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A QUESTION OF ERDŐS AND GRAHAM ON COVERING SYSTEMS

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Abstract

Erdős and Graham asked if there exists an n such that the divisors of n greater than 1 are the moduli of a distinct covering system with the following property: if there exists an integer which satisfies two congruences in the system, $a \pmod{d}$ and $a' \pmod{d'}$, then $\gcd(d, d') = 1$. We show that such an n does not exist. This appears as part of Problem #204 on the website www.erdosproblems.com, compiled and maintained by Thomas Bloom.

1. Introduction

A *covering system* is a set of congruences, $\{a_1 \pmod{d_1}, \ldots, a_t \pmod{d_t}\}$, such that every integer is equivalent to some $a_i \pmod{d_i}$ for some $1 \leq i \leq t$. A covering system is *distinct* if $1 < d_1 < d_2 < \ldots < d_t$. Two congruences *overlap* if there is an integer equivalent to both congruences.

There have been many questions asked about distinct covering systems, the most well-known being whether d_1 can be arbitrarily large (asked by Erdős [4]), whether $d_i | d_j$ must occur for some $i \neq j$ (asked by Schinzel [10]), and whether it is possible for all d_i to be odd (asked by Erdős and Selfridge [6, Section F13]). Hough answered the first question in [7], proving that d_1 must be at most 10^{16}. This bound was improved to 616,000 in [2] and the second question was answered affirmatively in [2] as well. The third question is still open, however there has been much progress towards an answer. It is known that for any odd distinct covering system, there must be some d_i divisible by 3 [8]. In fact it has been shown that there must be some d_i divisible by 9, or some d_i divisible by 3 and some d_j divisible by 5 [2]. It is also known that not all d_i can be square-free [1] in such a covering system.

We note that the result in [8], namely that any distinct covering system must have a modulus divisible by 2 or 3, would shorten our proof. However, as the result

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does not require such a powerful tool and in the interests of having a self-contained proof, we avoid using it.

A set of congruences (or a congruence set), $\{a_1 \pmod{d_1}, \ldots, a_t \pmod{d_t}\}$, is *coprime disjoint* (CD) if whenever two congruences in the set, $a_i \pmod{d_i}$ and $a_j \pmod{d_j}$, overlap for $i \neq j$, we have $\gcd(d_i, d_j) = 1$. All congruence sets considered will be distinct. Furthermore, we say that n is *non-intersecting* if there exist integers $\{a_d : d|n, d > 1\}$ such that $\{a_d \pmod{d} : d|n, d > 1\}$ is a CD congruence set. We call such a set a *CD congruence set of* n. Finally, n is *CD covering* if there exist integers $\{a_d : d|n, d > 1\}$ such that $\{a_d \pmod{d} : d|n, d > 1\}$ is a CD distinct covering system. It is clear that any CD covering n is non-intersecting. For convenience, when referring to the divisors of n we will not include 1.

Erdős and Graham [5] posited that a CD covering n probably does not exist and this is posed in [3, Problem #204]. We prove that this is correct, that is, there is no CD covering n. We also provide a necessary condition for n to be non-intersecting and prove certain families of n are non-intersecting.

2. Preliminaries

Let $A = \{a_1 \pmod{d_1}, \ldots, a_t \pmod{d_t}\}$ be a congruence set. An integer, x, is *covered* by $a_i \pmod{d_i}$ if $x \equiv a_i \pmod{d_i}$. Similarly, x is *covered* by A if there exists $1 \leq i \leq t$ such that $x \equiv a_i \pmod{d_i}$. A residue class, $a \pmod{l}$, is covered by A if every integer $x \equiv a \pmod{l}$ is covered by A. Note that if $\operatorname{lcm}(d_1, \ldots, d_t)|l$ and some integer $x \equiv a \pmod{l}$ is covered by A, then the residue class $a \pmod{l}$ is covered by A. This follows as

$$a + lr = x = a_i + d_i r',$$

for some $1 \leq i \leq t$ and $r, r' \in \mathbb{Z}$. As $d_i|l$, we have

$$a + lk = a_i + d_i \left(r' + \frac{l}{d_i}(k - r) \right),$$

for any $k \in \mathbb{Z}$.

Let $A(n)$ be the number of integers that are less than or equal to n and covered by A. Then the *density* of the integers that are covered by A is

$$\delta(A) = \lim_{n \to \infty} \frac{A(n)}{n}.$$

Let $D = \operatorname{lcm}(d_1, \ldots, d_t)$ and m be the number of residue classes mod D that are covered by A. Then as $A(n) = m\frac{n}{D} + O(D)$, we see that

$$\delta(A) = \frac{m}{D}.$$

So the fraction of residue classes mod D that are covered by A is equal to $\delta(A)$. Clearly $\delta(A) = 1$ if and only if A is a covering set.

We adapt the proof of [11, Lemma 2.3] in order to obtain an expression for the density of a CD congruence set.

Lemma 2.1. *Let* $A = \{a_1 \pmod{d_1}, \ldots, a_t \pmod{d_t}\}$ *be a CD congruence set and let* $I_s = \{\{d_{i_1}, \ldots, d_{i_s}\} \mid \gcd(d_{i_j}, d_{i_k}) = 1 \text{ for all } 1 \leq j, k \leq s\}$*. Then*

$$\delta(A) = \sum_{i=1}^{t} \frac{1}{d_i} - \sum_{\{d_{i_1}, d_{i_2}\} \in I_2} \frac{1}{d_{i_1} \cdot d_{i_2}} + \sum_{\{d_{i_1}, d_{i_2}, d_{i_3}\} \in I_3} \frac{1}{d_{i_1} \cdot d_{i_2} \cdot d_{i_3}} - \ldots, \tag{1}$$

where the s*-th sum is over all sets in* I_s*.*

Note that we consider empty sums to be 0. In particular, if the moduli of A are the divisors of a non-intersecting n, then the number of sums in Equation (1) is equal to the number of distinct prime divisors of n.

Proof of Lemma 2.1. Let A_i be the set of residue classes mod D that are covered by $a_i \pmod{d_i}$. Let $N(i_1, \ldots, i_s) = |\bigcap_{j=1}^{s} A_{i_j}|$. By the inclusion-exclusion principle, the number of residue classes covered by A is

$$\sum_{i=1}^{t} N(i) - \sum_{1 \leq i < j \leq t} N(i,j) + \sum_{1 \leq i < j < k \leq t} N(i,j,k) + \ldots.$$

Let $e(i,j) = 1$ if $a_i \pmod{d_i}$ and $a_j \pmod{d_j}$ overlap and 0 otherwise. As A is a CD congruence set, $a_i \pmod{d_i}$ and $a_j \pmod{d_j}$ cannot overlap if $\gcd(d_i, d_j) > 1$. If $\gcd(d_i, d_j) = 1$, then by the Chinese Remainder Theorem, $a_i \pmod{d_i}$ and $a_j \pmod{d_j}$ must overlap. Therefore, $e(i,j) = 1$ if and only if $\gcd(d_i, d_j) = 1$. By the Chinese Remainder Theorem, if $\gcd(d_{i_j}, d_{i_k}) = 1$ for $1 \leq j < k \leq s$ then there is a unique residue class mod $d_{i_1} \cdot \ldots \cdot d_{i_s}$ that satisfies all congruences $a_{i_1} \pmod{d_{i_1}}, \ldots, a_{i_s} \pmod{d_{i_s}}$. It follows that there are exactly $\frac{D}{d_{i_1} \cdot \ldots \cdot d_{i_s}}$ residue classes mod D in $\bigcap_{j=1}^{s} A_{i_j}$.

Therefore, $\bigcap_{j=1}^{s} A_{i_j} \neq \emptyset$ if and only if $\prod_{1 \leq j < k \leq s} e(i_j, i_k) = 1$. Then the number of residue classes mod D that are covered by A is

$$\sum_{i=1}^{t} \frac{D}{d_i} - \sum_{\substack{1 \leq i_1 < i_2 \leq t, \\ \gcd(d_{i_1}, d_{i_2}) = 1}} \frac{D}{d_{i_1} \cdot d_{i_2}} + \sum_{\substack{1 \leq i_1 < i_2 < i_3 \leq t, \\ \prod_{1 \leq j < k \leq 3} \gcd(d_{i_j}, d_{i_k}) = 1}} \frac{D}{d_{i_1} \cdot d_{i_2} \cdot d_{i_3}} - \ldots,$$

and dividing by D gives Equation (1). □

Note that Lemma 2.1 shows that the density of the integers covered by a CD congruence set is entirely determined by the moduli and not the specific residues associated with them. For a non-intersecting n, all CD congruence sets of n have the same moduli, so all CD congruence sets of n cover the same density of the integers. This is given by Equation (1) where $\{d_1, \dots, d_t\}$ is the set of divisors of n.

It will often be convenient to have a simpler bound. The following result is often stated without proof; we include one for clarity.

Lemma 2.2. *For any covering set,* $A = \{a_1 \pmod{d_1}, \dots, a_t \pmod{d_t}\}$, *we have*

$$\sum_{i=1}^{t} \frac{1}{d_i} \geq 1.$$

Proof. The residue classes mod D covered by $a_i \pmod{d_i}$ are exactly $a_i + k \cdot d_i \pmod{D}$ for $0 \leq k \leq \frac{D}{d_i} - 1$. Therefore, there are $\frac{D}{d_i}$ residue classes mod D covered by $a_i \pmod{d_i}$. So there are at most $\sum_{i=1}^{t} \frac{D}{d_i}$ residue classes mod D covered by A. It follows that

$$\delta(A) \leq \sum_{i=1}^{t} \frac{1}{d_i}.$$

As A is a covering set, we have $\delta(A) = 1$ and so the result follows. □

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$. We will repeatedly make use of the following identity.

Proposition 2.3 (Folklore)**.** *For any positive integer* n,

$$\sum_{d|n, d>1} \frac{1}{d} = -1 + \sum_{d|n} \frac{1}{d} = -1 + \prod_{i=1}^{s} \sum_{j=0}^{\alpha_i} \frac{1}{p_i^j}. \tag{2}$$

Proof. The first equality of Equation (2) is clear to see, so it suffices to prove the second equality. Note that

$$\prod_{i=1}^{s} \sum_{j=0}^{\alpha_i} \frac{1}{p_i^j} = \prod_{i=1}^{s} \left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \dots + \frac{1}{p_i^{\alpha_i}}\right). \tag{3}$$

After expanding the bracket on the right-hand side of Equation (3), the terms produced are all the reciprocals of integers of the form $p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s}$ for $0 \leq \beta_i \leq \alpha_i$. Additionally, any two of these terms are distinct, as for any two of them there must exist a $1 \leq i \leq s$ such that different powers of p_i divide their denominators. Therefore, letting $S = \{p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s} \;:\; 0 \leq \beta_i \leq \alpha_i\}$, we see that

$$\prod_{i=1}^{s} \sum_{j=0}^{\alpha_i} \frac{1}{p_i^j} = \sum_{k \in S} \frac{1}{k}.$$

As the divisors of n, including 1, are precisely the integers of the form $p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s}$ for $0 \leq \beta_i \leq \alpha_i$, we have

$$\sum_{k \in S} \frac{1}{k} = \sum_{d|n} \frac{1}{d},$$

and this completes the proof. ☐

We also often use the formula for the sum of an infinite geometric series to bound the right-hand side of Equation (2).

3. CD Covering n

Lemma 3.1. *Let $n > 1$ be non-intersecting. If p is the smallest prime that divides n, then $\frac{n}{p}$ has less than p distinct prime divisors.*

Proof. Let n be non-intersecting and $\frac{n}{p}$ have at least p distinct prime divisors. So $q_1, \ldots q_p$ are distinct primes that divide $\frac{n}{p}$ and therefore $pq_1, \ldots pq_p$ all divide n.

We claim that $a_{pq_i} \equiv a_{pq_j} \pmod{p}$ for some $i \neq j$. If not, then all $a_{pq_i} \pmod{p}$ are distinct for $1 \leq i \leq p$. As there are only p distinct residues mod p, by the pigeonhole principle $a_p \equiv a_{pq_i} \pmod{p}$ for some $1 \leq i \leq p$. However, then $a_{pq_i} \equiv a_p \pmod{p}$ and $a_{pq_i} \equiv a_{pq_i} \pmod{pq_i}$, which contradicts n being non-intersecting as $\gcd(pq_i, p) = p > 1$.

So $a_{pq_i} \equiv a_{pq_j} \equiv b \pmod{p}$ for some $i \neq j$ and $b \in \{0, 1, \ldots, p-1\}$. By the Chinese Remainder Theorem, there exists c such that $c \equiv \frac{a_{pq_i} - b}{p} \pmod{q_i}$ and $c \equiv \frac{a_{pq_j} - b}{p} \pmod{q_j}$ as $\gcd(q_i, q_j) = 1$. So $pc + b \equiv a_{pq_i} \pmod{pq_i}$ and $pc + b \equiv a_{pq_j} \pmod{pq_j}$. This contradicts n being non-intersecting as $\gcd(pq_i, pq_j) = p > 1$. ☐

By Lemma 3.1, we have two possible cases for non-intersecting n, depending on whether $p \nmid \frac{n}{p}$ or $p | \frac{n}{p}$: (i) $n = pq_1^{\alpha_1} \ldots q_s^{\alpha_s}$ for $0 \leq s \leq p-1$ where $p < q_1 < \ldots < q_s$ are distinct primes and $\alpha_i \geq 1$ for $1 \leq i \leq s$; or (ii) $n = p^{\alpha_p} q_1^{\alpha_1} \ldots q_s^{\alpha_s}$ for $0 \leq s \leq p-2$ where $p < q_1 < \ldots < q_s$ are distinct primes, $\alpha_p \geq 2$ and $\alpha_i \geq 1$ for $1 \leq i \leq s$.

We can now prove that there is no CD covering n.

Theorem 3.2. *There does not exist any n such that a congruence set of the form $\{a_d \pmod{d} : d | n, d > 1\}$, where $a_d \pmod{d}$ and $a_{d'} \pmod{d'}$ overlap only if $\gcd(d, d') = 1$, is a covering set.*

Proof. Let p be the smallest prime that divides n. As noted above, there are two cases for a non-intersecting n, depending on whether $p^2 | n$.

`Case 1:` $n = pq_1^{\alpha_1} \ldots q_s^{\alpha_s}$. We split into two subcases based on the parity of n.

`Subcase (i):` n odd. Observe that $\sum_{d|n, d>1} \frac{1}{d} = -1 + \sum_{d|n} \frac{1}{d}$. Thus, using Equation (2),

$$\sum_{d|n, d>1} \frac{1}{d} = -1 + \left(1 + \frac{1}{p}\right) \prod_{i=1}^{s} \sum_{j=0}^{\alpha_i} \frac{1}{q_i^j} < -1 + \frac{p+1}{p} \prod_{i=1}^{s} \frac{q_i}{q_i - 1}.$$

Note that $\frac{x}{x-1}$ is a decreasing function for $x > 1$. As n is odd, we have $q_i > p \geq 3$ and so $q_i > p + 1$ for $1 \leq i \leq s$. We can then bound $\frac{q_i}{q_i - 1}$ above by $\frac{p+1+i}{p+i}$ and as $s \leq p - 1$,

$$\sum_{d|n, d>1} \frac{1}{d} < -1 + \frac{p+1}{p} \cdot \frac{p+2}{p+1} \cdot \frac{p+3}{p+2} \cdot \ldots \cdot \frac{p+(p-1)+1}{p+(p-1)} = -1 + 2 = 1.$$

Therefore, by Lemma 2.2, n is not a covering set.

`Subcase (ii):` n even. In this case, $p = 2$ and either $n = 2$ or $n = 2q^\alpha$ for a prime $q \geq 3$. If $n = 2$ then clearly

$$\sum_{d|n, d>1} \frac{1}{d} = \frac{1}{2} < 1,$$

and it follows from Lemma 2.2 that n is not a covering set.

If $n = 2q^\alpha$ then, by Lemma 2.1, the density of the integers covered by a CD congruence set, A, of n, is

$$\delta(A) = \left(\frac{1}{2} + \sum_{i=1}^{\alpha} \frac{1}{q^i} + \sum_{i=1}^{\alpha} \frac{1}{2q^i}\right) - \sum_{i=1}^{\alpha} \frac{1}{2q^i} = \frac{1}{2} + \sum_{i=1}^{\alpha} \frac{1}{q^i}.$$

This is true because the only pairs of divisors, (d_1, d_2), of n such that $\gcd(d_1, d_2) = 1$ are $(2, q^i)$ for $1 \leq i \leq \alpha$. Then,

$$\delta(A) < \frac{1}{2} + \sum_{i=1}^{\infty} \frac{1}{q^i} = \frac{1}{2} + \frac{1}{q-1} \leq \frac{1}{2} + \frac{1}{3-1} = 1.$$

As $\delta(A) < 1$, we see that n is not a covering set.

`Case 2:` $n = p^{\alpha_p} q_1^{\alpha_1} \ldots q_s^{\alpha_s}$. Using Equation (2) as before,

$$\sum_{d|n, d>1} \frac{1}{d} = -1 + \left(\sum_{j=0}^{\alpha_p} \frac{1}{p^j}\right) \prod_{i=1}^{s} \sum_{j=0}^{\alpha_i} \frac{1}{q^j} < -1 + \left(\sum_{j=0}^{\infty} \frac{1}{p^j}\right) \prod_{i=1}^{s} \sum_{j=0}^{\infty} \frac{1}{q^j}.$$

As $\frac{x}{x-1}$ is a decreasing function and $q_i > p$ for $1 \leq i \leq s$, we can bound $\frac{q_i}{q_i-1}$ above by $\frac{p+i}{p+i-1}$. Therefore, $s \leq p - 2$ implies

$$\sum_{d|n, d>1} \frac{1}{d} < -1 + \frac{p}{p-1} \prod_{i=1}^{s} \frac{q_i}{q_i - 1} \leq -1 + \frac{p}{p-1} \cdot \frac{p+1}{p} \cdot \ldots \cdot \frac{p+(p-2)}{p+(p-2)-1} = 1.$$

So, by Lemma 2.2, n is not a covering set. □

4. Non-Intersecting n

In [5], the authors posed the following problem. For a given non-intersecting n and a CD congruence set, S, of n, let the density of the integers not satisfying any of the congruences in S be $\delta'(S)$. What is $\delta'(n) := \min_S \delta'(S)$ where the minimum is taken over all CD congruence sets, S, of n? Using $\delta(S)$ as defined in Lemma 2.1, it is clear that $\delta(S) = 1 - \delta'(S)$. Equation (1) implies that $\delta(S)$ is solely determined by the moduli in S, therefore $\delta'(S)$ is solely determined by the moduli in S. As all CD congruence sets of n have the same moduli, we see that $\delta'(S) = \delta'(n)$. It follows that $\delta'(n) = \delta'(S) = 1 - \delta(S)$ where $\delta(S)$ is given by Equation (1).

It remains to confirm which n are non-intersecting. We answer this question for certain families of integers. Lemma 3.1 goes part of the way to answering this, giving a necessary condition for n to be non-intersecting. We show that if $n = p^k$ or $n = qp^k$, for primes p, q and $k \geq 1$, and n satisfies the conditions of Lemma 3.1 then n is non-intersecting.

Proposition 4.1. *Let $n = p^k$ for a prime p and $k \geq 1$. Then n is non-intersecting, and the density of any CD congruence set of n is $\frac{p^k-1}{p^k(p-1)}$.*

Proof. The set $\{p^{i-1} \pmod{p^i} : 1 \leq i \leq k\}$ is CD as no integer is equivalent to two congruences from the set. To see this, let $x \equiv p^{i-1} \pmod{p^i}$ and $x \equiv p^{j-1} \pmod{p^j}$ for $i \neq j$. The first congruence implies that p^{i-1} is the highest power of p that divides x and similarly, the second congruence implies that p^{j-1} is the highest power of p that divides x. This means that $i = j$, which is a contradiction.

By putting the divisors of n into Equation (1), we see that the result equals

$$\sum_{i=1}^{k} \frac{1}{p^i} = \frac{p^k - 1}{p^k(p-1)},$$

and it follows from Lemma 2.1 that this is the density of the integers that satisfy a congruence in the set. □

Proposition 4.2. *Let $n = qp^k$ for distinct primes p, q and $k \geq 1$. Then n is non-intersecting if and only if $k = 1$ or $p > 2$, and the density of any CD congruence set of n is $\frac{p^k-1}{p^k(p-1)} + \frac{1}{q}$.*

Proof. Let $n = qp^k$ for distinct primes p, q and $k \geq 1$. When $k \geq 2$ and $p = 2$, we see that $\frac{n}{2} = 2^{k-1}q$ has 2 distinct prime divisors. Therefore, by Lemma 3.1, it follows that n is not non-intersecting.

Now consider the case where $k = 1$ or $p > 2$. When $k = 1$, we have $n = pq$ and the statement is symmetric in p and q. Therefore, by relabeling the primes if necessary, we may assume $p > q \geq 2$. Thus, in either case, we assume that $p > 2$.

Let a be an integer such that $a \not\equiv 0, q^{-1} \pmod{p}$ (this is possible as $p \geq 3$). Then let the congruence set be

$$0 \pmod{q}, \{p^{i-1}+1 \pmod{p^i} : 1 \leq i \leq k\}, \{aqp^{j-1}+1 \pmod{qp^j} : 1 \leq j \leq k\}. \quad (4)$$

We claim that this is a CD congruence set. Clearly if $x \equiv aqp^{j-1}+1 \pmod{qp^j}$ and $x \equiv 0 \pmod{q}$ then $q|1$ which is a contradiction. As $\gcd(q, p^i) = 1$ for all $1 \leq i \leq k$, it remains to check whether any congruences in the last two sets of Equation (4) overlap with each other.

First, we assume that $p^{i-1}+1 \pmod{p^i}$ and $p^{j-1}+1 \pmod{p^j}$ overlap for $i \neq j$. Therefore,

$$p^{i-1} + rp^i = p^{j-1} + tp^j, \quad (5)$$

for $r, t \in \mathbb{Z}$. Then the highest power of p that divides the left-hand side of Equation (5) is p^{i-1} and the highest power of p that divides the right-hand side is p^{j-1}, which is a contradiction.

Next, we assume that $aqp^{i-1}+1 \pmod{qp^i}$ and $aqp^{j-1}+1 \pmod{qp^j}$ overlap for $i \neq j$. Therefore,

$$aqp^{i-1} + rqp^i = aqp^{j-1} + tqp^j, \quad (6)$$

for $r, t \in \mathbb{Z}$. Then, as $a \not\equiv 0 \pmod{p}$, the highest power of p that divides the left-hand side of Equation (6) is p^{i-1} and the highest power of p that divides the right-hand side is p^{j-1}. This is a contradiction.

Finally, we assume that $p^{i-1}+1 \pmod{p^i}$ and $aqp^{j-1}+1 \pmod{qp^j}$ overlap. Then

$$p^{i-1} + rp^i = aqp^{j-1} + tqp^j, \quad (7)$$

for $r, t \in \mathbb{Z}$. So $p^{i-1}(1+rp) = qp^{j-1}(a+tp)$. Then the highest power of p that divides the left-hand side of Equation (7) is p^{i-1} and the highest power of p that divides the right-hand side is p^{j-1}. It follows that $i = j$ and so $(1+rp) = q(a+tp)$. However, this implies that $1 \equiv qa \pmod{p}$, which contradicts $a \not\equiv q^{-1} \pmod{p}$.

So the divisors of qp^k are the moduli of a CD congruence set. By putting the divisors of n into Equation (1), we see that the result equals

$$\begin{aligned} -1 + \left(1 + \frac{1}{q}\right) \sum_{i=0}^{k} \frac{1}{p^i} - \sum_{i=1}^{k} \frac{1}{qp^i} &= -1 + \frac{q+1}{q} + \left(1 + \frac{1}{q}\right) \sum_{i=1}^{k} \frac{1}{p^i} - \sum_{i=1}^{k} \frac{1}{qp^i} \\ &= \frac{1}{q} + \frac{p^k - 1}{p^k(p-1)}, \end{aligned}$$

and it follows from Lemma 2.1 that this is the density of the integers that satisfy a congruence in the set. □

By Lemma 3.1, this covers all even n whose divisors are the moduli of a CD congruence set.

5. Conclusion

The next step would be to provide a necessary and sufficient characterization for n to be non-intersecting. By Theorem 3.2, if the divisors of n are used in Equation (1) and the answer is greater than or equal to 1, then n is not non-intersecting. However, this is not an if and only if condition as can be seen by considering the divisors of $n = 20$. Using $\{2, 4, 5, 10, 20\} = \{d_1, d_2, d_3, d_4, d_5\}$ in Equation (1) gives a result of $\frac{19}{20}$. At the same time, Lemma 3.1 demonstrates that as $\frac{20}{2} = 10$ has more than 1 prime factor, n cannot be non-intersecting.

We believe that Lemma 3.1 is a sufficient condition for n to be non-intersecting, as well as a necessary one.

Conjecture 5.1. Let p be the smallest prime that divides n. If $\frac{n}{p}$ has less than p distinct prime divisors, then n is non-intersecting.

A proof of this conjecture has been claimed in [9]. A further avenue to explore is to characterize when $\{d_1, \dots, d_t\}$ are the moduli of a CD congruence set. An argument analogous to Lemma 3.1 implies that if $D = \{d_1, \dots, d_t\}$ is the set of moduli of a CD congruence set, then for any integer m there is no subset $A \subseteq D$ such that $|A| = m + 1$ and $\gcd(d_i, d_j) = m$ for all distinct $d_i, d_j \in A$. Stijn Cambie has constructed examples that show this is not a sufficient condition for $\{d_1, \dots, d_t\}$ to be non-intersecting, e.g. $\{3, 6, 12, 18, 30, 42\}$.

Another direction is the following. For any n, what is the maximum density of the integers covered by a congruence set whose moduli are the divisors of n? We have answered this for non-intersecting n. A more general question is asked in [5]: what is the maximum density of the integers covered by a congruence set with a given set of moduli $\{d_1, \dots, d_t\}$?

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