A NOTE ON HYPER-LEONARDO NUMBERS

Elen Viviani Pereira Spreafico
Institute of Mathematics, University of Mato Grosso do Sul, Campo Grande, Brazil
elen.spreafico@ufms.br

Paula Maria Machado Cruz Catarino
Department of Mathematics, Universidade de Trás-os-Montes e Alto Douro, Vila Real, Portugal
pcatarin@utad.pt

Received: 8/13/24, Accepted: 3/23/26, Published: 4/3/26

Abstract

This work explores a new generalization of the Leonardo sequence called the hyper-Leonardo numbers. Some algebraic properties of this sequence are studied by means of the generating function. In addition, these sequences' properties of convexity, concavity, log-concavity, and log-convexity are established.

1. Introduction

Recently, numerous researchers have been focusing their attention on several number sequences, such as Fibonacci, Pell, Lucas, Jacobsthal, Padovan, and Perrin numbers, as well as their generalizations. In particular, the well-known Fibonacci sequence has motivated the study of many other numerical sequences. One of these Fibonacci-related sequences is the Leonardo numbers, $\{Le_n\}_{n\geq 0}$, defined by the recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \ (n \geq 2), \tag{1}$$

with initial conditions $Le_0 = Le_1 = 1$. The sequence of Leonardo numbers, as well as some of its generalizations, has been studied studied from an analytic and matrix perspective (see more in [18], [5], [14], [13], [9], [10], [4], and the references therein).

Consider the generalizations of Fibonacci and Lucas numbers given by the hyper-Fibonacci $\{F_n^{(r)}\}_{n\geq 0}$ and the hyper-Lucas numbers $\{L_n^{(r)}\}_{n\geq 0}$. These sequences of integers were introduced in [12] and defined by the recurrence relations with initial

DOI: 10.5281/zenodo.19403822

conditions

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}, F_n^{(0)} = F_n, F_0^{(r)} = 0, F_1^{(r)} = 1, \tag{2}$$

$$L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)}, L_n^{(0)} = L_n, L_0^{(r)} = 2, L_1^{(r)} = 2r+1,$$

respectively, where r and n are non-negative integers, and F_n and L_n are the classical Fibonacci and Lucas numbers. In [12], the generating functions of these numbers were provided, namely

$$\sum_{n=0}^{\infty} F_n^{(r)} t^n = \frac{t}{(1-t-t^2)(1-t)^r}, \tag{3}$$

$$\sum_{n=0}^{\infty} L_n^{(r)} t^n = \frac{2-t}{(1-t-t^2)(1-t)^r}$$

(see more about these sequences for $r = 1$ and $r = 2$ in [17], whose sequence numbers are A000071, A001924, A001610 and A023548). In [11] and [7], some identities for the hyper-Fibonacci and the hyper-Lucas numbers were established. The combinatorial aspect of these sequences was explored in [3] and [6].

The k-hyper-Pell, k-hyper-Pell–Lucas, and k-hyper-modified-Pell sequences are generalizations of the k-Pell, k-Pell-Lucas, and modified k-Pell sequences, respectively. These generalizations were introduced in [8], where the authors explore the properties of these sequences and also establish their convexity, concavity, log-concavity, and log-convexity properties.

In [16], the hyper-Leonardo numbers are defined, and some properties of this new sequence are provided. Our main focus is to explore this sequence of numbers including a study of its log-convexity, and to provide some new properties. The next section will be dedicated to the introduction of this generalization of the Leonardo sequence. In addition, some properties are recovered, and new ones are provided. Section 3 is devoted to exploring the relationship between hyper-Fibonacci numbers and hyper-Leonardo numbers, and recovering the generating function for the hyper-Leonardo sequence of numbers. In Section 4, the Binet formula will be stated and several identities will be established. Moreover, in Section 5, the convexity, concavity, log-concavity, log-convexity, and log-balanced properties of the hyper-Leonardo numbers are also studied.

2. The Hyper-Leonardo Numbers and Some Properties

This section introduces the hyper-Leonardo sequence, defined in [[16], Definition 2.1], and provides some properties of this new sequence of numbers. Consider the following definition.

Definition 1. For integers $n \geq 0$ and $r \geq 0$, the *hyper-Leonardo sequence* is recursively defined by

$$Le_n^{(r)} = \sum_{k=0}^{n} Le_k^{(r-1)}, \text{ for } r \geq 1,$$

with initial conditions given by $Le_n^{(0)} = Le_n$, $Le_0^{(r)} = 1$, $Le_1^{(r)} = r+1$, where Le_n is the n-th Leonardo number given in Equation (1).

Table 1 shows us the first elements of the hyper-Leonardo sequence $\{Le_n^{(r)}\}_{n\geq 0}$, for $r = 0, 1, 2$ and 3, and $0 \leq n \leq 11$.

r \ n	0	1	2	3	4	5	6	7	8	9	10	11
$Le_n^{(0)}$	1	1	3	5	9	15	25	41	67	109	177	287
$Le_n^{(1)}$	1	2	5	10	19	34	59	100	167	276	453	740
$Le_n^{(2)}$	1	3	8	18	37	71	130	230	397	673	1126	1866
$Le_n^{(3)}$	1	4	12	30	67	138	268	498	895	1568	2694	4560

Table 1: The first elements of hyper-Leonardo sequences $\{Le_n^{(r)}\}_{n\geq 0}$, $r = 0, 1, 2, 3$.

The recurrence relation of the hyper-Leonardo numbers $\{Le_n^{(1)}\}_{n\geq 0}$ is derived below given the recurrence relation of Leonardo numbers (1), namely, $Le_n^{(0)} = Le_n = Le_{n-1} + Le_{n-2} + 1$, $(n \geq 2)$. The recurrence relation of the hyper-Leonardo numbers $\{Le_n^{(1)}\}_{n\geq 0}$ is given by

$$\begin{aligned}
Le_n^{(1)} &= \sum_{k=0}^{n} Le_k \\
&= Le_0 + Le_1 + \sum_{k=2}^{n} (Le_{k-1} + Le_{k-2} + 1) \\
&= Le_0 + Le_1 + \sum_{k=2}^{n} Le_{k-1} + \sum_{k=2}^{n} Le_{k-2} + (n-1) \\
&= Le_1 + \sum_{k=1}^{n} Le_{k-1} + \sum_{k=2}^{n} Le_{k-2} + n - 1 \\
&= Le_{n-1}^{(1)} + Le_{n-2}^{(1)} + n - 1 + 1 \\
&= Le_{n-1}^{(1)} + Le_{n-2}^{(1)} + \binom{n-1}{1} + 1.
\end{aligned}$$

Similarly, we obtain the recurrence relation of the hyper-Leonardo numbers $\{Le_n^{(2)}\}_{n\geq 0}$ as follows:

$$\begin{aligned}
Le_n^{(2)} &= \sum_{k=0}^{n} Le_k^{(1)} \\
&= Le_0^{(1)} + Le_1^{(1)} + \sum_{k=2}^{n} Le_k^{(1)} \\
&= Le_0^{(1)} + Le_1^{(1)} + \sum_{k=2}^{n} (Le_{k-1}^{(1)} + Le_{k-2}^{(1)} + (k-1) + 1) \\
&= Le_1^{(1)} + \sum_{k=1}^{n} Le_{k-1}^{(1)} + \sum_{k=2}^{n} Le_{k-2}^{(1)} + \sum_{k=2}^{n} (k-1) + \sum_{k=2}^{n} 1 \\
&= Le_{n-1}^{(2)} + Le_{n-2}^{(2)} + \frac{n(n-1)}{2} + n - 1 + 2 \\
&= Le_{n-1}^{(2)} + Le_{n-2}^{(2)} + \binom{n}{2} + \binom{n-1}{1} + 2.
\end{aligned}$$

In summary, we state the following result.

Proposition 1. *For $n \geq 2$, the Leonardo sequence $\{Le_n\}_{n\geq 0}$ satisfies the recurrence relation*

$$Le_n = Le_{n-1} + Le_{n-2} + 1,$$

with $Le_0 = Le_1 = 1$. For $n \geq 2$, the hyper-Leonardo sequence $\{Le_n^{(1)}\}_{n\geq 0}$ satisfies the recurrence relation

$$Le_n^{(1)} = Le_{n-1}^{(1)} + Le_{n-2}^{(1)} + \binom{n-1}{1} + 1,$$

with $Le_0^{(1)} = 1$ and $Le_1^{(1)} = 2$. For $n \geq 2$, the hyper-Leonardo sequence $\{Le_n^{(2)}\}_{n\geq 0}$ satisfies the recurrence relation

$$Le_n^{(2)} = Le_{n-1}^{(2)} + Le_{n-2}^{(2)} + \binom{n}{2} + \binom{n-1}{1} + 2,$$

with $Le_0^{(2)} = 1$ and $Le_1^{(2)} = 3$.

Consider the following lemma for the statement of the general case.

Lemma 1. *Consider the arithmetic progression $\{a_n\}_{n\geq 0} = \{a_n^{(0)}\}_{n\geq 0}$ defined by $a_n = 1$, for all non-negative integers n. Consider the arithmetic progression of order $r \geq 1$, $\{a_n^{(r)}\}_{n\geq r}$, defined by the partial sums of the arithmetic progression of* *order* $r-1$, $\{a_n^{(r-1)}\}_{n\geq r-1}$. *Then the sequence* $\{a_n^{(r)}\}_{n\geq r}$ *is given by the polynomial of degree* r

$$a_n^{(r)} = \sum_{k=1}^{n} a_k^{(r-1)} = \binom{n}{r} = \frac{(n)(n-1)\cdots(n-r+1)}{r!},$$

for $n \geq r$ *and* $r \geq 1$.

Proof. We prove the result by induction on r. For $r=1$, we have

$$a_n^{(1)} = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} 1 = n = \binom{n}{1},$$

for $n \geq 1$. For $r=2$, we obtain

$$a_n^{(2)} = \sum_{k=1}^{n} a_k^{(1)} = \sum_{k=1}^{n} k = \frac{(n-1)(n)}{2} = \binom{n}{2},$$

for $n \geq 2$. Suppose that the result is true for all positive integers less than or equal to r. The result can be verified for $r+1$. In fact,

$$a_n^{(r+1)} = \sum_{k=1}^{n} a_k^{(r)} = \binom{n}{r+1},$$

as required. □

With the previous lemma, we provide the next result.

Proposition 2. *For* $n \geq 0$ *and* $r \geq 0$ *the hyper-Leonardo sequence* $\{Le_n^{(r)}\}_{n\geq 0}$ *satisfies the recurrence relation*

$$Le_n^{(r)} = r + Le_{n-1}^{(r)} + Le_{n-2}^{(r)} + \binom{n+r-2}{r} + \sum_{j=1}^{r-1} j\binom{n+r-2-j}{r-j}, \text{ for } n \geq 2,$$

with $Le_0^{(r)} = 1$ *and* $Le_1^{(r)} = r+1$.

Proof. We prove the result by induction on r. For $r=0, r=1$, and $r=2$, the statement was proved in Proposition 1. Suppose that the result is true for all positive integers less than or equal to r, i.e.,

$$Le_n^{(r)} = Le_{n-1}^{(r)} + Le_{n-2}^{(r)} + \binom{n+r-2}{r} + \sum_{j=1}^{r-1} j\binom{n+r-2-j}{r-j} + r,$$

for $n \geq 2$, with $Le_0^{(r)} = 1$ and $Le_1^{(r)} = r+1$. The result can be verified for $r+1$. In fact,

$$\begin{aligned}
Le_n^{(r+1)} &= \sum_{k=0}^{n} Le_k^{(r)} \\
&= Le_0^{(r)} + Le_1^{(r)} + \sum_{k=2}^{n} Le_k^{(r)} \\
&= Le_0^{(r)} + Le_1^{(r)} \\
&\quad + \sum_{k=2}^{n} \left(Le_{k-1}^{(r)} + Le_{k-2}^{(r)} + \binom{k+r-2}{r} + \sum_{j=1}^{r-1} j \binom{k+r-2-j}{r-j} + r \right) \\
&= Le_1^{(r)} + \sum_{k=1}^{n} Le_{k-1}^{(r)} \\
&\quad + \sum_{k=2}^{n} Le_{k-2}^{(r)} + \sum_{k=2}^{n} \left(\binom{k+r-2}{r} + \sum_{j=1}^{r-1} j \binom{k+r-2-j}{r-j} + r \right).
\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}
\sum_{k=2}^{n} \left(\binom{k+r-2}{r} + \sum_{j=1}^{r-1} j \binom{k+r-2-j}{r-j} + r \right) &= \binom{n+r-1}{r+1} \\
&\quad + \sum_{j=1}^{r} j \binom{n+r-1-j}{r-j+1}.
\end{aligned}$$

Then

$$\begin{aligned}
Le_n^{(r+1)} &= Le_1^{(r)} + \sum_{k=1}^{n} Le_{k-1}^{(r)} + \sum_{k=2}^{n} Le_{k-2}^{(r)} + \sum_{k=2}^{n} \left(\binom{k-1}{r} + \sum_{j=1}^{r-1} j \binom{k-1}{r-j} + r \right) \\
&= (r+1) + Le_{n-1}^{(r+1)} + Le_{n-2}^{(r+1)} + \binom{n+r-1}{r+1} + \sum_{j=1}^{r} j \binom{n+r-1-j}{r-j+1} \\
&= Le_{n-1}^{(r+1)} + Le_{n-2}^{(r+1)} + \binom{n+r-1}{r+1} + \sum_{j=1}^{r} j \binom{n+r-1-j}{r-j+1} + (r+1).
\end{aligned}$$

Therefore, the identity is verified for every $n \geq 0$ and $r \geq 0$. □

The following proposition, given in [[16], Identity (22)], shows a useful identity which is a relationship between the elements of the sequences $\{Le_n^{(r)}\}_{n\geq 0}$ and$\{Le_n^{(r-1)}\}_{n\geq 0}$.

Proposition 3. *For non-negative integers r and n, the hyper-Leonardo sequence $\{Le_n^{(r)}\}_{n\geq 0}$ satisfies the following identity*

$$Le_n^{(r)} = Le_{n-1}^{(r)} + Le_n^{(r-1)}. \tag{4}$$

By replacing $r = 1$ in Equation (4) we get the following result.

Corollary 1. *For the hyper-Leonardo sequence $\{Le_n^{(1)}\}_{n\geq 0}$, the identity*

$$Le_n^{(1)} = Le_{n-1}^{(1)} + Le_n \tag{5}$$

holds, where Le_n is the n-th Leonardo number given in Equation (1).

Proposition 3 allows us to establish the following new identity.

Proposition 4. *Consider the hyper-Leonardo sequence $\{Le_n^{(r)}\}_{n\geq 0}$, $r \geq 1$. We have*

$$Le_{n-1}^{(r)} Le_{n+1}^{(r)} - (Le_n^{(r)})^2 = Le_n^{(r)} Le_{n+1}^{(r-1)} - Le_n^{(r-1)} Le_{n+1}^{(r)}.$$

Proof. By Proposition 3, we have the relations

$$Le_{n-1}^{(r)} = Le_n^{(r)} - Le_n^{(r-1)}$$

and

$$Le_{n+1}^{(r)} = Le_n^{(r)} + Le_{n+1}^{(r-1)}.$$

Then,

$$\begin{aligned} Le_{n-1}^{(r)} Le_{n+1}^{(r)} &= (Le_n^{(r)} - Le_n^{(r-1)})(Le_n^{(r)} + Le_{n+1}^{(r-1)}) \\ &= (Le_n^{(r)})^2 + Le_n^{(r)} Le_{n+1}^{(r-1)} - Le_n^{(r-1)}(Le_n^{(r)} + Le_{n+1}^{(r-1)}) \\ &= (Le_n^{(r)})^2 + Le_n^{(r)} Le_{n+1}^{(r-1)} - Le_n^{(r-1)} Le_{n+1}^{(r)}, \end{aligned}$$

as required. □

The next result is obtained by considering Propositions 2 and 3.

Proposition 5. *For integers $r \geq 1$ and $n \geq 2$, the hyper-Leonardo sequence $\{Le_n^{(r)}\}_{n\geq 0}$ satisfies the following identity*

$$Le_{n-2}^{(r)} = Le_{n-1}^{(r-1)} + \binom{n+r-2}{r} + \sum_{j=1}^{r-1} j \binom{n+r-2-j}{r-j} + r.$$

According to Proposition 2.2 in [10], the relationship between Leonardo and Fibonacci numbers is given by

$$Le_n = 2F_{n+1} - 1. \tag{6}$$

Since $Le_n = Le_n^{(0)}$ and $F_n = F_n^{(0)}$, Equation (6) can be written in the form

$$Le_n^{(0)} = 2F_{n+1}^{(0)} - 1, \tag{7}$$

for $n \geq 0$. Note that $F_0^{(r)} = 0$ for all non-negative integer r. Then $F_{n+1}^{(r)} = \sum_{k=0}^{n+1} F_k^{(r-1)} = \sum_{k=1}^{n+1} F_k^{(r-1)} = \sum_{k=0}^{n} F_{k+1}^{(r-1)}$. By using Equation (2), Definition 1, and Equation (7), for $r = 1$, we obtain

$$\begin{aligned} Le_n^{(1)} &= \sum_{k=0}^{n} Le_k^{(0)} \\ &= 2\sum_{k=0}^{n} F_{k+1}^{(0)} - (n+1) \\ &= 2F_{n+1}^{(1)} - (n+1), \end{aligned} \tag{8}$$

for $n \geq 0$. In general, we have the following result given in [16, Proposition 6].

Proposition 6. *The hyper-Leonardo numbers $\{Le_n^{(r)}\}_{n\geq 0}$ satisfy the following property, for any integers $n \geq 0$ and $r \geq 0$:*

$$Le_n^{(r)} = 2F_{n+1}^{(r)} - \binom{n+r}{r}. \tag{9}$$

Moreover, substituting the expression for $Le_n^{(1)}$ given in Equation (8) into Equation (5) given by Corollary 1, we obtain an elementary identity for hyper-Fibonacci numbers, namely, $F_{n+1}^{(1)} = F_n^{(1)} + F_{n+1}$, where F_n is the n-th Fibonacci number.

3. The Generating Function for the Hyper-Leonardo Sequence

Generating functions are power series where the coefficients give us information about the associated sequence. In general, the exponent of the variable in the series quantifies some property that we are interested in. In this section, we establish the generating function for the hyper-Leonardo sequence by considering the generating function for the hyper-Fibonacci sequence given by Equation (3), namely,

$$\sum_{n=0}^{\infty} F_n^{(r)} t^n = \frac{t}{(1-t-t^2)(1-t)^r}.$$

To establish the generating function for the hyper-Leonardo sequence, consider the lemma below.

Lemma 2. *For any positive integer r, we have*

$$\sum_{n=0}^{\infty}\binom{n+r}{r}t^n = \frac{1}{(1-t)^{r+1}}.$$

Proof. Note that, for $r \geq 0$, the expression $\sum_{n=0}^{\infty}\binom{n+r}{r}t^n$ can be written as

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{n+r}{r}t^n &= \frac{1}{r!}\sum_{n=0}^{\infty}(n+r)(n+r-1)\cdots(n+1)t^n\\ &= \frac{1}{r!}\sum_{n=0}^{\infty}\frac{d^r}{dt^r}\left(t^{n+r}\right)\\ &= \frac{1}{r!}\frac{d^r}{dt^r}\sum_{n=0}^{\infty}\left(t^{n+r}\right)\\ &= \frac{1}{r!}\frac{d^r}{dt^r}\left(\frac{t^r}{(1-t)^{r+1}}\right)\\ &= \frac{1}{(1-t)^{r+1}},\end{aligned}$$

as required. □

As a consequence of Lemma 2, we have the next result.

Theorem 1. *For $n \geq 0$, the generating function for the hyper-Leonardo number $\{Le_n^{(r)}\}_{n\geq 0}$, is given by*

$$\sum_{n=0}^{\infty} Le_n^{(r)}t^n = \frac{1-t+t^2}{(1-2t+t^3)(1-t)^r}. \tag{10}$$

Proof. By Equation (9), we have

$$\begin{aligned}t\sum_{n=0}^{\infty} Le_n^{(r)}t^n &= 2\sum_{n=0}^{\infty}F_{n+1}^{(r)}t^{n+1} - \sum_{n=0}^{\infty}\binom{n+r}{r}t^{n+1}\\ &= \frac{2t}{(1-t-t^2)(1-t)^r} - \sum_{n=0}^{\infty}\binom{n+r}{r}t^{n+1}.\end{aligned}$$

Then

$$\sum_{n=0}^{\infty} Le_n^{(r)}t^n = \frac{2}{(1-t-t^2)(1-t)^r} - \sum_{n=0}^{\infty}\binom{n+r}{r}t^n.$$

Therefore, for $r \geq 0$, the generating function for the hyper-Leonardo numbers $\{Le_n^{(r)}\}_{n\geq 0}$, is given by

$$\sum_{n=0}^{\infty} Le_n^{(r)} t^n = \frac{2}{(1-t-t^2)(1-t)^r} - \frac{1}{(1-t)^{r+1}}$$
$$= \frac{1-t+t^2}{(1-2t+t^3)(1-t)^r},$$

as required. □

As a consequence of Theorem 1, for $r = 0$, we obtain the generating function of Leonardo numbers $\{Le_n^{(0)}\}_{n\geq 0}$ established in [[10], Proposition 5.1].

Corollary 2. *For $n \geq 0$, the Leonardo number Le_n is the coefficient of t^n in the expansion of*

$$F(t) = \frac{2}{(1-t-t^2)} - \frac{1}{1-t} = \frac{1-t+t^2}{(1-2t+t^3)}.$$

Similarly, for $r = 1$ and $r = 2$ we obtain the generating function of the hyper-Leonardo numbers $Le_n^{(1)}$ and $Le_n^{(2)}$, given in the next result.

Corollary 3. *For $n \geq 0$, the hyper-Leonardo number $Le_n^{(1)}$ is the coefficient of t^n in the expansion of*

$$F(t) = \frac{2}{(1-t-t^2)(1-t)} - \frac{1}{(1-t)^2} = \frac{1-t+t^2}{(1-2t+t^3)(1-t)^2}.$$

Also, the hyper-Leonardo number $Le_n^{(2)}$ is the coefficient of t^n in the expansion of

$$F(t) = \frac{2}{(1-t-t^2)} - \frac{1}{(1-t)^3} = \frac{1-t+t^2}{(1-2t+t^3)(1-t)^3}.$$

Identity (10) was established in [[16], Theorem 1] and proved using induction arguments.

4. The Binet Formula and Some Identities

It is well-known that the Binet formula allows for finding the element of order n without the use of other terms of the sequence. In this section, we establish the Binet formula for the hyper-Leonardo numbers.

Consider the hyper-Leonardo sequence $\{Le_n^{(r)}\}_{n\geq 0}$ satisfying the recurrence relation (2). Then the solutions of $\{Le_n^{(r)}\}_{n\geq 0}$ can be given as the sum of a solution of the homogeneous part of (2) and a particular solution of (2). Considering the homogeneous part of (2), its characteristic polynomial is given by $P(z) = z^2 - z - 1$, with simple roots given by $\frac{1-\sqrt{5}}{2}$ and $\frac{1+\sqrt{5}}{2}$. These are the roots of the characteristic equation associated with the Fibonacci sequence. Therefore, the solution of the homogeneous part of (2) is given by $C_1\left(\frac{1-\sqrt{5}}{2}\right)^n + C_2\left(\frac{1+\sqrt{5}}{2}\right)^n$, where $C_1 = \frac{1}{2\sqrt{5}}(\sqrt{5}-2r-1)$ and $C_2 = \frac{1}{2\sqrt{5}}(\sqrt{5}+2r+1)$.

By fixing r and by letting $B_r(n) = \binom{n+r-2}{r} + \sum_{j=1}^{r-1} j\binom{n+r-2-j}{r-j} + r$, we get a particular solution $\sum_{j=0}^{r} a_j n^j$, where the a_j are constants for each $0 \le j \le r$, obtained by solving the equation

$$\sum_{j=0}^{r} a_j n^j = \sum_{j=0}^{r} a_j (n-1)^j + \sum_{j=0}^{r} a_j (n-2)^j + B_r(n).$$

Theorem 2 (Binet's formula)**.** *For $n \ge 0$, the n-th hyper-Leonardo number $Le_n^{(r)}$ is given by*

$$Le_n^{(r)} = C_1\left(\frac{1-\sqrt{5}}{2}\right)^n + C_2\left(\frac{1+\sqrt{5}}{2}\right)^n + \sum_{j=0}^{r} a_j n^j,$$

where $C_1 = \frac{1}{2\sqrt{5}}(\sqrt{5}-1)$, $C_2 = \frac{1}{2\sqrt{5}}(\sqrt{5}+1)$, and the a_j are constants for each $0 \le j \le r$, obtained by solving the equation

$$\sum_{j=0}^{r} a_j n^j = \sum_{j=0}^{r} a_j (n-1)^j + \sum_{j=0}^{r} a_j (n-2)^j + B_r(n),$$

for each fixed r.

For $r = 0$, we obtain the function $B_0(n) = 1$. Then, a particular solution is given by the constant -1. Hence, the next result is verified.

Proposition 7. *For $n \ge 0$, the n-th Leonardo number is given by*

$$Le_n = C_1\left(\frac{1-\sqrt{5}}{2}\right)^n + C_2\left(\frac{1+\sqrt{5}}{2}\right)^n - 1, \tag{11}$$

where $C_1 = \frac{1}{2\sqrt{5}}(\sqrt{5}-1)$ and $C_2 = \frac{1}{2\sqrt{5}}(\sqrt{5}+1)$.

The Binet formula for the n-th Leonardo number was provided in [[10], Proposition 2.4]. Now, in the case $r = 1$, we have the function defined on the set of natural numbers $B_1(n) = n$, and then a particular solution is given by $a_0 + a_1 n$, where a_0 and a_1 are constants. By plugging a particular solution into the recurrence relation $Le_n^{(1)} = Le_{n-1}^{(1)} + Le_{n-2}^{(1)} + n$, we obtain the system

$$\begin{cases} -a_0 + a_1 &= 0 \\ a_1 + 1 &= 0. \end{cases}$$

Then $a_0 = a_1 = -1$ and $-n-1$ is a particular solution. Thus, the next result is verified.

Proposition 8. *For $n \geq 0$, the n-th hyper-Leonardo number $Le_n^{(1)}$ is given by*

$$Le_n^{(1)} = C_1 \left(\frac{1-\sqrt{5}}{2} \right)^n + C_2 \left(\frac{1+\sqrt{5}}{2} \right)^n - n - 1, \tag{12}$$

where $C_1 = \frac{1}{2\sqrt{5}}(\sqrt{5}-3)$ and $C_2 = \frac{1}{2\sqrt{5}}(\sqrt{5}+3)$.

Similarly, by considering the case $r = 2$, we have the function defined on the set of natural numbers $B_2(n) = \binom{n}{2} + \binom{n-1}{1} + 2 = \frac{n^2-n+2}{2}$ and then a particular solution given by $a_0 + a_1 n + a_2 n^2$, where a_0, a_1 and a_2 are constants. By replacing a particular solution in the recurrence relation $Le_n^{(1)} = Le_{n-1}^{(1)} + Le_{n-2}^{(1)} + \frac{n^2-n+2}{2}$, we obtain the system

$$\begin{cases} a_0 + 3a_1 - 5a_2 - 1 & = & 0 \\ -a_1 + 6a - 2 + \frac{1}{2} & = & 0 \\ a_2 + \frac{1}{2} & = & 0. \end{cases}$$

Then $a_0 = -6, a_1 = -\frac{5}{2}$, and $a_2 = -\frac{1}{2}$. Thus, the next result is verified.

Proposition 9. *For $n \geq 0$, the n-th hyper-Leonardo number $Le_n^{(2)}$, is given by*

$$Le_n^{(2)} = C_1 \left(\frac{1-\sqrt{5}}{2} \right)^n + C_2 \left(\frac{1+\sqrt{5}}{2} \right)^n - (6 + \frac{5n}{2} + \frac{1}{2}n^2),$$

where $C_1 = \frac{1}{2\sqrt{5}}(\sqrt{5}-5)$ and $C_2 = \frac{1}{2\sqrt{5}}(\sqrt{5}+5)$.

5. The Concavity, Convexity, Log-Concavity, and Log-Convexity Properties

Log-convexity and log-concavity are significant characteristics of combinatorial sequences holding considerable importance in various fields, including quantum physics, white noise theory, probability, economics, and mathematical biology (see more in [19], [2], [1], [8], and the references therein). In [19], the author investigated the log-concavity and log-convexity of the hyper-Fibonacci numbers and hyper-Lucas numbers. The study was extended to hyper-Pell numbers and hyper-Pell–Lucas numbers in [2]. In [8], the authors studied the log-concavity and log-convexity cases for k-hyper-Pell numbers, and k-hyper-Pell–Lucas numbers. Also, in [2], the authors considered the log-concavity and log-convexity of hyper-Jacobsthal and hyper-Jacobsthal–Lucas sequences.

This section is concerned with the study of the log-concavity and log-convexity of the hyper-Leonardo numbers. To reach our goal, consider the following definitions given in [15, 19]. Let $\{a_n\}_{n\geq 0}$ be a sequence of positive numbers. If for all $j \geq 1$,

$a_j^2 \geq a_{j-1}a_{j+1}$ (respectively, $a_{j-1}a_{j+1} \geq a_j^2$), then the sequence $\{a_n\}_{n\geq 0}$ is called *log-concave* (respectively, *log-convex*). It is derived by this definition that $\{a_n\}_{n\geq 0}$ is log-convex (respectively, log-concave) if and only if its quotient sequence $\left\{\frac{a_{n+1}}{a_n}\right\}_{n\geq 0}$ is nondecreasing (respectively, nonincreasing). Also, by the definition given in [8], the sequence $\{a_n\}_{n\geq 0}$ is *convex* (respectively, *concave*) if for $n \geq 1$, $a_{n-1} + a_{n+1} \geq 2a_n$ (respectively, $a_{n-1}+a_{n+1} \leq 2a_n$). A sequence $\{a_n\}_{n\geq 0}$ of positive real numbers is *log-balanced* if $\{a_n\}_{n\geq 0}$ is log-convex and the sequence $\{\frac{a_n}{n!}\}_{n\geq 0}$ is log-concave.

The next result is about the convexity property related to the sequence $\{Le_n^{(r)}\}_{n\geq 0}$.

Theorem 3. *For $r \geq 0$ and $n \geq 1$, the sequence $\{Le_n^{(r)}\}_{n\geq 0}$ is convex.*

Proof. Proposition 2 shows us that

$$Le_n^{(r)} = Le_{n-1}^{(r)} + Le_{n-2}^{(r)} + \binom{n+r-2}{r} + \sum_{j=1}^{r-1} j\binom{n+r-2-j}{r-j} + r$$

for $r \geq 0$, $n \geq 0$, and with $Le_0^{(r)} = 1$, $Le_1^{(r)} = r + 1$. Then, we obtain

$$\begin{aligned}
Le_{n-1}^{(r)} + Le_{n+1}^{(r)} &= Le_{n-1}^{(r)} + Le_n^{(r)} + Le_{n-1}^{(r)} + \binom{n+1+r-2}{r} \\
&\quad + \sum_{j=1}^{r-1} j\binom{n+1+r-2-j}{r-j} + r \\
&= 2Le_n^{(r)} + Le_{n-1}^{(r)} + \binom{n+1+r-2}{r} \\
&\quad + \sum_{j=1}^{r-1} j\binom{n+1+r-2-j}{r-j} + r \\
&\quad - Le_{n-2}^{(r)} - \binom{n+r-2}{r} - \sum_{j=1}^{r-1} j\binom{n+r-2-j}{r-j} - r.
\end{aligned}$$

Since $Le_n^{(r)} \geq Le_{n-1}^{(r)}$ for every $r \geq 0$ and $n \geq 1$, and also $\binom{n}{j} \geq \binom{n-1}{j}$ for all $n \geq 1$, $j \geq 0$, and $n - 1 \geq j$, it follows that

$$\begin{aligned}
Le_{n-1}^{(r)} + \binom{n+1+r-2}{r} + \sum_{j=1}^{r-1} j\binom{n+1+r-2-j}{r-j} - Le_{n-2}^{(r)} - \binom{n+r-2}{r} \\
- \sum_{j=1}^{r-1} j\binom{n+r-2-j}{r-j} \geq 0.
\end{aligned}$$

Therefore, $Le_{n-1}^{(r)} + Le_{n+1}^{(r)} \geq 2Le_n^{(r)}$, as required. □

Next, the log-concavity and log-convexity will be considered for some values of r. For $r = 0$, we have $Le_n^{(0)} = Le_n$, where Le_n is the n-th Leonardo number given in Equation (1). Recall Proposition 2.2 in [10], where the relationship between Leonardo and Fibonacci numbers $Le_n = 2F_{n+1} - 1$ was established.

For $n = 1$, we have $(Le_1)^2 - Le_2Le_0 = -2 \leq 0$. For $n = 2$, $(Le_2)^2 - Le_3Le_1 = 4 \geq 0$. For $n = 3$, $(Le_3)^2 - Le_2Le_4 = -2 \leq 0$. Considering $n = 4$ we obtain $(Le_4)^2 - Le_3Le_5 = 6 \geq 0$. Similarly, for $n = 5$, $(Le_5)^2 - Le_4Le_6 = 0$. In general, we get

$$\begin{aligned}(Le_n)^2 - Le_{n+1}Le_{n-1} &= (2F_{n+1} - 1)^2 - (2F_{n+2} - 1)(2F_n - 1)\\ &= 4(F_{n+1})^2 - 4F_{n+1} + 1 - 4F_{n+2}F_n + 2F_{n+2} + 2F_n - 1\\ &= 4(F_{n+1})^2 - 4F_{n+2}F_n - 2F_{n+1} + 4F_n\\ &= 4(-1)^n + 2(2F_n - F_{n+1}).\end{aligned}$$

For $n \geq 1$, we have $F_{n-1} \leq F_n$, or equivalently $2F_n - F_{n+1} \geq 0$. Moreover, $2F_1 - F_2 = 1$, and for $n \geq 2$, $2F_n - F_{n+1} = F_{n-2}$. Then, for $n \geq 5$, $2F_n - F_{n+1} = F_{n-2} \geq F_{5-2} = 2$, and $4(-1)^n + 2(2F_n - F_{n+1}) \geq 0$. By the previous discussion, we can establish the following result.

Proposition 10. *For $n = 1$ and $n = 3$, the sequence $\{Le_n\}_{n\geq 0}$ of Leonardo numbers given in Equation (1) is log-convex. For $n = 5$, the sequence $\{Le_n\}_{n\geq 0}$ of Leonardo numbers given in Equation (1) is log-convex and log-concave. For $n = 2$, $n = 4$ and $n \geq 6$, the sequence $\{Le_n\}_{n\geq 0}$ of Leonardo numbers given in Equation (1) is log-concave.*

Also, Proposition 10 can be proved using the Cassini identity for Leonardo numbers established in [18]. As a consequence of Proposition 10, we can study the log-balanced characteristic for the sequence $\{Le_n\}_{n\geq 0}$ of Leonardo numbers.

Proposition 11. *For $n = 3$, the sequence $\{Le_n\}_{n\geq 0}$ of Leonardo numbers given in Equation (1) is log-balanced.*

Proof. Recall that for $n = 3$, the sequence $\{Le_n\}_{n\geq 0}$ of Leonardo numbers given in Equation (1) is log-convex. Consider the sequence $\{\frac{Le_n}{n!}\}_{n\geq 0}$. Since $(\frac{Le_3}{3!})^2 - \frac{Le_2}{2!}\frac{Le_4}{4!} = \frac{-19}{144} \leq 0$, for $n = 3$, the sequence $\{Le_n\}_{n\geq 0}$ of Leonardo numbers given in Equation (1) is log-balanced. □

The following proposition provides a general result about the log-convexity of the hyper-Leonardo numbers when $n = 1$ and $r \geq 1$. Note that the case $r = 0$ was covered in Proposition 10.

Proposition 12. *For $n = 1$ and $r = 1$, the hyper-Leonardo sequence $\{Le_n^{(r)}\}_{n\geq 0}$ given in Equation* (1) *is log-convex. For $n = 1$ and $r \geq 2$, the hyper-Leonardo sequence $\{Le_n^{(r)}\}_{n\geq 0}$ given in Equation* (1) *is log-concave.*

Proof. By replacing $n = 2$ in the result of Proposition 2, we obtain

$$Le_2^{(r)} = Le_1^{(r)} + Le_0^{(r)} + r + 1 + \frac{r(r-1)}{2},$$

with $Le_0^{(r)} = 1$ and $Le_1^{(r)} = r + 1$. Then, we have

$$\begin{aligned}(Le_1^{(r)})^2 - Le_0^{(r)} Le_2^{(r)} &= (r+1)^2 - \left(Le_1^{(r)} + Le_0^{(r)} + r + 1 + \frac{r(r-1)}{2}\right)\\ &= (r+1)^2 - \left(2r + 3 + \frac{r(r-1)}{2}\right)\\ &= \frac{r(r+1)}{2} - 2.\end{aligned}$$

Observe that if $r = 1$,

$$(Le_1^{(1)})^2 - Le_0^{(1)} Le_2 = \frac{1(2)}{2} - 2 = -1 \leq 0.$$

Also, if $r \geq 2$ then $\dfrac{r(r+1)}{2} - 2 \geq 0$. Thus, the result is verified. □

Proposition 13. *For $n \geq 2$ and $r = 1$, the hyper-Leonardo sequence $\{Le_n^{(1)}\}_{n\geq 0}$ given in Equation* (1) *is log-concave.*

Proof. Proposition 4 gives us that

$$Le_{n-1}^{(1)} Le_{n+1}^{(1)} - (Le_n^{(1)})^2 = Le_n^{(1)} Le_{n+1} - Le_n Le_{n+1}^{(1)}.$$

By Equations (11) and (12), namely,

$$Le_n = C_1\left(\frac{1-\sqrt{5}}{2}\right)^n + C_2\left(\frac{1+\sqrt{5}}{2}\right)^n - 1$$

and

$$Le_n^{(1)} = C_1\left(\frac{1-\sqrt{5}}{2}\right)^n + C_2\left(\frac{1+\sqrt{5}}{2}\right)^n - n - 1,$$

where $C_1 = \frac{1}{2\sqrt{5}}(\sqrt{5}-1)$ and $C_2 = \frac{1}{2\sqrt{5}}(\sqrt{5}+1)$, we obtain

$$Le_n^{(1)} Le_{n+1} - Le_n Le_{n+1}^{(1)} = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\left(\frac{n(1+\sqrt{5})}{2}+1\right) + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\left(\frac{n(1-\sqrt{5})}{2}+1\right) - 1.$$

Observe that $\frac{(1-\sqrt{5})}{2} < 0$. Therefore, for every integer $n \geq 2$, we can show by induction that

$$-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\left(\frac{n(1+\sqrt{5})}{2}+1\right) < 1.$$

Hence, we obtain that $Le_{n-1}^{(1)} Le_{n+1}^{(1)} - (Le_n^{(1)})^2 \leq 0$. □

Now, consider the next result.

Lemma 3 ([1])**.** *If both $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are log-concave, then so is their ordinary convolution defined by $z_n = \sum_{k=0}^{n} x_k y_{n-k}, n = 0, 1, 2, \ldots$.*

Therefore, by Proposition 13, the induction hypotheses, and Lemma 3, it follows that the hyper-Leonardo sequence $\{Le_n^{(r)}\}_{n\geq 0}$ given in Equation (1) is log-concave for $r \geq 0$ and $n \geq 2$. We can state this result as follows.

Proposition 14. *For $n \geq 2$ and $r \geq 0$, the hyper-Leonardo sequence $\{Le_n^{(r)}\}_{n\geq 0}$ given in Equation (1) is log-concave.*

6. Conclusion

In this paper, we presented a study of a generalization of the Leonardo sequence, the hyper-Leonardo numbers. Moreover, the algebraic properties of this sequence were studied, the generating function was recovered, and several identities were provided. In addition, the convexity, concavity, log-concavity, and log-convexity properties for these sequences were established.

It seems that not all of the results given here are currently in the literature. This sequence of numbers can still be studied from several perspectives, such as combinatorial, analytic, and matrix perspectives.

Acknowledgement. The first author is supported by Fundect, TO number 1405/2025. The first author expresses their sincere thanks to the Federal University of Mato Grosso do Sul – UFMS/MEC – Brazil for their valuable support. The second author is a member of the Research Centre CMAT-UTAD (Polo of Research Centre CMAT - Centre of Mathematics of University of Minho) and thanks the Portuguese Funds through FCT – Fundação para a Ciência e a Tecnologia, within the Projects UIDB/00013: Research Center CMA-UTAD. This project received financial support from Fundect, TO number 1405/2025.

References

[1] M. Ahmia and H. Belbachir, The log-concavity property associated to hyperjacobsthal and hyperjacobsthal-Lucas Sequences, *Turkish Journal of Analysis and Number Theory* **6** (3) (2018), 107-110.

[2] M. Ahmia, H. Belbachir, and A. Belkhir, The log-concavity and log-convexity properties associated to hyperpell numbers and hyperpell-Lucas numbers, *Ann. Math. Inform.* **43** (2014), 3-12.

[3] L. Ait-Amrane and D. Behloul, Generalized hyper-Lucas numbers and applications, *Indian J. Pure Appl. Math.* **53** (2022), 62-75.

[4] Y. Alp and E. G. Koçer, Some properties of Leonardo numbers, *Konuralp J. Math.* **9** (1) (2021), 183-189.

[5] F. R. V. Alves and R. P. M. Vieira, The Newton fractal's Leonardo sequence study with the Google Colab, *International Electronic Journal of Mathematics Education* **15** (2020), 9 pp.

[6] M. Bahsi, I. Mező, and S. Solak, A symmetric algorithm for hyper-Fibonacci and hyper-Lucas numbers, *Ann. Math. Inform.* **43** (2014), 19-27.

[7] N. N. Cao and F. Z. Zhao, Some properties of hyper-Fibonacci and hyper-Lucas numbers, *J. Integer Seq.* **13** (10.8.8) (2010), 11 pp.

[8] P. Catarino, F. Alves, and H. Campos, On hyper k-Pell, hyper k-Pell Lucas and hyper modified k-Pell sequences, *Integers* **22** (2022), #A60, 14 pp.

[9] P. Catarino and A. Borges, A note on incomplete Leonardo numbers, *Integers* **20** (2020), #A43, 7 pp.

[10] P. Catarino and A. Borges, On Leonardo numbers, *Acta Math. Univ. Comenian. (N.S.)* **89** (1) (2019), 75-86.

[11] L. L. Cristea, I. Martinjak, and I. Urbiha, Hyperfibonacci sequences and polytopic numbers, *J. Integer Seq.* **19** (16.7.6) (2016), 13 pp.

[12] A. Dil and I. Mező, A symmetric algorithm for hyperharmonic and Fibonacci numbers. *Appl. Math. Comput.* **206** (2008), 942-951.

[13] N. Kara and F. Yilmaz, On hybrid numbers with Gaussian Leonardo coefficients, *Mathematics* **11** (2023), 12 pp.

[14] K. Kuhapatanakul and J. Chobsorn, On the generalized Leonardo numbers, *Integers* **22** (2022), #A48, 7 pp.

[15] L. L. Liu and Y. Wang, On the log-convexity of combinatorial sequences, *Adv. in Appl. Math.* **39** (2007), 453-476.

[16] E. O. Mersin and M. Bahsi, Hyper-Leonardo numbers, *Conference Proceedings of Science and Technology* **5** (1) (2022), 14-20.

[17] OEIS Foundation Inc., The Online Encyclopedia of Integer Sequence, `https://oeis.org`.

[18] E.V.P. Spreafico and P. M. M. Catarino, On Leonardo numbers and Fibonacci fundamental system, in *Mathematical Methods for Engineering Applications, Springer Proceedings in Mathematics & Statistics* **439**, 2023, 67-80.

[19] L. N. Zheng and R. Liu, On the log-concavity of the hyperfibonacci numbers and the hyperlucas numbers, *J. Integer Seq.* **17** (14.1.4) (2014), 9 pp.