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A NEW ASYMPTOTIC FORMULA FOR $\pi(N)$ WITH THE DIVISOR COUNTING FUNCTION

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Abstract

This paper concerns the prime counting function $\pi(n)$ and the divisors counting function $\tau(n)$ of integers n. The main result is $\left|\pi(n) - \overline{\pi}(n)\right| < 0.7129749140934$, for all positive integers $n \geqslant 2$, where

$$\overline{\pi}(n) = \sum_{k=2}^{n} \exp\left(-k\left(\overline{\tau}(k) - 2\right)\right)$$

and

$$\overline{\tau}(k) = \sum_{j=1}^{k} \exp\left(-\left(\frac{\pi}{2}\right)^2 k j^3 \sin^2\left(\frac{k\pi}{j}\right)\right)$$

for all positive integers k. We have

$$\overline{\tau}(k) = \tau(k) + O\left(\frac{1}{e^k - 1}\right) \quad \text{as } k \to \infty.$$

1. Introduction

Throughout this paper, let $\log n$ denote the natural logarithm, with base e. The divisor function $\tau(n)$ and the prime-counting function $\pi(n)$ are essential in number theory, each playing distinct yet related roles in examining the distribution of divisors and prime numbers. The *divisor function*, often denoted by $d(n)$, $\sigma_0(n)$, or

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$\tau(n)$, counts the positive divisors of a positive integer n. For example, for $n = 6$, the divisors are $1, 2, 3$, and 6, so $\tau(6) = 4$. If n has a prime factorization

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k},$$

where p_i are distinct primes and $e_i \geqslant 1$, then

$$\tau(n) := \prod_{j=1}^{k} (e_j + 1) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1).$$

This follows from the fact that each exponent e_i provides $e_i + 1$ options for the power of divisors. The function is multiplicative, meaning that for coprime m and n, or $\gcd(m, n) = 1$, $\tau(mn) = \tau(m)\tau(n)$. It plays a key role in the study of the Riemann zeta-function, with its Dirichlet series expressed by $\zeta^2(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$ for $\Re(s) > 1$, linking it to the Riemann zeta-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$; [5]. Asymptotically, the average order is $\log n$, and the summatory function $\sum_{k=1}^{n} \tau(k)$ is $n \log n + (2\gamma - 1)n$ where γ is the Euler–Mascheroni constant. This reflects the typical growth, with $\tau(n)$ often close to $(\log n)^{\delta}$ for some δ. In other words, the normal values of $\tau(n)$ are $(\log n)^{\log 2 + o(1)}$. The maximal order is given by

$$\limsup_{n \to \infty} \frac{\log \tau(n)}{\log n / \log \log n} = \log 2,$$

and $\liminf_{n \to \infty} \tau(n) = 2$. An important inequality is $\tau(n) \leqslant 2\sqrt{n}$, proven by pairing divisors d and n/d, with equality only at $n = 1$. Additionally, the only k with $\tau(k) \geqslant k/2$ are $1, 2, 3, 4, 6, 8, 12$, highlighting the rarity of numbers with many divisors relative to their value. The growth of $\tau(n)$ is slow compared to n. Formally, for any $\epsilon > 0$, we have $\tau(n) = o(n^{\epsilon})$ as $n \to \infty$, meaning $\tau(n)$ grows slower than any positive power of n.

Jean-Marie De Konincks's 2020 paper [4] provides refined upper bounds. For example, for $n \geqslant 2$,

$$\tau(n) \leqslant \left(\eta_2 \frac{\log n}{\omega(n) \log_+ \omega(n)} \right)^{\omega(n)},$$

where $\eta_2 = 2.0907132\ldots$, and for $k = \omega(n) \geqslant 74$, it follows that $\tau(n) \leqslant \left(1 + \frac{\log n}{k \log k}\right)^k$. The notation $\log_+ x$ is defined as $\log \max(2, x)$. This ensures that the argument of the logarithm is at least 2, thereby circumventing issues such as $\log 1 = 0$ or negative values when $\omega(n) = 1$ (noting that the theorems in the paper [4] apply for $n \geqslant 2$, with $\log_+$ employed for $\omega(n) \geqslant 1$). For instance, if $\omega(n) \geqslant 2$, then $\log_+ \omega(n) = \log \omega(n)$; otherwise (when $\omega(n) = 1$), $\log_+ \omega(n) = \log 2$. In this case, $\omega(n)$ denotes the number of distinct prime factors of n.

This improves earlier bounds like Wigert's [10]

$$\tau(n) \leqslant 2 \log n \log \log n (1 + o(1)).$$

The *prime-counting function* $\pi(n)$ counts the number of primes $p \leqslant n$. The prime number theorem, a fundamental result, states that

$$\pi(n) \sim \frac{n}{\log n} \quad \text{as } n \to \infty,$$

meaning the ratio $\pi(n)/(n/\log n) \to 1$. A more refined approximation is $\pi(n) \sim \mathrm{li}(n)$, where $\mathrm{li}(n) = \int_2^n \frac{dt}{\log t}$ is the logarithmic integral. The error term of the prime number theorem is a focus of research. De la Vallée Poussin in 1899 [3] gave $\pi(n) = \mathrm{li}(n) + O\left(ne^{-a\sqrt{\log n}}\right)$ for some $a > 0$, and Kevin Ford in 2002 [6] improved it to

$$\pi(n) = \mathrm{li}(n) + O\left(n \exp\left(-0.2098 \left(\log n\right)^{3/5} \left(\log\log n\right)^{-1/5}\right)\right) \quad \text{as } n \to \infty.$$

Mossinghoff and Trudgian in 2015 [7] provided

$$\Big|\pi(n) - \mathrm{li}(n)\Big| \leqslant 0.2593 \left(n/\left(\log n\right)^{3/4}\right) \exp\left(-\sqrt{\log n/6.315}\right)$$

for $n \geqslant 229$. Whether or not the Riemann hypothesis implies that the error term in the prime number theorem satisfies $\big|\pi(n) - \mathrm{li}(n)\big| = O(\sqrt{n}\log n)$ remains one of the most significant open questions in analytic number theory [2,8,9]. For $n > 1$, one of the most precise formulae for the prime-counting function $\pi(n)$ is $R(n) - \sum_\rho R(n^\rho)$, where

$$R(n) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \mathrm{li}\left(n^{1/k}\right),$$

$\mu(k)$ denotes the Möbius function, and the sum is taken over the non-trivial zeros ρ of the Riemann zeta-function, ordered by increasing imaginary part.

The most direct connection between $\tau(n)$ and $\pi(n)$ arises from the definition of prime numbers. It is known that $\tau(n) = 2$ if and only if n is prime. This yields the following straightforward relationship [1]: the prime counting function $\pi(n)$ can be expressed in terms of the divisor function as

$$\pi(n) = \sum_{m=1}^{n} 1_{\tau(m)=2},$$

where $1_{\tau(m)=2}$ is the indicator function defined by

$$1_{\tau(m)=2} := \begin{cases} 1, & \text{if } \tau(m) = 2, \text{ i.e., } m \text{ is prime;} \\ 0, & \text{otherwise.} \end{cases}$$

This shows that $\pi(n)$ counts the integers up to n with exactly two divisors, aligning with the definition of primes.

Our main results are the following.

Theorem 1. *Let* $\tau(k)$ *be the number of the positive divisors of a positive ineteger* k*. Define the function* $\overline{\tau}(k)$ *by*

$$\overline{\tau}(k) := \sum_{j=1}^{k} \exp\left(-\left(\frac{\pi}{2}\right)^2 kj^3 \sin^2\left(\frac{k\pi}{j}\right)\right), \tag{1}$$

for all positive integers k*. We have*

$$\overline{\tau}(k) = \tau(k) + O\left(\frac{1}{e^k - 1}\right) \quad \textit{as } k \to \infty.$$

Using $\overline{\tau}(k)$ given in Theorem 1, we have the following result.

Theorem 2. *Let* $\pi(n)$ *be the prime counting function of* n*. Define the function* $\overline{\pi}(n)$ *by*

$$\overline{\pi}(n) := \sum_{k=2}^{n} e^{-k(\overline{\tau}(k)-2)},$$

where $\overline{\tau}(k)$ *is given in Equation* (1)*. We have*

$$\Big|\pi(n) - \overline{\pi}(n)\Big| < 0.7129749140934,$$

for all positive integers $n \geqslant 2$*.*

Theorem 3. *Let* $\sigma(n)$ *be the sum of the positive divisors of a positive integer* n *given by*

$$\sigma(n) = \sum_{k=1}^{n} k[k|n],$$

where $[k|n]$ *is the divisibility indicator function defined in Equation* (2)*, with* $d = k$*, for all positive integers* k, n *such that* $1 \leqslant k \leqslant n$*. Suppose there exists a function* $A : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_+$*, with* $A(k, n)\sin^2(\frac{n\pi}{k}) \geqslant 1$ *at* $k \nmid n$*, for all* k, n *such that* $1 \leqslant k \leqslant n$*. If* $\overline{\sigma}(n)$ *is given by*

$$\overline{\sigma}(n) = \sum_{k=1}^{n} k \exp\left(-nkA(k, n)\sin^2\left(\frac{n\pi}{k}\right)\right).$$

Then, for all $n \geqslant 1$*, the following inequality holds:*

$$\overline{\sigma}(n) - \sigma(n) < \frac{n}{e^n - 1}.$$

2. Notation and Preliminaries

Throughout this paper, we denote by $\mathbb{R}$, $\mathbb{Z}_+$ or $\mathbb{N}$, and $\mathbb{C}$ the sets of real numbers, positive integers, and complex numbers, respectively. We let $\mathcal{P}$ denote the set of prime numbers. Let $\mathbb{R}_+$ denote the set of positive real numbers.

The big-O notation is defined as follows: for functions f_n and g_n, we get $f_n = O(g_n)$ if there exist constants $C > 0$ and $n_0 > 0$ such that for all $n > n_0$, we obtain $|f_n| \leqslant C|g_n|$. The same convention applies to the asymptotic notation $X \sim Y$, which means $\lim_{x\to\infty} \frac{X(x)}{Y(x)} = 1$ or $X = (1 + o(1))Y$. The little-o notation, $f_n = o(g_n)$, means $\lim_{n\to\infty} \frac{f_n}{g_n} = 0$, indicating f_n grows stricly slower than g_n.

For $d, n \in \mathbb{Z}$, *the divisibility indicator function* is defined by

$$[d|n] := \begin{cases} 1, & \text{if } d \text{ divides } n; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Let X be a set and $\mathcal{A} \subseteq X$ a subset. The indicator function given in Equation (2) (also called the charateristic function) of the set $\mathcal{A}$ is the function

$$1_{\mathcal{A}} : X \longrightarrow \{0, 1\}$$

defined by

$$1_{\mathcal{A}(x)} := \begin{cases} 1, & \text{if } x \in \mathcal{A}; \\ 0, & \text{if } x \notin \mathcal{A}. \end{cases}$$

For sets $\mathcal{A}$ and $\mathcal{B}$, the indicator functions satisfy the following identities:

$$1_{\mathcal{A}\cap\mathcal{B}(x)} = 1_{\mathcal{A}(x)} \cdot 1_{\mathcal{B}(x)},$$

$$1_{\mathcal{A}\cup\mathcal{B}(x)} = 1_{\mathcal{A}(x)} + 1_{\mathcal{B}(x)} - 1_{\mathcal{A}(x)} \cdot 1_{\mathcal{B}(x)},$$

$$1_{\mathcal{A}^{\complement}(x)} = 1 - 1_{\mathcal{A}(x)},$$

$$1_{\mathcal{A}\setminus\mathcal{B}(x)} = 1_{\mathcal{A}(x)} - 1_{\mathcal{B}(x)}.$$

Let X be a finite set and $\mathcal{A} \subseteq X$. Then, the cardinality of $\mathcal{A}$ is given by

$$|\mathcal{A}| = \sum_{x\in X} 1_{\mathcal{A}(x)}.$$

Let $X \subseteq \mathbb{Z}$ be a finite set and let $\mathcal{A} \subseteq X$. Suppose $[a, b] \subseteq X$, where $a, b \in \mathbb{Z}$, with $a \leqslant b$, denotes the closed integer interval from a to b. Thus, the cardinality of the set $\mathcal{A} \cap [a, b]$ is given by

$$\big|\mathcal{A} \cap [a, b]\big| = \sum_{x=a}^{b} 1_{\mathcal{A}(x)},$$

where $1_{\mathcal{A}} : X \to \{0, 1\}$, i.e.,

$$1_{\mathcal{A}(x)} = \begin{cases} 1, & \text{if } x \in \mathcal{A}; \\ 0, & \text{if } x \notin \mathcal{A}. \end{cases}$$

3. Lematta

In this section, our goal is to prove the lemmas needed to obtain the main results.

Lemma 1. *Let $k, n \in \mathbb{N}$ with $1 \leqslant k \leqslant n$, and let $[k|n]$ be the divisibility indicator function. Suppose there exists a function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_+$, with $A(k,n)\sin^2(\frac{n\pi}{k}) \geqslant 1$ at $k \nmid n$, for all k and n such that $1 \leqslant k \leqslant n$. If $\overline{[k|n]}$ is given by*

$$\overline{[k|n]} := \exp\left(-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)\right).$$

Then, for all k and n, the following inequality holds:

$$\left|[k|n] - \overline{[k|n]}\right| \leqslant e^{-nk}. \tag{3}$$

Proof. Consider

$$\overline{[k|n]} = \exp\left(-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)\right)$$

as an approximation function of $[k|n]$. The proof proceeds by case analysis, considering the scenarios in which k divides n and those in which it does not.

`Case 1:` $k|n$. In this case, the argument of the sine is $m\pi$, where m is an integer, and hence, $\sin^2\left(\frac{n\pi}{k}\right) = 0$. Consequently, the approximation $\overline{[k|n]}$ becomes

$$\overline{[k|n]} = \exp\left(-nkA(k,n)\cdot 0\right) = \exp(0) = 1.$$

Given that $\overline{[k|n]} = 1$ when $k|n$, we compute the difference

$$\left|[k|n] - \overline{[k|n]}\right| = |1-1| = 0.$$

Now, since $1 \leqslant k \leqslant n$, it follows that $nk \geqslant 1$, so

$$e^{-nk} > 0.$$

Therefore, Inequality (3) holds for this case.

`Case 2:` $k \nmid n$. If k does not divide n, then $[k|n] = 0$. We need to show that

$$\left|0 - \overline{[k|n]}\right| \leqslant e^{-nk}.$$

First, note that $\overline{[k|n]} = \exp\left(-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)\right)$. Since the exponential function $\exp(x)$ is positive for all real x, and the exponent $-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)$ is real with $A(k,n)\sin^2\left(\frac{n\pi}{k}\right) \geqslant 1$ and $nk \geqslant 1$, it follows that

$$\exp\left(-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)\right) \leqslant \exp\left(-nk\right),$$

for $k \nmid n$. Therefore, Inequality (3) holds in this case as well. Combining both cases, for all k and n such that $1 \leqslant k \leqslant n$, we conclude that

$$\left|[k|n] - \overline{[k|n]}\right| \leqslant e^{-nk}.$$

This completes the proof. □

Lemma 2. *Let $\tau(n)$ be the number of the positive divisors of a positive integer n. Suppose there exists a function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_+$, with $A(k,n)\sin^2(\frac{n\pi}{k}) \geqslant 1$ at $k \nmid n$, for all positive integers k and n such that $1 \leqslant k \leqslant n$. If $\overline{\tau}(n)$ is given by*

$$\overline{\tau}(n) := \sum_{k=1}^{n} \exp\left(-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)\right).$$

Then, as $n \to \infty$, we have

$$\overline{\tau}(n) = \tau(n) + O\left(\frac{1}{e^n - 1}\right).$$

Proof. The proof uses Lemma 1, which states that under the same conditions on $A(k,n)$, we have $\left|[k|n] - \overline{[k|n]}\right| \leqslant e^{-nk}$, where $\overline{[k|n]} = \exp\left(-nkA(k,n)\sin^2(\frac{n\pi}{k})\right)$. Moreover, Lemma 1 implies that $[k|n] \leqslant \overline{[k|n]}$ for all positive integers k and n with $1 \leqslant k \leqslant n$. We analyze the difference $\overline{\tau}(n) - \tau(n)$ and bound it using exponential decay. Now, express the difference

$$\tau(n) - \overline{\tau}(n) = \sum_{k=1}^{n}[k|n] - \sum_{k=1}^{n}\overline{[k|n]} = \sum_{k=1}^{n}\left([k|n] - \overline{[k|n]}\right).$$

From the proof of Lemma 1, recall the following:

- When $k|n$, we have $[k|n] = 1$ and $\overline{[k|n]} = \exp\left(-nkA(k,n)\cdot 0\right) = 1$. Since $\sin(\frac{n\pi}{k}) = 0$, we obtain $[k|n] - \overline{[k|n]} = 0$.
- When $k \nmid n$, we have $[k|n] = 0$ and $\overline{[k|n]} = \exp\left(-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)\right) > 0$, so $[k|n] - \overline{[k|n]} = -\overline{[k|n]} < 0$.

Thus, the sum

$$\sum_{k=1}^{n}\left([k|n] - \overline{[k|n]}\right) = \sum_{k\nmid n}\left(-\overline{[k|n]}\right) = -\sum_{k\nmid n}\overline{[k|n]},$$

which is negative, implying $\overline{\tau}(n) > \tau(n)$.

To bound the error, consider

$$\begin{aligned}\overline{\tau}(n) - \tau(n) &= \sum_{k=1}^{n}\overline{[k|n]} - \sum_{k=1}^{n}[k|n] \\ &= \sum_{k=1}^{n}\left(\overline{[k|n]} - [k|n]\right) \\ &= \sum_{k\nmid n}\overline{[k|n]}.\end{aligned}$$

If $k|n$, then $\overline{[k|n]} - [k|n] = 0$. Now, for $k \nmid n$, by the condition $A(k,n)\sin^2\left(\frac{n\pi}{k}\right) \geqslant 1$, it follows that

$$\overline{[k|n]} = \exp\left(-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)\right) \leqslant \exp(-nk \cdot 1) = \exp(-nk).$$

Therefore,

$$\sum_{k \nmid n} \overline{[k|n]} \leqslant \sum_{\substack{k=1 \\ k \nmid n}}^{n} e^{-nk} \leqslant \sum_{k=1}^{n} e^{-nk}. \tag{4}$$

The middle expression of Inequality (4) receives only contributions of k that are not divisors of n, for all k and n such that $1 \leqslant k \leqslant n$. Computing the right-hand side of Inequality (4) yields the upper bound we are looking for. Now, computing the right-hand side of Inequality (4), we have

$$\sum_{k=1}^{n} e^{-nk} = \sum_{k=1}^{n} \left(e^{-n}\right)^k.$$

This is a geometric series with the first term e^{-n} and common ratio e^{-n}. Summing from $k = 1$ to $k = n$, it follows that

$$\sum_{k=1}^{n} \left(e^{-n}\right)^k = e^{-n}\frac{1-\left(e^{-n}\right)^n}{1-e^{-n}} = \frac{e^{-n} - e^{-n(n+1)}}{1-e^{-n}}. \tag{5}$$

Since $0 < e^{-n(n+1)} < e^{-n}$ and $e^{-n} \to 0$ as $n \to \infty$, by Equation (5) we have

$$\sum_{k=1}^{n} e^{-nk} < \frac{e^{-n}}{1-e^{-n}} = \frac{1}{e^n - 1}.$$

However, the sum is $\frac{e^{-n}-e^{-n(n+1)}}{1-e^{-n}}$. Since $e^{-n(n+1)}$ is negligible for large n, it is asymptotically $\frac{e^{-n}}{1-e^{-n}} = \frac{1}{e^n-1}$. The inequality holds for finite $n \geqslant 1$. Therefore,

$$\overline{\tau}(n) - \tau(n) \leqslant \sum_{k=1}^{n} e^{-nk} < \frac{1}{e^n - 1}.$$

Since $\overline{\tau}(n) - \tau(n) < \frac{1}{e^n-1}$, we have

$$\overline{\tau}(n) = \tau(n) + O\left(\frac{1}{e^n - 1}\right) \quad \text{as } n \to \infty.$$

$\square$

4. Proof of Theorem 3

Proof of Theorem 3. The proof proceeds by leveraging Lemmas 1 and 2, which provide sharp bounds on certain related indicator and divisor functions. We analyze the difference $\overline{\sigma}(n)-\sigma(n)$ and bound it by exploiting exponential decay in appropriate terms.

First, let us define the sum of the positive divisors of a positive integer n by

$$\sigma(n) := \sum_{k=1}^{n} k[k|n],$$

where $[k|n]$ denotes the divisibility indicator function defined in Equation (2) with $d=k$ for all positive integers k, n satisfying $1 \leqslant k \leqslant n$.

Second, we define the function $\overline{\sigma}(n)$ by

$$\overline{\sigma}(n) := \sum_{k=1}^{n} k \exp\left(-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)\right),$$

where $A(k,n)$ satisfies the conditions stated in Lemma 1.

Now, express the difference:

$$\overline{\sigma}(n)-\sigma(n) = \sum_{k=1}^{n} k \exp\left(-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)\right) - \sum_{k=1}^{n} k[k|n].$$

Factoring k out of the expression on the right-hand side of the previous equation, we obtain

$$\overline{\sigma}(n)-\sigma(n) = \sum_{k=1}^{n} k\left(\exp\left(-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)\right) - [k|n]\right).$$

From the proof of Lemma 1, recall the following:

- When $k|n$, we have $[k|n]=1$ and $\sin(\frac{n\pi}{k})=0$, so $\overline{[k|n]}=e^0=1$; thus, $\overline{[k|n]}-[k|n]=0$.
- When $k \nmid n$, we have $[k|n]=0$ and $\overline{[k|n]}=\exp\left(-nkA(k,n)\sin^2\left(\frac{n\pi}{k}\right)\right)>0$, so $\overline{[k|n]}-[k|n]=\overline{[k|n]}>0$. Since $A(k,n)\sin^2\left(\frac{n\pi}{k}\right)\geqslant 1$, it follows that $\overline{[k|n]}\leqslant e^{-nk}$.

Therefore,

$$\overline{\sigma}(n)-\sigma(n) = \sum_{k=1}^{n} k\left(\overline{[k|n]}-[k|n]\right) = \sum_{\substack{k=1\\ k\nmid n}}^{n} k\overline{[k|n]},$$

as the terms where $k|n$ contribute 0. Now, bounding each term for $k \nmid n$, we have $\overline{[k|n]} \leqslant e^{-nk}$, so

$$k\overline{[k|n]} \leqslant ke^{-nk}.$$

Thus,

$$\overline{\sigma}(n) - \sigma(n) \leqslant \sum_{\substack{k=1 \\ k \nmid n}}^{n} ke^{-nk} \leqslant \sum_{k=1}^{n} ke^{-nk}.$$

The inequalities above hold for all k and n such that $1 \leqslant k \leqslant n$. The right-hand side of the above inequality is an upper bound, as it collects contributions from the terms where $k|n$. However, those terms in the original sum are equal to 0, so the upper bound remains valid. Now, we need to bound the sum $\sum_{k=1}^{n} ke^{-nk}$. Suppose that $x = e^{-n}$ and consider the sum $S = \sum_{k=1}^{n} kx^k$. Since $k \leqslant n$ for all $k \in \{1, 2, 3, \ldots, n\}$, we can bound S as follows:

$$S \leqslant n \sum_{k=1}^{n} x^k.$$

It is known that $\sum_{k=1}^{n} x^k = x \cdot \frac{1-x^n}{1-x}$. Since $0 < x^n < 1$, we have $\sum_{k=1}^{n} x^k < \frac{x}{1-x}$. Therefore, we obtain

$$S < n \cdot \frac{x}{1-x} = n \cdot \frac{e^{-n}}{1-e^{-n}} = \frac{n}{e^n - 1}.$$

Consequently,

$$\overline{\sigma}(n) - \sigma(n) < \frac{n}{e^n - 1}.$$

This establishes the desired strict inequality for all $n \geqslant 1$. The inequality is strict because the finite sum is less than the infinite series bound. Additionally, the original sum excludes terms where $k|n$ in the differences, which ensures that the inequality holds. □

Lemma 3. *Let $\pi(n)$ be the prime counting function of n, and let $\overline{\pi}_0(n)$ be an approximation function of $\pi(n)$ defined by*

$$\overline{\pi}_0(n) := \sum_{m=2}^{n} \exp\left(-m\left(\tau(m) - 2\right)\right),$$

where $\tau(m)$ is the number of the positive divisors of a positive integer m. Therefore, for all integers $n \geqslant 2$, we have

$$\overline{\pi}_0(n) - \pi(n) < \frac{1}{e^3(e-1)}.$$

Proof. First, consider the expression

$$\overline{\pi}_0(n) = \sum_{m=2}^{n} \exp(-m(\tau(m) - 2)),$$

where $\tau(m)$ is the number of the positive divisors of a positive integer m. For prime m, we have $\tau(m) = 2$, so $e^{-m(2-2)} = e^0 = 1$. For composite m, we obtain $\tau(m) > 2$, so $e^{-m(\tau(m)-2)} < 1$. Thus,

$$\begin{aligned}\overline{\pi}_0(n) &= \sum_{m=2}^{n} \exp\left(-m\left(\tau(m) - 2\right)\right) \\ &= \sum_{\substack{m=2 \\ m \text{ prime}}}^{n} 1 + \sum_{\substack{m=2 \\ m \text{ composite}}}^{n} e^{-m(\tau(m)-2)} \\ &= \pi(n) + \sum_{\substack{m=4 \\ m \text{ composite}}}^{n} e^{-m(\tau(m)-2)}.\end{aligned}$$

We proceed by bounding the difference $\overline{\pi}_0(n) - \pi(n)$. As established, it follows that

$$\overline{\pi}_0(n) - \pi(n) = \sum_{\substack{m=4 \\ m \text{ composite}}}^{n} e^{-m(\tau(m)-2)}.$$

For composite $m \geqslant 4$, we get $\tau(m) \geqslant 3$, so $\tau(m) - 2 \geqslant 1$, and thus $e^{-m(\tau(m)-2)} \leqslant e^{-m}$. Therefore,

$$\overline{\pi}_0(n) - \pi(n) = \sum_{\substack{m=4 \\ m \text{ composite}}}^{n} e^{-m(\tau(m)-2)} < \sum_{m=4}^{n} e^{-m},$$

as the sum over composites is a subset of the sum over all $m \geqslant 4$. The inequality is strict for finite n due to the presence of primes.

Now, compute the infinite sum

$$\sum_{m=4}^{\infty} e^{-m} = e^{-4} + e^{-5} + e^{-6} + \cdots = e^{-4}(1 + e^{-1} + e^{-2} + \cdots) = \frac{e^{-4}}{1 - e^{-1}} = \frac{e^{-3}}{e^1 - 1},$$

as it is a geometric series with first term e^{-4} and common ratio $e^{-1} < 1$. Thus,

$$\overline{\pi}_0(n) - \pi(n) < \frac{1}{e^3(e-1)}.$$

This bound holds for all $n \geqslant 2$. For $n = 2, 3$, the difference is 0, which is less than $\frac{1}{e^3(e-1)} \approx 0.029$, and for $n \geqslant 4$, the sum is less than this value, as shown by exponential decay. □

Lemma 4. *Let* $A : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_+$ *be a function defined for all positive integers* k *and* n *with* $1 \leqslant k \leqslant n$*. If* $A(k,n)\sin^2\left(\frac{n\pi}{k}\right) \geqslant 1$ *for all integers* k *and* n *such that* $k \nmid n$*. Then, one possible expression for* $A(k,n)$ *that satisfies these conditions is*

$$A(k,n) = \left(\frac{\pi}{2}\right)^2 k^2.$$

Proof. Consider the expression

$$\left|\sin\left(\frac{n\pi}{k}\right)\right|.$$

Assume that $k \nmid n$ with $1 \leqslant k \leqslant n$ for $k, n \in \mathbb{N}$. Thus, there exist integers $m, q \in \mathbb{N}$ such that $n = mk + q$. From this, we get

$$\begin{aligned}\left|\sin\left(\frac{n\pi}{k}\right)\right| &= \left|\sin\left(m\pi + \frac{q\pi}{k}\right)\right| \\ &= \left|\sin(m\pi)\cdot\cos\left(\frac{q\pi}{k}\right) + \sin\left(\frac{q\pi}{k}\right)\cdot\cos(m\pi)\right| \\ &= \left|0\cdot\cos\left(\frac{q\pi}{k}\right) \pm \sin\left(\frac{q\pi}{k}\right)\right| \\ &= \left|\sin\left(\frac{q\pi}{k}\right)\right|. \qquad (6)\end{aligned}$$

In Equation (6), it is evident that $\frac{q}{k} \in (0,1)$. Consider the inequality

$$\left|\sin(x\pi)\right| \geqslant \frac{2}{\pi}\text{dist}(x,\mathbb{Z}) \qquad (7)$$

where $\text{dist}(x,\mathbb{Z}) := \min_{m\in\mathbb{Z}}\left|x - m\right|$.

For $\frac{q}{k} = \frac{1}{2}$, we have

$$\left|\sin\left(\frac{q\pi}{k}\right)\right| = \left|\sin\left(\frac{\pi}{2}\right)\right| = |1| = 1.$$

This condition does not violate Inequality (9), which remains essential for the validity of $A(k,n)$. We now consider two cases.

`Case 1:` $\frac{q}{k} \in \left(0, \frac{1}{2}\right)$. Under this condition we have $\text{dist}(\frac{q}{k},\mathbb{Z}) = \frac{q}{k}$. Combining Equation (6) with Inequality (7) and setting $x = \frac{q}{k}$, we obtain

$$\left|\sin\left(\frac{q\pi}{k}\right)\right| \geqslant \frac{2}{\pi}\cdot\frac{q}{k} \geqslant \frac{2}{\pi}\cdot\frac{1}{k} = \frac{2}{\pi k},$$

for all $k \in \mathbb{N}$.

`Case 2:` $\frac{q}{k} \in \left(\frac{1}{2}, 1\right)$.

Assume that $q = k - q_0$, where $q_0 \in \mathbb{N}$. Therefore, using Equation (6), we get

$$\begin{aligned} \left|\sin\left(\frac{q\pi}{k}\right)\right| &= \left|\sin\left((k-q_0)\frac{\pi}{k}\right)\right| \\ &= \left|\sin\left(\pi - \frac{q_0\pi}{k}\right)\right| \\ &= \left|\sin(\pi)\cdot\cos\left(\frac{q_0\pi}{k}\right) - \sin\left(\frac{q_0\pi}{k}\right)\cdot\cos(\pi)\right| \\ &= \left|0\cdot\cos\left(\frac{q_0\pi}{k}\right) + \sin\left(\frac{q_0\pi}{k}\right)\right| \\ &= \left|\sin\left(\frac{q_0\pi}{k}\right)\right|. \end{aligned} \tag{8}$$

Under the assumption that $\frac{q}{k} \in (\frac{1}{2}, 1)$ and $q = k - q_0$, we obtain

$$\begin{aligned} 1/2 &< q/k < 1, \\ k/2 &< q < k, \\ k/2 &< k - q_0 < k, \\ 0 &< q_0 < \frac{k}{2}. \end{aligned}$$

This implies that $\frac{q_0}{k} < \frac{1}{2}$. Combining Inequality (7) with Equation (8), we get

$$\left|\sin\left(\frac{q_0\pi}{k}\right)\right| \geqslant \frac{2}{\pi}\operatorname{dist}\left(\frac{q_0}{k}, \mathbb{Z}\right),$$

with $\operatorname{dist}\left(\frac{q_0}{k}, \mathbb{Z}\right) = \frac{q_0}{k}$. From this, we deduce that

$$\left|\sin\left(\frac{q_0\pi}{k}\right)\right| \geqslant \frac{2}{\pi}\cdot\frac{q_0}{k} \geqslant \frac{2}{\pi k}.$$

From `Cases 1` and `2`, we have

$$\left|\sin\left(\frac{q\pi}{k}\right)\right| \geqslant \frac{2}{\pi k}.$$

Since

$$\left|\sin\left(\frac{n\pi}{k}\right)\right| = \left|\sin\left(\frac{q\pi}{k}\right)\right|,$$

we obtain

$$\left|\sin\left(\frac{n\pi}{k}\right)\right| \geqslant \frac{2}{\pi k}. \tag{9}$$

Squaring both sides of Inequality (9) yields

$$\sin^2\left(\frac{n\pi}{k}\right) \geqslant \left(\frac{2}{\pi k}\right)^2.$$

Consequently,

$$\left(\frac{\pi}{2}\right)^2 k^2 \sin^2\left(\frac{n\pi}{k}\right) \geqslant 1.$$

Therefore, we conclude that

$$A(k,n) = \left(\frac{\pi}{2}\right)^2 k^2.$$

□

5. Proofs of Theorems 1 and 2

The proof of Theorem 1 follows immediately by combining Lemma 4 with Lemma 2.

Theorem 2 claims that

$$\Big|\pi(n) - \overline{\pi}(n)\Big| < 0.7129749140934,$$

for all integers $n \geqslant 2$, where $\overline{\pi}(n)$ is given by

$$\overline{\pi}(n) = \sum_{k=1}^{n} \exp\left(-k\left(\overline{\tau}(k) - 2\right)\right).$$

This is a strong claim as $\pi(n)$ is an integer. This implies that $\overline{\pi}(n)$ must lie within the interval $(\pi(n) - 1, \pi(n) + 1)$. Since both endpoints of the interval bound an integer value, this suggests that $\overline{\pi}(n)$ is very close to $\pi(n)$.

Proof of Theorem 2. We will analyze the behavior of $\overline{\pi}(n)$ by breaking it down into contributions from prime and composite numbers. First, let us recall Equation (1):

$$\overline{\tau}(k) = \sum_{j=1}^{k} \exp\left(-\left(\frac{\pi}{2}\right)^2 kj^3 \sin^2\left(\frac{k\pi}{j}\right)\right).$$

Consider $P = \{p \leqslant n |\ p \text{ is prime}\}$ and $C = \{c \leqslant n |\ c \text{ is composite}\}$. Then we can express $\overline{\pi}(n)$ as follows:

$$\overline{\pi}(n) = \sum_{k \in P \cup C, 2 \leqslant k \leqslant n} \exp\left(-k\left(\overline{\tau}(k) - 2\right)\right) = \sum_{p \in P} e^{-p(\overline{\tau}(p)-2)} + \sum_{c \in C} e^{-c(\overline{\tau}(c)-2)}.$$

We need to show $\overline{\pi}(n)$ is within 1 of $\pi(n)$, the number of primes up to n, that is, $\pi(n) = |P|$. For k a prime, say p, we have $\tau(p) = 2$ and $\overline{\tau}(p) = 2 + \epsilon_p$, where $0 < \epsilon_p = O\left(\frac{1}{e^p - 1}\right)$. Thus,

$$e^{-p(\overline{\tau}(p)-2)} = e^{-p\epsilon_p} = 1 - p\epsilon_p + O\left((p\epsilon_p)^2\right),$$

and since $\epsilon_p \approx e^{-bp}$ for some $b > 0$, we have $p\epsilon_p \approx pe^{-bp}$, which is very small. For example, when $p = 3$, we have $\epsilon_3 \approx e^{-6\pi^2} \approx 10^{-25.7}$, so $3\epsilon_3 \approx 3 \times 10^{-25.7}$, which is negligible.

For k a composite, say c, we have $\tau(c) \geqslant 3$, so $\overline{\tau}(c) = \tau(c) + \epsilon_c \geqslant 3 + \epsilon_c$ (as with ϵ_p, ϵ_c is negligible), and

$$e^{-c(\overline{\tau}(c)-2)} = e^{-c(\tau(c)-2)-c\epsilon_c},$$

which is exponentially smaller than e^{-c} by the fact that $\tau(c) - 2 \geqslant 1$ for composite c.

Let

$$T = \sum_{p \leqslant n} \left(1 - e^{-p\epsilon_p}\right),$$

be the loss of primes, and

$$U = \sum_{c \in C} e^{-c(\overline{\tau}(c)-2)},$$

be the gain of composites. It follows that

$$\overline{\pi}(n) = \sum_{p \leqslant n} e^{-p\epsilon_p} + U = \pi(n) - T + U,$$

so $\Big|\pi(n) - \overline{\pi}(n)\Big| \leqslant \big|T - U\big|$. We need to show that $\Big|T - U\Big| < 0.7129749140934$.

We begin with the function T given by

$$T = \sum_{p \leqslant n} \left(1 - e^{-p\epsilon_p}\right),$$

where, for each prime $p \leqslant n$, the expression ϵ_p is a parameter satisfying $0 < \epsilon_p < \frac{1}{e^p - 1}$.

Using the Taylor series expansion for the exponential function,

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k!},$$

it follows that

$$1 - e^{-x} = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k!} = -\sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!}.$$

Substituing $x = p\epsilon_p$, we obtain

$$1 - e^{-p\epsilon_p} = -\sum_{k=1}^{\infty} (-1)^k \frac{(p\epsilon_p)^k}{k!}.$$

Thus,

$$T = \sum_{p \leqslant n} (1 - e^{-p\epsilon_p}) = -\sum_{p \leqslant n} \sum_{k=1}^{\infty} (-1)^k \frac{(p\epsilon_p)^k}{k!}.$$

Previously we noted that

$$U = \sum_{c \in C} e^{-c(\overline{\tau}(c)-2)},$$

which is a positive function satisfying $0 < U < \frac{1}{e^3(e-1)} \approx 0.0289749140934$; see Lemma 3. We consider the absolute difference $|T - U|$ as follows:

$$\begin{aligned}\Big|T - U\Big| &= \Big| - \sum_{p \leqslant n} \sum_{k=1}^{\infty} (-1)^k \frac{(p\epsilon_p)^k}{k!} - U \Big| \\ &= \Big|U + \sum_{p \leqslant n} \sum_{k=1}^{\infty} (-1)^k \frac{(p\epsilon_p)^k}{k!}\Big|.\end{aligned}$$

Given the alternating nature of the series, we bound the expression by considering the non-alternating series

$$\Big|U + \sum_{p \leqslant n} \sum_{k=1}^{\infty} (-1)^k \frac{(p\epsilon_p)^k}{k!}\Big| \leqslant \Big|U + \sum_{p \leqslant n} \sum_{k=1}^{\infty} \frac{(p\epsilon_p)^k}{k!}\Big|.$$

It is known that

$$\sum_{k=1}^{\infty} \frac{(p\epsilon_p)^k}{k!} = e^{p\epsilon_p} - 1,$$

with $p\epsilon_p < \frac{p}{e^p - 1} < 1$ for all $p \in \mathcal{P}$. For $0 < x < 1$, it holds that $e^x - 1 < \frac{x}{1-x}$, from this we get

$$e^{p\epsilon_p} - 1 < \frac{p\epsilon_p}{1 - p\epsilon_p} < \frac{p}{e^p - p - 1}.$$

Consequently, we obtain

$$\Big|U + \sum_{p \leqslant n} \sum_{k=1}^{\infty} \frac{(p\epsilon_p)^k}{k!}\Big| \ll \Big|U + \sum_{p \leqslant n} \sum_{k=1}^{\infty} (p\epsilon_p)^k\Big| = \Big|U + \sum_{p \leqslant n} \frac{p\epsilon_p}{1 - p\epsilon_p}\Big| < U + \sum_{p \leqslant n} \frac{p}{e^p - p - 1}.$$

By routine computation we have

$$\sum_{p \in \mathcal{P}} \frac{p}{e^p - p - 1} \approx 0.68393482759 < 0.684,$$

where $\mathcal{P}$ denotes the set of all primes. Since $U < \frac{1}{e^3(e-1)} \approx 0.0289749140934$, we compute

$$U + \sum_{p \leqslant n} \frac{p}{e^p - p - 1} < 0.684 + 0.0289749140934 = 0.7129749140934.$$

Since

$$\Big|T - U\Big| < 0.7129749140934$$

and

$$\left|\pi(n) - \overline{\pi}(n)\right| \leqslant \left|T - U\right|,$$

it follows that

$$\left|\pi(n) - \overline{\pi}(n)\right| < 0.7129749140934$$

for all $n \geqslant 2$. Therefore, Theorem 2 holds. □

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