=1}^{\infty} -\frac{2^{-t}q^{1-t}\psi^t(q)\log\left(2^{-t}q^{1-t}\psi^t(q)\right)}{q}.$$

The right-hand side diverges to $+\infty$. By the divergent case of Theorem 4,

$$\mathcal{L}(L(\theta_t)) = 1. \tag{27}$$

By the assumption that $\underline{\tau}_\psi > 1$ and Proposition 7,

$$L(\psi) = \limsup_{k\to+\infty} \left(\frac{P_k}{Q_k}, \frac{P_k}{Q_k} + \frac{\psi(Q_k)}{Q_k}\right).$$

By direct computation,

$$L(\theta_t) \subset \limsup_{k\to+\infty} \left(\frac{P_k}{Q_k}, \frac{P_k}{Q_k} + \frac{\psi(Q_k)}{Q_k}\right)^t.$$

By (27) and the monotonicity of Lebesgue measure,

$$\mathcal{L}\left([0,1] \cap \limsup_{k\to+\infty} \left(\frac{P_k}{Q_k}, \frac{P_k}{Q_k} + \frac{\psi(Q_k)}{Q_k}\right)^t\right) = 1.$$

Since $Q_k \to +\infty$ as $k \to \infty$, we have

$$\lim_{k\to+\infty} \operatorname{diam} \left(\frac{P_k}{Q_k}, \frac{P_k}{Q_k} + \frac{\psi(Q_k)}{Q_k}\right) = 0.$$

By Proposition 3,

$$\dim(L(\psi)) = \dim\left(\limsup_{k\to+\infty} \left(\frac{P_k}{Q_k}, \frac{P_k}{Q_k} + \frac{\psi(Q_k)}{Q_k}\right)\right) \geq t = \frac{1}{1+\underline{\tau}_\psi}.$$

This completes the proof. □

References

[1] V. Beresnevich and S. Velani, A mass transference principle and the Duffin–Schaeffer conjecture for Hausdorff measures, *Ann. of Math. (2)* **164** (3) (2006), 971–992.

[2] A. S. Besicovitch, Sets of fractional dimensions (IV): on rational approximation to real numbers, *J. Lond. Math. Soc.* **9** (2) (1934), 126–131.

[3] C. Cao, J. Wu, and Z. Zhang, The efficiency of approximating real numbers by Lüroth expansion, *Czechoslovak Math. J.* **63** (2) (2013), 497–513.

[4] M. M. Dodson, Hausdorff dimension, lower order and Khintchine's theorem in metric Diophantine approximation, *J. Reine Angew. Math.* **432** (1992), 69–76.

[5] V. Jarník, Zur metrischen Theorie der diophantischen Approximationen, *Prace Mat.-Fiz.* **36** (1) (1928–1929), 91–106.

[6] A. Khintchine, Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der diophantischen Approximationen, *Math. Ann.* **92** (1924), 115–125.

[7] B. Tan and Q. Zhou, Approximation properties of Lüroth expansions, *Discrete Contin. Dyn. Syst. Ser. A* **41** (6) (2021), 2873–2890.

[8] B. Tan and Q. Zhou, Dimension theory of the product of partial quotients in Lüroth expansions, *Int. J. Number Theory* **17** (5) (2021), 1139–1154.