#A5 INTEGERS 26 (2026)

LINEAR RECURRENCE RELATIONS SATISFIED BY THE GENERAL CONDITIONAL SEQUENCES

Emre Güday[1]
Department of Mathematics, Bilecik Seyh Edebali University, Bilecik, Turkiye
and
Graduate School of Natural and Applied Science, Ankara University, Ankara, Turkiye
emre.guday@bilecik.edu.tr; emreguday@gmail.com

Murat Sahin
Department of Mathematics, Ankara University, Ankara, Turkiye
msahin@ankara.edu.tr

Received: 6/3/25, Accepted: 11/24/25, Published: 1/5/26

Abstract

In this paper, we consider an mr-th order linear recurring sequence s_n over $\mathbb{R}$, which is a generalization of the general conditional sequences defined by Panario et al. The sequence s_n contains infinitely many sequences that are not included in the conditional sequences, for example, inverses of cyclotomic polynomials, the Kronecker symbol, etc. We calculate the Binet form of the sequence s_n by using r-decimation sequences, that is, the sequence obtained by taking every r-th term of s_n. The paper also provides the Binet form of the general conditional sequences for given m and r by using the successor operator method. Finally, some open problems and conjectures are given. One of the conjectures presents the relationship among the successor operator, the coefficients of the linear recurrence relations satisfied by the conditional sequences, and the integer partitions.

1. Introduction

In [10], Edson and Yayenie introduced a generalization of the Fibonacci sequence defined for all $n \geq 2$ by $q_n = a_i q_{n-1} + q_{n-2},\ n \equiv i \pmod 2$, where $a_i \in \mathbb{R}$ for $i = 0, 1$, and proposed an open problem of finding a closed form of the generating function and the Binet form for a sequence satisfying the relation $q_n = a_i q_{n-1} + q_{n-2},\ n \equiv i \pmod r$ for $i = 0, 1, \ldots, r-1$, where $n \geq 2$. Sahin [12, 13] solved this problem by using continuants. Also, Edson et al. [11] solved this problem independently.

DOI: 10.5281/zenodo.18154045
[1]Corresponding Author

In addition to these, Panario et al. defined a sequence satisfying the relation $q_n = a_{i,1}q_{n-1} + a_{i,2}q_{n-2},\ n \equiv i \pmod{r}$ with real coefficients for $i = 0, 1, \ldots, r-1$, where $n \geq 2$, and found the Binet form of this sequence (Theorem 8 in [3]). Then, Panario et al. [5] introduced a new generalization, namely the general conditional recurrence sequences q_n, defined by

$$q_n = a_{j,1}q_{n-1} + \cdots + a_{j,m}q_{n-m},\ n \equiv j \pmod{r},\ (n \geq m) \tag{1}$$

with real coefficients and arbitrary initial values $q_0, q_1, \ldots, q_{m-1} \in \mathbb{R}$, where $0 \leq j \leq r-1$. Panario et al. obtained the generating function of (1) (Theorem 3 in [5]) and the Binet form of (1) for a special case, namely for given m and $r = 2$ (Theorem 9 in [5]). As a result of [3] and [5], the problem of finding the Binet form of (1) remains unsolved for given m and r such that $m > 2$ and $r > 2$. The main objective of this paper is to obtain the Binet form of the general conditional sequences satisfying recurrence (1). To achieve this purpose, we will define a linear recurring sequence s_n including the general conditional sequences and obtain the Binet form of s_n. Then, we will solve the problem of finding the Binet form of (1) for arbitrary m and r. The reason for defining the sequence s_n can be given as follows. Panario et al. showed that the sequence q_n satisfying the recurrence (1) satisfies a linear recurrence relation (Theorem 2, [5]) of the form

$$q_{n+kr^2} = a_0 q_{n+(kr-1)r} + a_1 q_{n+(kr-2)r} + \cdots + a_{kr-1}q_n,\ (n \geq kr^2) \tag{2}$$

with special coefficients $a_0, a_1, \ldots, a_{kr-1}$ and special initial values q_0, q_1, $\ldots$, q_{kr^2-1}, where k is the least positive integer such that $m \leq kr$. We will generalize the sequence q_n by considering Relation (2) with arbitrary coefficients and arbitrary initial values. Thus, finding the Binet form of a generalization of Relation (2) solves the problem of calculating the Binet form of the general conditional recurrences for arbitrary r and m. Moreover, relation (2) means that the characteristic polynomial corresponding to the general conditional sequence q_n has degree kr^2. In this work, we will find a characteristic polynomial of degree less than kr^2, if possible. For further details on conditional recurrences, see [10], [11], [5], [4], [3], [13], [12], [14]. Let us explain what we mean by the special coefficients in Relation (2). In the linear recurrence relation (2), the coefficients a_i are dependent on the coefficients $a_{i,j}$ of (1); the initial values q_0, q_1, $\ldots$, q_{m-1} are arbitrary, but the remaining initial values q_m, q_{m+1}, $\ldots$, q_{kr^2-1} are dependent on the first m initial values. That is, the coefficients a_i and initial values q_m, q_{m+1}, $\ldots$, q_{kr^2-1} are not arbitrary. By choosing arbitrary coefficients and arbitrary initial values, one can find infinitely many sequences that are not conditional sequences. Example 1 is one such sequence. This point naturally suggests the following question: What is the Binet form and generating function of recurrence (2) for arbitrary coefficients and arbitrary initial values? The paper focuses on this problem, since solving it enables the calculation of the Binet form of (1).

Now, we define the sequences which are a generalization of the general conditional sequences. Let $A_0, A_1, \ldots, A_{m-1} \in \mathbb{R}$ be arbitrary, m and r be integers such that $m \geq 2$ and $r \geq 2$, and let s_n denote the sequence satisfying the mr-th order linear recurrence relation

$$s_{n+mr} = A_{m-1}s_{n+(m-1)r} + A_{m-2}s_{n+(m-2)r} + \cdots + A_0 s_n,\ (n \geq 0) \tag{3}$$

with arbitrary initial values $s_0, s_1, \ldots, s_{mr-1} \in \mathbb{R}$. Example 1 shows that the sequence s_n satisfying the linear recurrence relation (3) is a generalization of the conditional recurrence sequence q_n, but the converse is not true.

Example 1. Consider the conditional recurrence sequence $\{q_n\}$ satisfying the relation

$$q_0, q_1 \in \mathbb{R}, \quad q_n = \begin{cases} aq_{n-1} + bq_{n-2}, & n \equiv 0 \pmod 2 \\ cq_{n-1} + dq_{n-2}, & n \equiv 1 \pmod 2 \end{cases} \tag{4}$$

for all $n \geq 2$. We know from [3] that this sequence satisfies the linear recurrence relation

$$q_{n+4} = (ac + b + d)q_{n+2} + (-bd)q_n$$

with the initial values q_0, q_1, $q_2 = aq_1 + bq_0$, $q_3 = cq_2 + dq_1$. Obviously, for $m = r = 2$ in Relation (3), if the coefficients are taken as

$$A_1 = ac + b + d, \quad A_0 = -bd$$

and the initial values are taken as $s_0 = q_0$, $s_1 = q_1$, $s_2 = aq_1 + bq_0$, $s_3 = cq_2 + dq_1$, the sequence $\{q_n\}$ satisfies the 4-th order linear recurrence

$$s_{n+4} = A_1 s_{n+2} + A_0 s_n.$$

However, the converse is not true. To show this, we construct a counterexample by considering the sequence

$$s_{n+4} = s_{n+2} - s_n$$

with the initial conditions $s_0 = 1$, $s_1 = 0$, $s_2 = 1$, $s_3 = 0$. This sequence corresponds to coefficients of the inverse of the 12th cyclotomic polynomial ($A014021$). To prove that this sequence does not satisfy the conditional recurrence (4), assume the contrary. Then the following system of equations must have a solution

$$\begin{aligned} ac + b + d &= 1, \\ bd &= 1, \\ b &= 1, \\ cb &= 0. \end{aligned}$$

However, the above system of equations has no solution, contradicting our assumption that the sequence satisfies a conditional recurrence, and thus proving that the sequence satisfies no conditional recurrence. Moreover, one can construct infinitely many such counterexamples.

In addition to the above example, not only does the sequence $\{s_n\}$ include many well-known sequences that are part of the conditional sequences, but it also contains some special sequences, such as the *inverse of cyclotomic polynomials* and the *Kronecker symbol*, which appear in Sloane's On-Line Encyclopedia of Integer Sequences [15] and are not included in the general conditional sequences. For inverses of cyclotomic polynomials, see [18]. In Table 1, some known sequences of the form (3) for given coefficients and initial values are listed. Some of these sequences correspond to a conditional sequence, while others do not.

Coefficients	**Initial values**	**Sequence**	**Type**
$(-1, k^2+2)$	$(0, 1, k, k^2+1)$	k-Fibonacci [20]	conditional
$(-k^2, 2k+4)$	$(0, 1, 2, 4k)$	k-Pell [16]	conditional
$(-1, k^2+2)$	$(2, k, k^2+2, k^3+3k)$	k-Lucas [19]	conditional
$(-4t^2, k^2+4t)$	$(0, 1, k, k^2+2t)$	(k, t)-Jacobsthal [21]	conditional
$(-k^2, 2k+4)$	$(2, 2, 2k+4, 6k+8)$	k-Pell-Lucas [17]	conditional
$(-4t^2, k^2+4t)$	$(2, k, k^2+4t, k^3+6kt)$	(k, t)-Jacobsthal-Lucas [21]	conditional
$(-1, 34)$	$(0, 1, 6, 35)$	Balancing number [1]	conditional
$(-1, 34)$	$(1, 3, 17, 99)$	Lucas-Balancing number [6]	conditional
$(-1, -1)$	$(1, 1, 0, -1)$	inverse of 6th cyclotomic polynomial	non-conditional
$(1, -1)$	$(1, 0, 1, 0)$	inverse of 12th cyclotomic polynomial	non-conditional
$(-1, -1, -1)$	$(1, 0, 0, 0, -1, 0)$	inverse of 8th cyclotomic polynomial	non-conditional
$(-1, -1)$	$(1, 0, 0, -1, 0, 0)$	inverse of 9th cyclotomic polynomial	non-conditional
$(1, -1)$	$(0, 1, 0, 0)$	Kronecker symbol $(-6/n)$	non-conditional

Table 1: Some Known Sequences

2. Generating Function of the Sequence $\{s_n\}$

This section is devoted to finding the generating function of the sequence $\{s_n\}$ defined in (3). The following theorem is obtained as a special case of the generating function in [2]. For completeness and to introduce some notation for use in subsequent sections, we will prove the theorem.

Theorem 1. *Let $\{s_n\}$ be the sequence satisfying the Relation* (1)*. Then $\{s_n\}$ has the generating function*

$$G(x) = \frac{\sum_{i=0}^{r-1} \left(v_0^{(i)} x^i + v_1^{(i)} x^{r+i} + v_2^{(i)} x^{2r+i} + \cdots + v_{m-1}^{(i)} x^{(m-1)r+i} \right)}{1 - A_{m-1}x^r - A_{m-2}x^{2r} - \cdots - A_1 x^{mr} - A_0} \tag{5}$$

where

$$\begin{aligned}
v_0^{(i)} &= s_i,\\
v_1^{(i)} &= s_{r+i} - A_{m-1}s_i,\\
v_2^{(i)} &= s_{2r+i} - A_{m-1}s_{r+i} - A_{m-2}s_i,\\
&\vdots\\
v_{m-1}^{(i)} &= s_{(m-1)r+i} - A_{m-1}s_{(m-2)r+i} - \cdots - A_1 s_i.
\end{aligned}$$

Proof. Let $G(x) = \sum_{n=0}^{\infty} s_n x^n$ be the formal expansion of the sequence, and note that

$$\begin{aligned}
x^{-mr}G(x) &= \sum_{n=0}^{\infty} s_n x^{n-mr} = \sum_{n=-mr}^{\infty} s_{n+mr}x^n,\\
-A_{m-1}x^{-(m-1)r}G(x) &= \sum_{n=0}^{\infty} -A_{m-1}s_n x^{n-(m-1)r} = \sum_{n=-(m-1)r}^{\infty} -A_{m-1}s_{n+(m-1)r}x^n,\\
-A_{m-2}x^{-(m-2)r}G(x) &= \sum_{n=0}^{\infty} -A_{m-2}s_n x^{n-(m-2)r} = \sum_{n=-(m-2)r}^{\infty} -A_{m-2}s_{n+(m-2)r}x^n,\\
&\vdots\\
-A_1x^{-r}G(x) &= \sum_{n=0}^{\infty} -A_1 s_n x^{n-r} = \sum_{n=-r}^{\infty} -A_1 s_{n+r}x^n,\\
-A_0G(x) &= \sum_{n=0}^{\infty} -A_0 s_n x^n.
\end{aligned}$$

Then, we obtain that

$$G(x) = \frac{\sum_{i=0}^{r-1}\left(v_0^{(i)}x^i + v_1^{(i)}x^{r+i} + v_2^{(i)}x^{2r+i} + \cdots + v_{m-1}^{(i)}x^{(m-1)r+i}\right)}{1 - A_{m-1}x^r - A_{m-2}x^{2r} - \cdots - A_1x^{mr} - A_0},$$

where

$$\begin{aligned}
v_0^{(i)} &= s_i,\\
v_1^{(i)} &= s_{r+i} - A_{m-1}s_i,\\
v_2^{(i)} &= s_{2r+i} - A_{m-1}s_{r+i} - A_{m-2}s_i,\\
&\vdots\\
v_{m-1}^{(i)} &= s_{(m-1)r+i} - A_{m-1}s_{(m-2)r+i} - \cdots - A_1 s_i.
\end{aligned}$$

□

Remark 1. In [5], the generating function of the general conditional recurrences is presented. Theorem 1 is a generalization of the result, since the coefficients A_i, $0 \le i \le m-1$, and the initial values s_j, $0 \le j \le mr-1$, are arbitrary.

We present the following illustrative example to illustrate the above remark for a special case, that is, for $m = 2$.

Example 2. The general conditional sequence $\{q_n\}$ with arbitrary initial values q_0, and q_1 is defined for all $n \ge 2$ by

$$q_n = \begin{cases} a_{0,1}q_{n-1} + a_{0,2}q_{n-2}, & n \equiv 0 \pmod r \\ a_{1,1}q_{n-1} + a_{1,2}q_{n-2}, & n \equiv 1 \pmod r \\ & \vdots \\ a_{r-1,1}q_{n-1} + a_{r-1,2}q_{n-2}, & n \equiv r-1 \pmod r \end{cases} \tag{6}$$

where $a_{i,j}$ are arbitrary real numbers for $0 \le i \le r-1$ and $1 \le j \le 2$ (see [3]). This sequence satisfies the linear recurrence relation

$$s_{n+2r} = A_0 s_{n+r} + A_1 s_n, \ (n \ge 0)$$

if the initial values are chosen as $s_0 = q_0$, $s_1 = q_1$, $s_2 = a_{2,1}q_1 + a_{2,2}q_0$, $s_3 = a_{3,1}q_2 + a_{3,2}q_1, \ldots$, $s_{2r-1} = a_{r-1,1}q_{2r-2} + a_{r-1,2}q_{2r-3}$, and the coefficients A_0 and A_1 are taken as

$$\begin{aligned} A_0 &= (-1)^{r+1} a_{0,2}a_{1,2}a_{2,2}\ldots a_{r-1,2}, \text{ and} \\ A_1 &= K_0^{(r-1)} + a_{0,2}K_1^{(r-2)}, \end{aligned}$$

where $K_0^{(r-1)}$ and $K_1^{(r-2)}$ are the generalized continuant defined in [3]. When $m = 2$ in (5), Theorem 1 gives Theorem 6 of [3], that is, the generating function of the sequence (6)

$$G(x) = \frac{\sum_{i=0}^{r-1} v_0^{(i)} x^i + v_1^{(i)} x^{r+i}}{1 - A_1 x^r - A_0 x^{2r}}$$

can be obtained, where $v_0^{(i)} = s_i$, $v_1^{(i)} = s_{r+i} - A_1 s_i$.

3. The Binet Form of the Sequence $\{s_n\}$

Let $g(x)$ be the characteristic polynomial of the sequence $\{s_n\}$. That is,

$$g(x) = x^{mr} - A_{m-1}x^{(m-1)r} - A_{m-2}x^{(m-2)r} - \cdots - A_0.$$

Let f be a polynomial defined by

$$f(x) = x^m - A_{m-1}x^{m-1} - A_{m-2}x^{m-2} - \cdots - A_0.$$

In this section, the Binet form of the sequence $\{s_n\}$ is calculated via the m-th degree polynomial f which is the characteristic polynomial of the r-decimation sequences of $\{s_n\}$. Suppose that the polynomial f has m distinct roots. The Binet form is independent of whether $g(x)$ has multiple roots or not. Before finding the Binet form of $\{s_n\}$, we need the following definition.

Definition 1. Let r and τ be two positive integers. The r-decimation and the translation (or shift) of a sequence $\{s_n\}$ are defined by $\{s_{rn}\}$ and $\{s_{n+\tau}\}$, respectively [9].

Example 3. Let $\{s_n\}$ be a sequence. Then, 2-decimation of $\{s_n\}$ is the sequence $\{s_{2n}\}$, that is the sequence $s_0,\ s_2,\ s_4,\ \ldots$. Also, one translation (or one shift) of the sequence $\{s_{2n}\}$ is the sequence $\{s_{2n+1}\}$, that is the sequence $s_1,\ s_3,\ s_5,\ \ldots$.

The following theorem solves the problem of finding the Binet form of a general conditional sequence for arbitrary r and m. This fact will be shown in the next section.

Theorem 2. *Let $f(x) = x^m - A_{m-1}x^{m-1} - A_{m-2}x^{m-2} - \cdots - A_0$ be the characteristic polynomial of an m-th order linear recurrence relation and have m distinct roots $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m$. The Binet form of the sequence $\{s_n\}$ is*

$$s_n = v_0^{(\xi(n))} \sum_{t=1}^{m} \frac{\alpha_t^{m-1+\lfloor \frac{n}{r} \rfloor}}{f'(\alpha_t)} + v_1^{(\xi(n))} \sum_{t=1}^{m} \frac{\alpha_t^{m-2+\lfloor \frac{n}{r} \rfloor}}{f'(\alpha_t)} + \cdots + v_{m-1}^{(\xi(n))} \sum_{t=1}^{m} \frac{\alpha_t^{\lfloor \frac{n}{r} \rfloor}}{f'(\alpha_t)},\ n \geq 0$$

where

$$\begin{aligned}
v_0^{(\xi(n))} &= s_{\xi(n)}, \\
v_1^{(\xi(n))} &= s_{r+\xi(n)} - A_{m-1}s_{\xi(n)}, \\
v_2^{(\xi(n))} &= s_{2r+\xi(n)} - A_{m-1}s_{r+\xi(n)} - A_{m-2}s_{\xi(n)}, \\
&\vdots \\
v_{m-1}^{(\xi(n))} &= s_{(m-1)r+\xi(n)} - A_{m-1}s_{(m-2)r+\xi(n)} - \cdots - A_1 s_{\xi(n)},
\end{aligned}$$

and

$$\xi(n) = \begin{cases} 0, & n \equiv 0 \pmod{r}, \\ 1, & n \equiv 1 \pmod{r}, \\ & \vdots \\ r-1, & n \equiv r-1 \pmod{r}. \end{cases}$$

Proof. Consider the r-decimation sequences $\{s_{rk}\}$, $\{s_{rk+1}\}$,..., $\{s_{rk+(r-1)}\}$ of $\{s_n\}$, and denote the terms of these sequences by $d_k^{(0)}$, $d_k^{(1)}$, ..., $d_k^{(r-1)}$, respectively. That is,

$$d_k^{(0)} := s_{rk},\ d_k^{(1)} := s_{rk+1},\ \ldots,\ d_k^{(r-1)} := s_{rk+(r-1)},\ k \geq 0.$$

Note that

$$\begin{aligned} s_{rk+mr} &= A_{m-1}s_{rk+(m-1)r} + A_{m-2}s_{rk+(m-2)r} + \cdots + A_0 s_{rk}, \\ s_{rk+mr+1} &= A_{m-1}s_{rk+(m-1)r+1} + A_{m-2}s_{rk+(m-2)r+1} + \cdots + A_0 s_{rk+1}, \\ &\vdots \\ s_{rk+mr+(r-1)} &= A_{m-1}s_{rk+(m-1)r+(r-1)} + \cdots + A_0 s_{rk+(r-1)}. \end{aligned}$$

Some manipulations of the indices of the above linear recurrence relations yield

$$\begin{aligned} s_{r(k+m)} &= A_{m-1}s_{r(k+m-1)} + A_{m-2}s_{r(k+m-2)} + \cdots + A_0 s_{rk}, \\ s_{r(k+m)+1} &= A_{m-1}s_{r(k+m-1)+1} + A_{m-2}s_{r(k+m-2)+1} + \cdots + A_0 s_{rk+1}, \\ &\vdots \\ s_{r(k+m)+(r-1)} &= A_{m-1}s_{r(k+m-1)+(r-1)} + \cdots + A_0 s_{rk+(r-1)}. \end{aligned}$$

Then, we obtain that

$$\begin{aligned} d_{k+m}^{(0)} &= A_{m-1}d_{k+m-1}^{(0)} + A_{m-2}d_{k+m-2}^{(0)} + \cdots + A_0 d_k^{(0)}, \\ d_{k+m}^{(1)} &= A_{m-1}d_{k+m-1}^{(1)} + A_{m-2}d_{k+m-2}^{(1)} + \cdots + A_0 d_k^{(1)}, \\ &\vdots \\ d_{k+m}^{(r-1)} &= A_{m-1}d_{k+m-1}^{(r-1)} + A_{m-2}d_{k+m-2}^{(r-1)} + \cdots + A_0 d_k^{(r-1)}. \end{aligned}$$

It follows from these facts that the r-decimation sequences of $\{s_n\}$ satisfy the m-th order linear recurrence relations

$$d_{k+m}^{(i)} = A_{m-1}d_{k+m-1}^{(i)} + A_{m-2}d_{k+m-2}^{(i)} + \cdots + A_0 d_k^{(i)},\ 0 \leq i \leq r-1. \tag{7}$$

On the other hand, it is known from [2] that the Binet form of the m^{th} order linear recurring sequences $\{d_n^{(i)}\}$ satisfying Relation (7) whose characteristic polynomial f has m distinct roots α_1, α_2, ..., α_m, is

$$d_k^{(i)} = \sum_{j=0}^{m-1} v_j^{(i)} p_{k-j}$$

where

$$p_l = \sum_{t=1}^{m} \frac{\alpha_t^{m-1-l}}{f'(\alpha_t)},\ p_0 = 0,$$

and

$$\begin{aligned}
v_0^{(i)} &= s_i, \\
v_1^{(i)} &= s_{r+i} - A_{m-1}s_i, \\
v_2^{(i)} &= s_{2r+i} - A_{m-1}s_{r+i} - A_{m-2}s_i, \\
&\vdots \\
v_{m-1}^{(i)} &= s_{(m-1)r+i} - A_{m-1}s_{(m-2)r+i} - \cdots - A_1 s_i.
\end{aligned}$$

Then, we can write the Binet forms of the r-decimation sequences of $\{s_n\}$ as follows:

$$\begin{aligned}
s_{rk} &= v_0^{(0)}\sum_{t=1}^{m}\frac{\alpha_t^{m-1+k}}{f'(\alpha_t)} + v_1^{(0)}\sum_{t=1}^{m}\frac{\alpha_t^{m-2+k}}{f'(\alpha_t)} + \cdots + v_{m-1}^{(0)}\sum_{t=1}^{m}\frac{\alpha_t^{k}}{f'(\alpha_t)}, \\
s_{rk+1} &= v_0^{(1)}\sum_{t=1}^{m}\frac{\alpha_t^{m-1+k}}{f'(\alpha_t)} + v_1^{(1)}\sum_{t=1}^{m}\frac{\alpha_t^{m-2+k}}{f'(\alpha_t)} + \cdots + v_{m-1}^{(1)}\sum_{t=1}^{m}\frac{\alpha_t^{k}}{f'(\alpha_t)}, \\
&\vdots \\
s_{rk+(r-1)} &= v_0^{(r-1)}\sum_{t=1}^{m}\frac{\alpha_t^{m-1+k}}{f'(\alpha_t)} + v_1^{(r-1)}\sum_{t=1}^{m}\frac{\alpha_t^{m-2+k}}{f'(\alpha_t)} + \cdots + v_{m-1}^{(r-1)}\sum_{t=1}^{m}\frac{\alpha_t^{k}}{f'(\alpha_t)},
\end{aligned}$$

for all $k \geq 0$.

Now, define the function

$$\xi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{r}, \\ 1 & \text{if } n \equiv 1 \pmod{r}, \\ & \vdots \\ r-1 & \text{if } n \equiv r-1 \pmod{r}. \end{cases}$$

Therefore, the Binet form of the sequence $\{s_n\}$ is found as

$$s_n = v_0^{(\xi(n))}\sum_{t=1}^{m}\frac{\alpha_t^{m-1+\lfloor \frac{n}{r}\rfloor}}{f'(\alpha_t)} + v_1^{(\xi(n))}\sum_{t=1}^{m}\frac{\alpha_t^{m-2+\lfloor \frac{n}{r}\rfloor}}{f'(\alpha_t)} + \cdots + v_{m-1}^{(\xi(n))}\sum_{t=1}^{m}\frac{\alpha_t^{\lfloor \frac{n}{r}\rfloor}}{f'(\alpha_t)},\ n \geq 0,$$

where

$$\begin{aligned}
v_0^{(\xi(n))} &= s_{\xi(n)}, \\
v_1^{(\xi(n))} &= s_{r+\xi(n)} - A_{m-1}s_{\xi(n)}, \\
v_2^{(\xi(n))} &= s_{2r+\xi(n)} - A_{m-1}s_{r+\xi(n)} - A_{m-2}s_{\xi(n)}, \\
&\vdots \\
v_{m-1}^{(\xi(n))} &= s_{(m-1)r+\xi(n)} - A_{m-1}s_{(m-2)r+\xi(n)} - \cdots - A_1 s_{\xi(n)}.
\end{aligned}$$

□

As a particular case of Theorem 2, for $m = 2$ and arbitrary r, we give the following result which provides the Binet form of conditional recurrence $\{q_n\}$ given in (6). This formula is different from the Binet form of [3], since the polynomials used in here and [3] are different.

Corollary 1. *In Theorem 2, if the coefficients and the initial values are chosen as*

$$A_0 = (-1)^{r+1} a_{0,2} a_{1,2} a_{2,2} \dots a_{r-1,2},$$
$$A_1 = K_0^{(r-1)} + a_{0,2} K_1^{(r-2)},$$

$s_0 = q_0$, $s_1 = q_1$, $s_2 = a_{2,1}q_1 + a_{2,2}q_0$, $s_3 = a_{3,1}q_2 + a_{3,2}q_1, \dots$, $s_{2r-1} = a_{r-1,1}q_{2r-2} + a_{r-1,2}q_{2r-3}$, *we obtain the Binet form*

$$q_n = \frac{1}{\beta - \alpha}\{(q_{0+\xi(n)}\beta - q_{r+\xi(n)})\alpha^{\lfloor \frac{n}{r} \rfloor} + (q_{r+\xi(n)} - q_{0+\xi(n)}\alpha)\beta^{\lfloor \frac{n}{r} \rfloor}\}$$

of the general conditional sequence $\{q_n\}$ *for* $m = 2$*, where* α *and* β *are distinct roots of the polynomial* $x^2 - A_1 x - A_0$.

3.1. Conditional Recurrences for $r = 3$

The successor operator is a useful tool in order to obtain the linear recurrence relation satisfied by the general conditional sequences. Let $\{t_n\}$ denote any sequence defined by a linear recurrence relation. The successor operator, denoted by E, is the operator defined by $Et_n = t_{n+1}$ and $E^j t_n = t_{n+j}$.

In this subsection, the Binet form of the conditional sequence $\{q_n\}$ for $r = 3$ and $m = 4$ is calculated as an example by using the successor method (see [5]) and Theorem 2. Panario et al. (Theorem 6 in [5]) found the linear recurrence relation satisfied by the general conditional sequences for given m and $r = 2$. This result (Theorem 6 in [5]) has an importance, since the coefficients of the linear recurrence is a formulization of the successor method. In this subsection, we extend the result from $r = 2$ to $r = 3$. In Theorem 3, a linear recurrence relation whose order less than the order of Relation (2) satisfied by the sequence $\{q_n\}$ for $r = 3$ and arbitrary m is determined up to sign of its coefficients. The aim of Theorem 3 is to present a formula which can be used rather than the successor operator method. At the end of this section, we give some open problems and a conjecture to further researches. Because, solving these problems and proving the conjecture provide a formula which can be used rather than the successor operator method.

Example 4. Let $r = 3$ and $m = 4$. The sequence $\{q_n\}$ with initial values q_0, q_1, q_2, q_3 is defined for $n \geq 4$ by:

$$q_n = \begin{cases} a_{0,1}q_{n-1} + a_{0,2}q_{n-2} + a_{0,3}q_{n-3} + a_{0,4}q_{n-4}, & n \equiv 0 \pmod 3, \\ a_{1,1}q_{n-1} + a_{1,2}q_{n-2} + a_{1,3}q_{n-3} + a_{1,4}q_{n-4}, & n \equiv 1 \pmod 3, \\ a_{2,1}q_{n-1} + a_{2,2}q_{n-2} + a_{2,3}q_{n-3} + a_{2,4}q_{n-4}, & n \equiv 2 \pmod 3. \end{cases} \tag{8}$$

Let $a_{i,4} \neq 0$ for $0 \leq i \leq 2$. Define k as the smallest positive integer such that $4 \leq 3k$; that is, $k = 2$. For convenience, define $a_{i,0} = 1$ for $0 \leq i \leq 2$ and $a_{i,j} = 0$ if $4 < j \leq 6$. Equation (8) can be rewritten as

$$0 = \begin{cases} -a_{0,0}q_n + a_{0,1}q_{n-1} + a_{0,2}q_{n-2} + \cdots + a_{0,6}q_{n-6}, & n \equiv 0 \pmod 3, \\ -a_{1,0}q_n + a_{1,1}q_{n-1} + a_{1,2}q_{n-2} + \cdots + a_{1,6}q_{n-6}, & n \equiv 1 \pmod 3, \\ -a_{2,0}q_n + a_{2,1}q_{n-1} + a_{2,2}q_{n-2} + \cdots + a_{2,6}q_{n-6}, & n \equiv 2 \pmod 3. \end{cases} \tag{9}$$

If n in the ith equation of system (9) is replaced by $(n+2)3+i$ for all $0 \leq i \leq 2$, we obtain the following equations:

$$0 = \begin{cases} -a_{0,0}q_{(n+2)3} + a_{0,1}q_{(n+2)3-1} + \cdots + a_{0,6}q_{(n+2)3-6}, & n \equiv 0 \pmod 3, \\ -a_{1,0}q_{(n+2)3+1} + a_{1,1}q_{(n+2)3+1-1} + \cdots + a_{1,6}q_{(n+2)3+1-6}, & n \equiv 1 \pmod 3, \\ -a_{2,0}q_{(n+2)3+2} + a_{2,1}q_{(n+2)3+2-1} + \cdots + a_{2,6}q_{(n+2)3+2-6}, & n \equiv 2 \pmod 3. \end{cases} \tag{10}$$

Let $Q_n^{(t)} = q_{3n+t}$, for each $0 \leq t \leq 2$. Then, Equation (10) becomes

$$-a_{0,0}Q_{n+2}^{(0)} + a_{0,1}Q_{n+1}^{(2)} + a_{0,2}Q_{n+1}^{(1)} + a_{0,3}Q_{n+1}^{(0)} + a_{0,4}Q_n^{(2)} + a_{0,5}Q_n^{(1)} + a_{0,6}Q_n^{(0)} = 0$$

$$-a_{1,0}Q_{n+2}^{(1)} + a_{1,1}Q_{n+2}^{(0)} + a_{1,2}Q_{n+1}^{(2)} + a_{1,3}Q_{n+1}^{(1)} + a_{1,4}Q_{n+1}^{(0)} + a_{1,5}Q_n^{(2)} + a_{1,6}Q_n^{(1)} = 0$$

$$-a_{2,0}Q_{n+2}^{(2)} + a_{2,1}Q_{n+2}^{(1)} + a_{2,2}Q_{n+2}^{(0)} + a_{2,3}Q_{n+1}^{(2)} + a_{2,4}Q_{n+1}^{(1)} + a_{2,5}Q_{n+1}^{(0)} + a_{2,6}Q_n^{(2)} = 0$$

Then, by using the successor operator, the following equations

$$(-a_{0,0}E^2 + a_{0,3}E + a_{0,6})Q_n^{(0)} + (a_{0,2}E + a_{0,5})Q_n^{(1)} + (a_{0,1}E + a_{0,4})Q_n^{(2)} = 0$$

$$(a_{1,1}E^2 + a_{1,4}E)Q_n^{(0)} + (-a_{1,0}E^2 + a_{1,3}E + a_{1,6})Q_n^{(1)} + (a_{1,2}E + a_{1,5})Q_n^{(2)} = 0$$

$$(a_{2,2}E^2 + a_{2,5}E)Q_n^{(0)} + (a_{2,1}E^2 + a_{2,4}E)Q_n^{(1)} + (-a_{2,0}E^2 + a_{2,3}E + a_{2,6})Q_n^{(2)} = 0$$

are obtained. Let P be a 3×3 matrix defined by

$$\begin{bmatrix} (-a_{0,0}E^2 + a_{0,3}E + a_{0,6}) & (a_{0,2}E + a_{0,5}) & (a_{0,1}E + a_{0,4}) \\ (a_{1,1}E^2 + a_{1,4}E) & (-a_{1,0}E^2 + a_{1,3}E + a_{1,6}) & (a_{1,2}E + a_{1,5}) \\ (a_{2,2}E^2 + a_{2,5}E) & (a_{2,1}E^2 + a_{2,4}E) & (-a_{2,0}E^2 + a_{2,3}E + a_{2,6}) \end{bmatrix}$$

and let $D(E) = detP$. It is known from [5], the sequence $\{q_n\}$ given in (8) satisfies the linear recurrence relation whose characteristic polynomial is $D(x^3)$. Then

$$\begin{aligned}
D(E) = &- a_{0,0}a_{1,0}a_{2,0}E^6 \\
&+ (a_{0,0}a_{1,0}a_{2,3} + a_{0,0}a_{1,3}a_{2,0} + a_{0,0}a_{1,2}a_{2,1} + a_{0,3}a_{1,0}a_{2,0} + a_{0,2}a_{1,1}a_{2,0} \\
&+ a_{0,1}a_{1,1}a_{2,1} + a_{0,1}a_{1,0}a_{2,2})E^5 \\
&+ (a_{0,0}a_{1,0}a_{2,6} - a_{0,0}a_{1,3}a_{2,3} + a_{0,0}a_{1,6}a_{2,0} + a_{0,0}a_{1,2}a_{2,4} + a_{0,0}a_{1,5}a_{2,1} \\
&- a_{0,3}a_{1,0}a_{2,3} - a_{0,3}a_{1,3}a_{2,0} - a_{0,3}a_{1,2}a_{2,1} + a_{0,6}a_{1,0}a_{2,0} - a_{0,2}a_{1,1}a_{2,3} \\
&+ a_{0,2}a_{1,4}a_{2,0} + a_{0,2}a_{1,2}a_{2,2} + a_{0,5}a_{1,1}a_{2,0} + a_{0,1}a_{1,1}a_{2,4} + a_{0,1}a_{1,4}a_{2,1} \\
&+ a_{0,1}a_{1,0}a_{2,5} - a_{0,1}a_{1,3}a_{2,2} + a_{0,4}a_{1,1}a_{2,1} + a_{0,4}a_{1,0}a_{2,2})E^4 \\
&+ (-a_{0,0}a_{1,3}a_{2,6} - a_{0,0}a_{1,6}a_{2,3} + a_{0,0}a_{1,5}a_{2,4} - a_{0,3}a_{1,0}a_{2,6} + a_{0,3}a_{1,3}a_{2,3} \\
&- a_{0,3}a_{1,6}a_{2,0} - a_{0,3}a_{1,2}a_{2,4} - a_{0,3}a_{1,5}a_{2,1} - a_{0,6}a_{1,0}a_{2,3} - a_{0,6}a_{1,3}a_{2,0} \\
&- a_{0,6}a_{1,2}a_{2,1} - a_{0,2}a_{1,1}a_{2,6} - a_{0,2}a_{1,4}a_{2,3} + a_{0,2}a_{1,2}a_{2,5} + a_{0,2}a_{1,5}a_{2,2} \\
&- a_{0,5}a_{1,1}a_{2,3} + a_{0,5}a_{1,4}a_{2,0} + a_{0,5}a_{1,2}a_{2,2} + a_{0,1}a_{1,4}a_{2,4} - a_{0,1}a_{1,3}a_{2,5} \\
&- a_{0,1}a_{1,6}a_{2,2} + a_{0,4}a_{1,1}a_{2,4} + a_{0,4}a_{1,4}a_{2,1} + a_{0,4}a_{1,0}a_{2,5} - a_{0,4}a_{1,3}a_{2,2})E^3 \\
&+ (-a_{0,0}a_{1,6}a_{2,6} + a_{0,3}a_{1,3}a_{2,6} + a_{0,3}a_{1,6}a_{2,3} - a_{0,3}a_{1,5}a_{2,4} - a_{0,6}a_{1,0}a_{2,6} \\
&+ a_{0,6}a_{1,3}a_{2,3} - a_{0,6}a_{1,6}a_{2,0} - a_{0,6}a_{1,2}a_{2,4} - a_{0,6}a_{1,5}a_{2,1} - a_{0,2}a_{1,4}a_{2,6} \\
&+ a_{0,2}a_{1,5}a_{2,5} - a_{0,5}a_{1,1}a_{2,6} - a_{0,5}a_{1,4}a_{2,3} + a_{0,5}a_{1,2}a_{2,5} + a_{0,5}a_{1,5}a_{2,2} \\
&- a_{0,1}a_{1,6}a_{2,5} + a_{0,4}a_{1,4}a_{2,4} - a_{0,4}a_{1,3}a_{2,5} - a_{0,4}a_{1,6}a_{2,2})E^2 \\
&+ (a_{0,3}a_{1,6}a_{2,6} + a_{0,6}a_{1,3}a_{2,6} + a_{0,6}a_{1,6}a_{2,3} - a_{0,6}a_{1,5}a_{2,4} - a_{0,5}a_{1,4}a_{2,6} \\
&+ a_{0,5}a_{1,5}a_{2,5} - a_{0,4}a_{1,6}a_{2,5})E \\
&+ (a_{0,6}a_{1,6}a_{2,6})
\end{aligned}$$

and thus,

$$D(x^3) = A_0x^{18} + A_3x^{15} + A_6x^{12} + A_9x^9 + A_{12}x^6 + A_{15}x^3 + A_{18}$$

where

$$A_d = \sum_{\substack{d=d_1+d_2+d_3 \\ 0\le d_1,d_2,d_3\le 6}} \pm a_{0,d_1}a_{1,d_2}a_{2,d_3} \text{ for } d = 0, 3, \ldots, 18$$

such that $d_2 \not\equiv i+1 \pmod 3$ and $d_3 \not\equiv i+2 \pmod 3$ if $d_1 \equiv i \pmod 3$ for each $0 \le i \le 2$.

On the other hand, from the fact that $a_{i,5} = a_{i,6} = 0$ and $a_{i,4} \ne 0$ for $0 \le i \le 2$, one can obtain a linear recurrence relation whose order is less than 18 by omitting the zero terms:

$$\begin{aligned}
D(x^3) =& A_0x^{18} + A_3x^{15} + A_6x^{12} + A_9x^9 + A_{12}x^6 \\
=& x^6(A_0x^{12} + A_3x^9 + A_6x^6 + A_9x^3 + A_{12}).
\end{aligned}$$

Therefore, the sequence $\{q_n\}$ for $m = 4$ and $r = 3$ satisfies a 12^{th} order linear recurrence relation which has the characteristic polynomial

$$f(x) = A_0x^{12} + A_3x^9 + A_6x^6 + A_9x^3 + A_{12}.$$

$$q_n = v_0^{(\xi(n))}\sum_{t=1}^{4}\frac{\alpha_t^{3+\lfloor\frac{n}{3}\rfloor}}{f'(\alpha_t)} + v_1^{(\xi(n))}\sum_{t=1}^{4}\frac{\alpha_t^{2+\lfloor\frac{n}{3}\rfloor}}{f'(\alpha_t)} + v_2^{(\xi(n))}\sum_{t=1}^{4}\frac{\alpha_t^{1+\lfloor\frac{n}{3}\rfloor}}{f'(\alpha_t)} + v_3^{(\xi(n))}\sum_{t=1}^{4}\frac{\alpha_t^{\lfloor\frac{n}{3}\rfloor}}{f'(\alpha_t)}$$

where

$$\begin{aligned}
v_0^{(\xi(n))} &= q_{\xi(n)},\\
v_1^{(\xi(n))} &= q_{3+\xi(n)} - A_0q_{\xi(n)},\\
v_2^{(\xi(n))} &= q_{6+\xi(n)} - A_0q_{3+\xi(n)} - A_3q_{\xi(n)},\\
v_3^{(\xi(n))} &= q_{9+\xi(n)} - A_0q_{6+\xi(n)} - A_3q_{3+\xi(n)} - A_6q_{\xi(n)},
\end{aligned}$$

and

$$\xi(n) = \begin{cases} 0, & n \equiv 0 \pmod 3,\\ 1, & n \equiv 1 \pmod 3,\\ 2, & n \equiv 2 \pmod 3.\end{cases}$$

Theorem 3. *Let $m > 3$ be a positive integer and let $\{a_{i,j}\}$ be real numbers such that $a_{i,m} \neq 0$ for $0 \leq i \leq 2$ and $1 \leq j \leq m$. The sequence $\{q_n\}$ given in (1) becomes*

$$q_n = \begin{cases} a_{0,1}q_{n-1} + a_{0,2}q_{n-2} + \cdots + a_{0,m}q_{n-m}, & n \equiv 0 \pmod 3,\\ a_{1,1}q_{n-1} + a_{1,2}q_{n-2} + \cdots + a_{1,m}q_{n-m}, & n \equiv 1 \pmod 3,\\ a_{2,1}q_{n-1} + a_{2,2}q_{n-2} + \cdots + a_{2,m}q_{n-m}, & n \equiv 2 \pmod 3.\end{cases} \tag{11}$$

The sequence $\{q_n\}$ given in (11) satisfies the linear recurrence relation

$$q_n = A_3q_{n-3(m-1)} + A_6q_{n-3(m-2)} + \cdots + A_{3(m-1)}q_{n-3} + A_{3m}q_n \tag{12}$$

where

$$A_d = \sum_{\substack{d=d_1+d_2+d_3\\ 0\leq d_1,d_2,d_3\leq m}} \pm a_{0,d_1}a_{1,d_2}a_{2,d_3} \text{ for } d = 0, 3, \ldots, 3m$$

such that $d_2 \not\equiv i+1 \pmod 3$ and $d_3 \not\equiv i+2 \pmod 3$ if $d_1 \equiv i \pmod 3$ for each $0 \leq i \leq 2$.

Proof. Let $m = 3t + 1$ for some integer t. The proof is similar when $m = 3t$ and $m = 3t + 2$. By using the successor operator method, the following equations are obtained:

$$
\begin{aligned}
&(- a_{0,0}E^k + a_{0,3}E^{k-1} + \cdots + a_{0,3k})Q_n^{(0)} + (a_{0,2}E^{k-1} + \cdots + a_{0,3k-1})Q_n^{(1)} \\
&\quad + (a_{0,1}E^{k-1} + \cdots + a_{0,3k-2})Q_n^{(2)} = 0 \\
&(a_{1,1}E^k + \cdots + a_{1,3k-2}E)Q_n^{(0)} + (-a_{1,0}E^k + a_{1,3}E^{k-1} + \cdots + a_{1,3k})Q_n^{(1)} \\
&\quad + (a_{1,2}E^{k-1} + \cdots + a_{1,3k-1})Q_n^{(2)} = 0 \\
&(a_{2,2}E^k + \cdots + a_{2,3k-1}E)Q_n^{(0)} + (a_{2,1}E^k + \cdots + a_{2,3k-2}E)Q_n^{(1)} \\
&\quad + (-a_{2,0}E^k + a_{2,3}E^{k-1} + \cdots + a_{2,3k})Q_n^{(2)} = 0
\end{aligned}
$$

Then, the corresponding matrix P for the sequence $\{q_n\}$ is given by

$$
P = \begin{bmatrix} P_{00}(E) & P_{02}(E) & P_{01}(E) \\ P_{11}(E) & P_{10}(E) & P_{12}(E) \\ P_{22}(E) & P_{21}(E) & P_{20}(E) \end{bmatrix}
$$

where

$$
\begin{aligned}
P_{00}(E) =& - a_{0,0}E^k + a_{0,3}E^{k-1} + \cdots + a_{0,3k} \\
P_{02}(E) =& a_{0,2}E^{k-1} + \cdots + a_{0,3k-1} \\
P_{01}(E) =& a_{0,1}E^{k-1} + \cdots + a_{0,3k-2} \\
P_{11}(E) =& a_{1,1}E^k + \cdots + a_{1,3k-2}E \\
P_{10}(E) =& - a_{1,0}E^k + a_{1,3}E^{k-1} + \cdots + a_{1,3k} \\
P_{12}(E) =& a_{1,2}E^{k-1} + \cdots + a_{1,3k-1} \\
P_{22}(E) =& a_{2,2}E^k + \cdots + a_{2,3k-1}E \\
P_{21}(E) =& a_{2,1}E^k + \cdots + a_{2,3k-2}E \\
P_{20}(E) =& - a_{2,0}E^k + a_{2,3}E^{k-1} + \cdots + a_{2,3k}
\end{aligned}
$$

and k is the least positive integer such that $m \leq 3k$, that is, $k = \frac{m+2}{3}$. Thus,

$D = D(E) = detP$ is obtained as:

$$\begin{aligned}
D(E) = &- a_{0,0}a_{1,0}a_{2,0}E^{3k} \\
&+ (a_{0,0}a_{1,0}a_{2,3} + a_{0,0}a_{1,3}a_{2,0} + a_{0,0}a_{1,2}a_{2,1} + a_{0,3}a_{1,0}a_{2,0} + a_{0,2}a_{1,1}a_{2,0} \\
&+ a_{0,1}a_{1,1}a_{2,1} + a_{0,1}a_{1,0}a_{2,2})E^{3k-1} \\
&\vdots \\
&+ (a_{0,3k-6}a_{1,3k}a_{2,3k} + a_{0,3k-3}a_{1,3k-3}a_{2,3k} + a_{0,3k-3}a_{1,3k}a_{2,3k-3} \\
&- a_{0,3k-3}a_{1,3k-1}a_{2,3k-2} + a_{0,3k}a_{1,3k-6}a_{2,3k} + a_{0,3k}a_{1,3k}a_{2,3k-6} \\
&+ a_{0,3k}a_{1,3k-3}a_{2,3k-3} - a_{0,3k}a_{1,3k-1}a_{2,3k-5} - a_{0,3k}a_{1,3k-4}a_{2,3k-2} \\
&- a_{0,3k-4}a_{1,3k-2}a_{2,3k} + a_{0,3k-4}a_{1,3k-1}a_{2,3k-1} - a_{0,3k-1}a_{1,3k-5}a_{2,3k} \\
&+ a_{0,3k-1}a_{1,3k-1}a_{2,3k-4} - a_{0,3k-1}a_{1,3k-2}a_{2,3k-3} + a_{0,3k-1}a_{1,3k-4}a_{2,3k-1} \\
&- a_{0,3k-5}a_{1,3k}a_{2,3k-1} + a_{0,3k-2}a_{1,3k-2}a_{2,3k-2} - a_{0,3k-2}a_{1,3k-3}a_{2,3k-1} \\
&- a_{0,3k-2}a_{1,3k}a_{2,3k-4})E^2 \\
&+ (a_{0,3k-3}a_{1,3k}a_{2,3k} + a_{0,3k}a_{1,3k-3}a_{2,3k} + a_{0,3k}a_{1,3k}a_{2,3k-3} \\
&- a_{0,3k}a_{1,3k-1}a_{2,3k-2} - a_{0,3k-1}a_{1,3k-2}a_{2,3k} + a_{0,3k-1}a_{1,3k-1}a_{2,3k-1} \\
&- a_{0,3k-2}a_{1,3k}a_{2,3k-1})E \\
&+ a_{0,3k}a_{1,3k}a_{2,3k}.
\end{aligned}$$

Note that $a_{i,3k} = a_{i,3k-1} = 0$ for $i = 0, 1, 2$ by the successor method. Then,

$$\begin{aligned}
D(E) &= - E^2(A_0E^{3k-2} - A_3E^{3k-3} - \cdots - A_{9k-6}) \\
&= - E^2(E^m - A_3E^{m-1} - \cdots - A_{3m}).
\end{aligned}$$

Therefore, the sequence $\{q_n\}$ satisfies the linear recurrence relation

$$q_{n+3m} = A_3q_{n-3(m-1)} + A_6q_{n-3(m-2)} + \cdots + A_{3(m-1)}q_{n-3} + A_{3m}q_n$$

where

$$A_d = \sum_{\substack{d=d_1+d_2+d_3 \\ 0\le d_1,d_2,d_3\le m}} \pm a_{0,d_1}a_{1,d_2}a_{2,d_3} \text{ for } d = 0, 3, \ldots, 3m$$

such that $d_2 \not\equiv i + 1 \pmod 3$ and $d_3 \not\equiv i + 2 \pmod 3$ if $d_1 \equiv i \pmod 3$ for each $0 \le i \le 2$. The above restriction about d_1, d_2, and d_3 comes from the definition of the determinant. □

Open Problem 1. Determine the sign of the coefficients of Relation (12).

Note that the order of Relation (12) is $3m$ which is less than $9k$ where k is the smallest positive integer such that $m \le 3k$. Therefore, a linear recurrence relation is obtained whose order is less than the order of Relation (2) when $m \ne 3k$.

Open Problem 2. Let $r > 3$. Find a linear recurrence relation satisfied by the general conditional sequences whose order is less than the order of Relation (2) for $m < r$ and $m > r$.

Moreover, several numerical examples motivate this work to give the following conjecture.

Conjecture 1. Let m and r be two positive integers such that $m > r$. The general conditional sequence $\{q_n\}$ defined in (1) satisfies the linear recurrence

$$q_n = A_r q_{n-r} + A_{2r} q_{n-2r} + \cdots + A_{mr} q_{n-mr}$$

where

$$A_d = \sum_{\substack{d=d_1+\cdots+d_r \\ 0\le d_1,\ldots,d_r\le m}} \pm a_{0,d_1} \ldots a_{r-1,d_r} \text{ for } d = 0, r, \ldots, mr$$

such that $d_2 \not\equiv i+1 \pmod r$, $d_3 \not\equiv i+2 \pmod r$, $\ldots$, and $d_r \not\equiv r-1+i \pmod r$ if $d_1 \equiv i \pmod r$ for each $0 \le i \le r-1$.

3.2. A Link with the Integer Partitions

As we will see in Example 5, a relationship exists between the restricted integer partitions and the coefficients of the linear recurrence relations satisfied by the general conditional sequences. Finally, based on the observation and some numerical examples, we give a conjecture.

For $1 \le N \le n$, $m \ge 0$, let $w_{N,m}(n)$ be the number of ordered partitions of the integer n into N parts each of size at least 0 but no larger than m (see [8]). Let $w^*_{N,m}(n)$ and $\overline{w}_{N,m}(n)$ denote the number of ordered distinct and ordered non-distinct partitions of n into N parts each of size at least 0 but no larger than m, respectively.

Example 5. Recall from Example 4 that a conditional sequence $\{q_n\}$ for $r = 3$ and $m = 4$ satisfying the conditional recurrence relation

$$q_n = \begin{cases} a_{0,1}q_{n-1} + a_{0,2}q_{n-2} + a_{0,3}q_{n-3} + a_{0,4}q_{n-4}, & n \equiv 0 \pmod 3, \\ a_{1,1}q_{n-1} + a_{1,2}q_{n-2} + a_{1,3}q_{n-3} + a_{1,4}q_{n-4}, & n \equiv 1 \pmod 3, \\ a_{2,1}q_{n-1} + a_{2,2}q_{n-2} + a_{2,3}q_{n-3} + a_{2,4}q_{n-4}, & n \equiv 2 \pmod 3. \end{cases}$$

for all $n \ge 4$ satisfies the linear recurrence relation

$$q_n = A_3 q_{n-3(m-1)} + A_6 q_{n-3(m-2)} + \cdots + A_{3(m-1)} q_{n-3} + A_{3m} q_n$$

where

$$A_d = \sum_{\substack{d=d_1+d_2+d_3 \\ 0\le d_1,d_2,d_3\le m}} \pm a_{0,d_1} a_{1,d_2} a_{2,d_3} \text{ for } d = 0, 3, \ldots, 3m$$

and d_1, d_2, d_3 satisfy the condition

$$d_2 \not\equiv i+1 \pmod 3 \text{ and } d_3 \not\equiv i+2 \pmod 3 \text{ if } d_1 \equiv i \pmod 3,\ 0 \le i \le 2. \quad (13)$$

Note that $A_0 = 1$ which is the coefficient of the term q_n in the above relation.

Now, let $\mathcal{A}_d$ denote the number of the terms $a_{0,d_1} a_{1,d_2} a_{2,d_3}$ composing the coefficients A_d. Consider the number of ordered partitions of the integer d into 3 parts each of size at least 0 but no larger than 4.

- For $d = 0$: $w_{3,4}(0) = |\{(0,0,0)\}|$,
- For $d = 3$:

$$\begin{aligned} w_{3,4}(3) = |\{&(0,0,3),(0,3,0),(3,0,0),(0,1,2),(0,2,1),(1,0,2),(1,2,0),(2,0,1),\\ &(2,1,0),(1,1,1)\}|, \end{aligned}$$

- For $d = 6$:

$$\begin{aligned} w_{3,4}(6) = |\{&(0,4,2),(0,2,4),(2,0,4),(2,4,0),(4,0,2),(4,2,0),(1,1,4),(1,4,1),\\ &(4,1,1),(0,3,3),(3,0,3),(3,3,0),(2,2,2)\}|, \end{aligned}$$

- For $d = 9$:

$$\begin{aligned} w_{3,4}(9) = |\{&(1,4,4),(4,1,4),(4,4,1),(2,3,4),(2,4,3),(3,2,4),(3,4,2),(4,2,3),\\ &(4,3,2),(3,3,3)\}|, \end{aligned}$$

- For $d = 12$:

$$w_{3,4}(12) = |\{(4,4,4)\}|.$$

If the partitions that satisfy condition (13) are listed, it is concluded that

- For $d = 0$: $\mathcal{A}_0 = |\{(0,0,0)\}|$,
- For $d = 3$:

$$\begin{aligned} \mathcal{A}_3 &= |\{(0,0,3),(0,3,0),(3,0,0),(0,2,1),(1,0,2),(2,1,0),(1,1,1)\}| \\ &= \overline{w}_{3,4}(3) + \frac{w^*_{3,4}(3)}{2}, \end{aligned}$$

- For $d = 6$:

$$\begin{aligned} \mathcal{A}_6 = |\{&(0,2,4),(2,4,0),(4,0,2),(1,1,4),(1,4,1),(4,1,1),(0,3,3),(3,0,3),\\ &(3,3,0),(2,2,2)\}| \\ = \overline{w}&_{3,4}(6) + \frac{w^*_{3,4}(6)}{2}, \end{aligned}$$

- For $d = 9$:

$$\begin{aligned}\mathcal{A}_9 &= |\{(1,4,4),(4,1,4),(4,4,1),(2,4,3),(3,2,4),(4,3,2),(3,3,3)\}| \\ &= \overline{w}_{3,4}(9) + \frac{w^*_{3,4}(9)}{2},\end{aligned}$$

- For $d = 12$:

$$\mathcal{A}_{12} = |\{(4,4,4)\}|.$$

That is, it can be seen that the value of $\mathcal{A}_d$ for $d = 0, 3, \ldots, 12$ is written in terms of the restricted integer partition functions $w_{3,4}(d)$, $w^*_{3,4}(d)$, and $\overline{w}_{3,4}(d)$ as follows:

$$\mathcal{A}_d = w_{3,4}(d) - \frac{w^*_{3,4}(d)}{2}$$

or, equivalently,

$$\mathcal{A}_d = \overline{w}_{3,4}(d) + \frac{w^*_{3,4}(d)}{2}$$

for $d = 0, 3, \ldots, 12$.

Thus, we propose the following conjecture.

Conjecture 2. Let D be the set defined by

$$\begin{aligned}D = \{(d_1,d_2,d_3) \ : \ d = d_1 + d_2 + d_3,\ d_{j+1} \not\equiv i + j \pmod 3 \text{ if } d_1 \equiv i \pmod 3, \\ 1 \le j \le 2,\ 0 \le i \le 2\},\end{aligned}$$

and let

$$\mathcal{A}_d = \Big|\{a_{0,d_1} a_{1,d_2} a_{2,d_3} \ : \ 0 \le d_1, d_2, d_3 \le m,\ (d_1,d_2,d_3) \in D\}\Big|.$$

Then, for $d = 0, 3, \ldots, 3m$,

$$\mathcal{A}_d = w_{3,m}(d) - \frac{w^*_{3,m}(d)}{2}$$

or, equivalently,

$$\mathcal{A}_d = \overline{w}_{3,m}(d) + \frac{w^*_{3,m}(d)}{2}.$$

Acknowledgement. The corresponding author would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their financial support during his doctorate studies. This work has been supported by TUBITAK-1001 (Project No.124F034).

References

[1] A. Behera and G. K. Panda, On the square roots of triangular numbers, *Fibonacci Quart.* **37** (1999), 98–105.

[2] C. Levesque, On m-th order linear recurrences, *The Fibonacci Quart.* Vol.**23**, No.**4** (1985), 290-293.

[3] D. Panario, M. Sahin, and Q. Wang, A family of Fibonacci-like conditional sequences, *Integers* **13** (2013), #A78.

[4] D. Panario, M. Sahin, and Q. Wang, Non-homogeneous conditional recurrences, *Linear Multilinear Algebra* **66:10** (2018), 2089-2099.

[5] D. Panario, M. Sahin, Q. Wang, and W. Webb, General conditional recurrences, *Appl. Math. Comput.* **243** (2014), 220–231.

[6] G. K. Panda, Some fascinating properties of balancing numbers, *Congr. Numer.* **194** (2009), 185–189.

[7] J. L. Cereceda, Binet's formula for generalized tribonacci numbers, *International Journal of Mathematical Education in Science and Technology* **46:8**, https://doi.org/10.1080/0020739X.2015.1031837 (2015), 1235-1243.

[8] J. Ratsaby, Estimate of the number of restricted integer-partitions, *Appl. Anal. Discrete Math.* **2** (2008), 222-233.

[9] M. Buck and N. Zierler, Decimations of linear recurring sequences, *Comput. Math. Appl.* **39** (2000), 95-102.

[10] M. Edson and O. Yayenie, A new generalization of Fibonacci sequence and extended Binet's formula, *Integers* **9** (2009), 639–654.

[11] M. Edson, S. Lewis, and O. Yayenie, The k-periodic Fibonacci sequence and an extended Binet's formula, *Integers* **11** (2011), 739–751.

[12] M. Sahin, The Binet-like formula of a family of the conditional sequences by matrix methods, *Ars Combin.* **102** (2011), 393–398.

[13] M. Sahin, The generating function of a family of sequences in terms of the continuant, *Appl. Math. Comput.* **217(12)** (2011), 5416–5420.

[14] O. Yayenie, A note on generalized Fibonacci sequence, *Appl. Math. Comput.* **217** (2011), 5603–5611.

[15] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, https://oeis.org

[16] P. Catarino, On some identities and generating functions for k-Pell numbers, *International Journal of Mathematical Analysis* **7** (2013), 1877–1884.

[17] P. Catarino and P. Vasco, On some identities and generating functions for k-Pell-Lucas sequence, *Appl. Math. Sci.* **7** (2013), 4867–4873.

[18] P. Moree, Inverse cyclotomic polynomials, *J. Number Theory* **129** (2009), 667–680.

[19] S. Falcon, On the k-Lucas numbers, *Int. J. Contemp. Math. Sci.* **6** (2011), 1039–1050.

[20] S. Falcon and A. Plaza, On the Fibonacci k-numbers, *Chaos, Solitons and Fractals* **32** (2007), 1615–1624.

[21] S. Uygun, The (s,t)-Jacobsthal and (s,t)-Jacobsthal-Lucas sequences, *Appl. Math. Sci.* **9** (2015), 3467–3476.