



**COUNTING APPEARANCES OF INTEGERS IN SETS OF
ARITHMETIC PROGRESSIONS**

Florian Pausinger

*Department of Mathematics and CEMS.UL, Faculty of Sciences, University of
Lisbon, Portugal.*

fpausinger@fc.ul.pt

Received: 7/15/25, Accepted: 3/27/26, Published: 5/1/26

Abstract

The sequence A067549 of the On-Line Encyclopedia of Integer Sequences is defined as $(a_k)_{k \geq 1}$, with a_k being the determinant of the $k \times k$ matrix whose diagonal contains the first k prime numbers and all other elements are ones. We relate this sequence to a concrete counting problem. Choose an arbitrary residue class r_i for each prime p_i with $1 \leq i \leq k$ and set $P_k = \prod_{i=1}^k p_i$. We show that a_k is the number of integers in $[1, P_k]$ that are contained in at most one of the k chosen residue classes. Interestingly, we show that this sequence is closely related to the better known sequence A005867 for which we derive a novel characterization in terms of determinants and which gives the number of integers in $[1, P_k]$ that are not contained in any of the k residue classes. Our proof is purely structural and, therefore, it can be generalized to counting appearances of integers in residue classes of arbitrary arithmetic progressions generated by k different primes using the determinant of a matrix of ones having those k primes on its diagonal. The revealed structure also offers a fast way of calculating such determinants.

1. Introduction

The sequence A067549 of the On-Line Encyclopedia of Integer Sequences [3], is defined as $(a_k)_{k \geq 1}$ with a_k being the determinant of the $k \times k$ matrix whose diagonal contains the first k prime numbers and all other elements are ones, i.e.,

$$a_k = \det \begin{bmatrix} 2 & 1 & \dots & 1 \\ 1 & 3 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & p_k \end{bmatrix}.$$

The first five elements of this sequence, for $k = 1, \dots, 5$, are

$$2, 5, 22, 140, 1448, \dots;$$

see [3] for a table of the first 300 elements of this sequence. We relate this sequence to a concrete counting problem. Let p_1, \dots, p_k be the first k prime numbers and let $P_k := \prod_{i=1}^k p_i$ be their product. We choose an arbitrary residue class r_i for each prime p_i and consider k segments of arithmetic progressions restricted to the interval $[1, P_k]$, i.e., for p_i and r_i we consider

$$s_i = \{n : n = h \cdot p_i + r_i, h \in \mathbb{N}_0\} \cap [1, P_k].$$

For a given choice $R = (r_1, \dots, r_k)$ of residue classes modulo p_1, \dots, p_k , such that $0 \leq r_i \leq p_i - 1$ for all i , let

$$\gamma(n) = \gamma_R(n) = \#\{p_i : n \equiv r_i \pmod{p_i}\}$$

for $1 \leq n \leq P_k$ be the number of residue classes in which n is contained. In the following, we drop the subscript whenever the context is clear. We introduce some terminology:

- An integer n is *covered by a prime* p_i if it is contained in the residue class r_i .
- An integer n is *free* if it is not contained in any residue class, i.e., $\gamma(n) = 0$.
- An integer n is *available* if it is contained in at most one residue class, i.e., $\gamma(n) \leq 1$.
- An integer n is *occupied* if it is contained in at least two residue class, i.e., $\gamma(n) \geq 2$.

By definition, a free integer is also available. Moreover, we define a second sequence $(f_k)_{k \geq 1}$ by

$$f_k = (-1)^k \det \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 2 & 1 & \dots & 1 & 1 \\ 1 & 3 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & p_k & 1 \end{bmatrix}.$$

The first five elements of this sequence, for $k = 1, \dots, 5$, are

$$1, 2, 8, 48, 480 \dots$$

We have the following main result.

Theorem 1. *Let p_1, \dots, p_k be the first k prime numbers and let $P_k := \prod_{i=1}^k p_i$ be their product. For a given choice of residue classes, i.e., one residue class r_i for each prime p_i , the number of available elements in $[1, P_k]$ is a_k and the number of free elements is f_k . In other words, for every choice of residue classes, there are a_k integers n with $1 \leq n \leq P_k$ such that $\gamma(n) \leq 1$ and f_k integers with $\gamma(n) = 0$.*

Moreover, our method can be generalized to counting free and available integers in arbitrary sets of k arithmetic progressions generated by k different primes as shown in Section 5.

Remark 1. The problem of determining the number of free integers in the interval $[1, P_k]$ with residue classes $r_i = 0$ for all i is already mentioned by Dickson [1, p 439] in the first volume of his History of the Theory of Numbers. He refers to results by A. de Polignac, J. Deschamps, and H.J.S Smith for which Martin [2] provides proofs in his manuscript. Determining the number of available integers and the connection of the two counting problems to the determinants a_k and f_k are new to the best of our knowledge.

Remark 2. Lemma 1 below gives a new characterization of sequence A005867 from the On-Line Encyclopedia of Integer Sequences [3] in terms of determinants.

Remark 3. As a side product, the proof of Theorem 1 provides a quick way of calculating determinants of the form a_k and f_k .

The paper is organized as follows. In Section 2 we derive structural results about a_k and f_k . In Section 3 we analyze the counting problem and obtain an equation that relates the available numbers after considering k progressions to the available numbers after considering $k + 1$ progressions. Theorem 1 is proven in Section 4 and generalized in Section 5.

2. Preliminaries

We start with a structural lemma for f_k .

Lemma 1. *For $k \geq 2$ we have that*

$$f_k = (p_k - 1)f_{k-1}.$$

Proof. First assume that k is even. In the following we use the Laplace expansion of the initial matrix along its last row; see e.g. [4, Chapter 1]. Note that the first $k - 1$ terms of this expansion all have determinant 0. Since k is even, the sign of the element in row $k + 1$ and column k in the expansion is $(-1)^k(-1)^{k-1} = -1$ and the sign of the element in row $k + 1$ and column $k + 1$ is $(-1)^k(-1)^k = 1$. Hence,

we have that

$$\begin{aligned}
 f_k &= (-1)^k \det \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 2 & 1 & \dots & 1 & 1 \\ 1 & 3 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & p_k & 1 \end{bmatrix} \\
 &= (-1)^k \left(-p_k \det \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 2 & 1 & \dots & 1 & 1 \\ 1 & 3 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 \\ 1 & 1 & \dots & p_{k-1} & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 2 & 1 & \dots & 1 & 1 \\ 1 & 3 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & p_{k-1} & 1 \end{bmatrix} \right) \\
 &= (p_k - 1)f_{k-1}.
 \end{aligned}$$

The same argument works for odd k . □

We have a second structural lemma for a_k .

Lemma 2. *For $k \geq 2$ we have that*

$$a_k = f_{k-1} + (p_k - 1)a_{k-1}.$$

Proof. We use the rule for determinants

$$\det(v_1, v_2, \dots, v_n + w) = \det(v_1, v_2, \dots, v_n) + \det(v_1, v_2, \dots, w),$$

in which w, v_1, \dots, v_n are row (or column vectors). We have that

$$\begin{aligned}
 a_k &= \det \begin{bmatrix} 2 & 1 & \dots & 1 \\ 1 & 3 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & p_k \end{bmatrix} \\
 &= \det \begin{bmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 3 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & p_{k-1} & 1 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix} + \det \begin{bmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 3 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & p_{k-1} & 1 \\ 0 & 0 & \dots & 0 & (p_k - 1) \end{bmatrix}.
 \end{aligned}$$

Now observe that

$$\det \begin{bmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 3 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & p_{k-1} & 1 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix} = (-1)^{k-1} \det \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 2 & 1 & \dots & 1 & 1 \\ 1 & 3 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & p_{k-1} & 1 \end{bmatrix} = f_{k-1}$$

by the rules for determinants and

$$\det \begin{bmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 3 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & p_{k-1} & 1 \\ 0 & 0 & \dots & 0 & (p_k - 1) \end{bmatrix} = (p_k - 1)a_{k-1}$$

by the Laplace expansion along the last row. Note that the sign of diagonal elements is always positive in the Laplace expansion. Consequently, we have that

$$a_k = f_{k-1} + (p_k - 1)a_{k-1}. \quad \square$$

3. The Counting Problem

We aim to understand the systematic intersections in the interval $[1, P_k]$ of k arithmetic progressions generated by the first k prime numbers p_1, \dots, p_k and arbitrary choice of residue classes r_1, \dots, r_k with $P_k = \prod_{i=1}^k p_i$. In particular, we want to understand whether an integer $1 \leq n \leq P_k$ is not covered by any residue class, is contained in at most one or in at least two residue classes.

Assume we have generated all k arithmetic progressions and intersected them with the interval $[1, P_k]$ obtaining the sets $s_i, 1 \leq i \leq k$. For each integer $1 \leq n \leq P_k$ we can now ask in how many sets s_i it is contained, i.e., compute $\gamma(n)$. We denote the number of all available integers by $av(k)$, the number of free integers by $free(k)$, and the number of integers that are contained in two or more sets by $occ(k)$, i.e.,

$$\begin{aligned} av(k) &:= \#\{n \in [1, P_k] : \gamma(n) \leq 1\}, \\ free(k) &:= \#\{n \in [1, P_k] : \gamma(n) = 0\}, \\ occ(k) &:= \#\{n \in [1, P_k] : \gamma(n) > 1\}. \end{aligned}$$

Note that we always have $P_k = av(k) + occ(k)$. To illustrate the definitions, we consider the cases $k = 1$ and $k = 2$ with $p_1 = 2, p_2 = 3$ as well as $P_1 = 2$ and $P_2 = 6$. It is easy to see that $occ(1) = 0, free(1) = 1$, and $av(1) = 2$, because the arithmetic progression for $p_1 = 2$ is the first progression we consider and it contains one element in $[1, 2]$.

Independent of the choice of residue classes, the arithmetic progression for 3 will contain two elements in $[1, 6]$ and the progression for 2 will contain three elements. By the Chinese Remainder Theorem (CRT) the two arithmetic progressions intersect in exactly one point in the interval $[1, 6]$, again independent of the choice of r_1 and r_2 . Hence, three of the six integers in $[1, 6]$ are covered by 2 and two integers are covered by 3. Since there is exactly one intersection, five of the six integers are

available, i.e., $occ(2) = 1$, $av(2) = 5$. Furthermore, two of the six elements are not covered independent of the choice of residue classes, i.e., $free(2) = 2$ and we have that

$$\begin{aligned} free(2) &= 2 = (3 - 1) \cdot 1 = (p_2 - 1) \cdot free(1), \\ occ(2) &= 1 = 2 + (3 - 1) \cdot 0 - 1 = P_1 + (p_2 - 1) \cdot occ(1) - free(1). \end{aligned}$$

We have the following general equations.

Lemma 3. *For $k \geq 2$ we have that*

$$occ(k) = P_{k-1} + (p_k - 1) \cdot occ(k - 1) - free(k - 1)$$

and

$$free(k) = (p_k - 1) \cdot free(k - 1).$$

Proof. We prove the assertion by induction on k . It is true for $k = 2$ by the above example. Now assume it is true for $k - 1$. This means after considering the first $k - 1$ arithmetic progressions restricted to the interval $[1, P_{k-1}]$, we have $free(k - 1)$ free integers and $occ(k - 1)$ integers contained in two or more sets. Then, there are $av(k - 1) = P_{k-1} - occ(k - 1)$ many available integers.

Extending the $k - 1$ progressions from the interval $[1, P_{k-1}]$ to $[1, P_k]$ basically generates p_k copies of the pattern observed in $[1, P_{k-1}]$. Hence, we find $p_k \cdot free(k - 1)$ free integers, $p_k \cdot av(k - 1)$ available integers, and $p_k \cdot occ(k - 1)$ occupied integers in $[1, P_k]$ before considering the k -th progression.

Now take an arbitrary number $x \in [1, P_{k-1}]$ and consider its p_k translates

$$x, x + P_{k-1}, \dots, x + (p_k - 1)P_{k-1}$$

in $[1, P_k]$. The set of translates forms a permutation in \mathbb{F}_{p_k} . In other words, independent of the residue class we choose for p_k , exactly one of the p_k elements generated by x will be contained in the k -th progression.

The number x can be either free, available, or occupied, and hence, for every free, available, or occupied number from $[1, P_{k-1}]$ we get exactly one free, available, or occupied number in the k -th progression. In particular, this means that after adding the k -th progression, we find the already existing $p_k \cdot occ(k - 1)$ occupied integers in $[1, P_k]$ plus the newly occupied integers, generated by those integers that were available but not free in $[1, P_{k-1}]$. Therefore, we have that

$$occ(k) = p_k \cdot occ(k - 1) + av(k - 1) - free(k - 1).$$

In addition, before adding the k -th progression, we had $p_k \cdot free(k - 1)$ free integers in $[1, P_k]$. However, every free integer in $[1, P_{k-1}]$ generates one integer covered by the k -th progression. Hence, after adding the k -th progression we have that

$$free(k) = (p_k - 1) \cdot free(k - 1). \quad \square$$

We can apply Lemma 3 to get a similar formula for $av(k)$.

Lemma 4. For $k \geq 2$ we have that

$$av(k) = (p_k - 1) \cdot av(k - 1) + free(k - 1).$$

Proof. Using the relations $av(k) = P_k - occ(k)$ and $P_k = p_k P_{k-1}$, we can rewrite the equation as follows:

$$\begin{aligned} av(k) &= P_k - occ(k) \\ &= P_k - (P_{k-1} + (p_k - 1) \cdot occ(k - 1) - free(k - 1)) \\ &= P_k - (P_{k-1} + (p_k - 1) \cdot (P_{k-1} - av(k - 1)) - free(k - 1)) \\ &= (p_k - 1) \cdot av(k - 1) + free(k - 1). \quad \square \end{aligned}$$

4. Proof of Theorem 1

Proof of Theorem 1. We have already seen that $a_1 = 2 = av(1)$ and $f_1 = 1 = free(1)$. Furthermore, for $k = 2$ we have that

$$a_2 = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 2 \cdot 3 - 1 = 5$$

is the number of available integers, which we have determined in the example in Section 3 and

$$f_2 = (-1)^2 \det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} = 1 + 1 + 2 \cdot 3 - 2 - 3 - 1 = 2$$

gives exactly the number of free elements, i.e., those integers in $[1, 6]$ that are not covered by any of the two primes. To make things more interesting we consider one more example. We have that

$$a_3 = \det \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = 30 + 1 + 1 - 2 - 3 - 5 = 22$$

and we see that

$$a_3 = 22 = 2 + (5 - 1) \cdot 5 = f_2 + (p_3 - 1) \cdot a_2.$$

To turn to the general step, assume that a_k is the number of available elements in $[1, P_k]$ and f_k is the number of free elements. By Lemma 4 we know that

$$av(k + 1) = (p_{k+1} - 1) \cdot av(k) + free(k).$$

Furthermore, by Lemma 2 we know that

$$a_{k+1} = (p_{k+1} - 1)a_k + f_k$$

and, therefore, we conclude that $av(k + 1) = a_{k+1}$. Similarly, it was shown in Lemma 3 that

$$free(k + 1) = (p_{k+1} - 1) \cdot free(k)$$

and by Lemma 1 we have

$$f_{k+1} = (p_{k+1} - 1)f_k.$$

Hence, we conclude that $free(k + 1) = f_{k+1}$ and the theorem is proven. \square

5. Generalization

Note that our argument is purely structural and we did not use the explicit values of the first k primes. We only used explicit values for the base case of the theorem. However, this base case can easily be generalized as the next lemma shows. Moreover, we used the prime property when we considered permutations of \mathbb{F}_{p_k} . But again, no explicit value of p_k was used. Hence, it is possible to extend our method to count free and available integers in any set of k arithmetic progressions generated by k different primes and arbitrary residue classes. We define

$$A_k(p_{j_1}, \dots, p_{j_k}) = \det \begin{bmatrix} p_{j_1} & 1 & \dots & 1 \\ 1 & p_{j_2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & p_{j_k} \end{bmatrix}$$

and

$$F_k(p_{j_1}, \dots, p_{j_k}) = (-1)^k \det \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ p_{j_1} & 1 & \dots & 1 & 1 \\ 1 & p_{j_2} & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & p_{j_k} & 1 \end{bmatrix}.$$

Then we have the following two lemmas.

Lemma 5. *Let p_i, p_j be two distinct primes and r_i, r_j an arbitrary choice of residue classes for these primes. The number of available integers in $[1, p_i p_j]$ after the two progressions have been restricted to the interval is $A_2(p_i, p_j)$ and the number of free integers is $F_2(p_i, p_j)$.*

Proof. We have that $A_2(p_i, p_j) = p_i p_j - 1$ and the assertion follows by the same argument as in the example for $p_i = 2$ and $p_j = 3$. Similarly, $F_2(p_i, p_j) = p_i \cdot p_j - p_i - p_j + 1$ from which the assertion follows by the principle of inclusion-exclusion. \square

Lemma 6. *Let p_i, p_j, p_k be three distinct primes and r_i, r_j, r_k an arbitrary choice of residue classes for these primes. The number of available integers in $[1, p_i p_j p_k]$ after the 3 progressions have been restricted to the interval is $A_3(p_i, p_j, p_k)$.*

Proof. We have that

$$A_3(p_i, p_j, p_k) = \det \begin{bmatrix} p_i & 1 & 1 \\ 1 & p_j & 1 \\ 1 & 1 & p_k \end{bmatrix} = p_i \cdot p_j \cdot p_k + 1 + 1 - p_i - p_j - p_k.$$

We interpret this as follows for $P = p_i p_j p_k$. The interval $[1, P]$ can be partitioned in the following ways:

$$\begin{aligned} [1, P] &= [1, p_i p_j] \cup [p_i p_j + 1, 2p_i p_j] \cup \dots \cup [(p_k - 1)p_i p_j, P] \\ &= [1, p_i p_k] \cup [p_i p_k + 1, 2p_i p_k] \cup \dots \cup [(p_j - 1)p_i p_k, P] \\ &= [1, p_k p_j] \cup [p_k p_j + 1, 2p_k p_j] \cup \dots \cup [(p_i - 1)p_k p_j, P]. \end{aligned}$$

The first partition corresponds to the pair of primes (p_i, p_j) and by the CRT there is exactly one integer covered by both primes in each interval of the partition. Since there are p_k such intervals, there are p_k integers with $\gamma(n) > 1$. The next partition corresponds to the pair (p_i, p_k) contributing p_j integers with $\gamma(n) > 1$ and the third to the pair (p_k, p_j) contributing p_i such integers. Finally, by the CRT we know that there is exactly one point with $\gamma(n) = 3$ in $[1, P]$, which is counted in each of the above cases. Hence, by the principle of inclusion exclusion we get that there are

$$p_i p_j p_k - p_i - p_j - p_k + 1 + 1$$

available integers in the interval $[1, p_i p_j p_k]$. □

Hence, we have that

$$\begin{aligned} A_3(p_i, p_j, p_k) &= p_i p_j p_k + 1 + 1 - p_i - p_j - p_k \\ &= p_i p_j p_k + p_i p_j - p_i p_j + 1 + 1 - p_i - p_j - p_k \\ &= p_i p_j - p_i - p_j + 1 + (p_k - 1)(p_i p_j - 1) \\ &= F_2(p_i, p_j) + (p_k - 1)A_2(p_i, p_j). \end{aligned}$$

Furthermore, Lemmas 1, 2, 3, and 4 can be generalized to imply the following theorem.

Theorem 2. *Let p_{j_1}, \dots, p_{j_k} be k distinct prime numbers and let $P_k := \prod_{i=1}^k p_{j_i}$ be their product. For $k \geq 2$ and a given choice of residue classes, i.e., one residue class r_i for each prime p_i , the number of available elements in $[1, P_k]$ is $A_k(p_{j_1}, \dots, p_{j_k})$ and the number of free elements is $F_k(p_{j_1}, \dots, p_{j_k})$. In other words, for every choice of residue classes, there are $A_k(p_{j_1}, \dots, p_{j_k})$ integers n with $1 \leq n \leq P_k$ such that $\gamma(n) \leq 1$ and $F_k(p_{j_1}, \dots, p_{j_k})$ integers with $\gamma(n) = 0$.*

Acknowledgement. This work was supported by FCT - Fundação para a Ciência e a Tecnologia, I.P., through national funds, under the project UID/04561/2025.

References

- [1] L. E. Dickson, *History of the Theory of Numbers. Vol. I: Divisibility and Primality*, Dover Publications, Mineola, 2005.
- [2] D. R. Martin, Proofs regarding primorial patterns, preprint, <https://oeis.org/A005867/a005867.pdf>.
- [3] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, <https://oeis.org>.
- [4] G. Shilov, *Linear Algebra*, Prentice-Hall, Englewood Cliffs, 1971.