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THE LOG-BALANCEDNESS OF THE SEQUENCE FOR THE ALTERNATING SUMS OF MOTZKIN NUMBERS

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Abstract

Let $\{M_n\}_{n\geq 0}$ be the Motzkin sequence. For $n \geq 0$, put $T_n = \sum_{k=0}^{n}(-1)^{n-k}M_k$. The main purpose of this paper is to study the log-behavior of the sequence $\{T_n\}_{n\geq 0}$. We show that $\{T_n\}_{n\geq 8}$ is log-balanced. In addition, we discuss the log-behavior of some sequences involving T_n. For instance, we prove that the sequence $\{M_n + T_n\}_{n\geq 6}$ is log-balanced.

1. Introduction

Let $\{M_n\}_{n\geq 0}$ denote the Motzkin sequence in this paper. The Motzkin sequence $\{M_n\}_{n\geq 0}$ is sequence A001006 in the OEIS [9] and satisfies the recurrence relation

$$(n+3)M_{n+1} = (2n+3)M_n + 3nM_{n-1}, \tag{1}$$

with the initial conditions $M_0 = 1$ and $M_1 = 1$. Some values of $\{M_n\}_{n\geq 0}$ are listed in Table 1. The Motzkin numbers play an important role in enumerative

n	0	1	2	3	4	5	6	7	8	9	10	11	12
M_n	1	1	2	4	9	21	51	127	323	835	2188	5798	15511

Table 1: Some values of $\{M_n\}_{n\geq 0}$

combinatorics. The value of M_n is the number of lattice paths from $(0,0)$ to (n,n), with steps $(0,2)$, $(2,0$, and $(1,1)$, never rising above the line $y = x$. Consider the sequence for the alternating sums of Motzkin numbers. For $n \geq 0$, let

$$T_n = \sum_{k=0}^{n}(-1)^{n-k}M_k. \tag{2}$$

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The sequence $\{T_n\}_{n\geq 0}$ is sequence A187306 in the OEIS [9] and some values of $\{T_n\}_{n\geq 0}$ are given in Table 2. The value of T_n is the number of simple permutations

n	0	1	2	3	4	5	6	7	8	9	10	11	12
T_n	1	0	2	2	7	14	37	90	233	602	1586	4212	11299

Table 2: Some values of $\{T_n\}_{n\geq 0}$

of each length that avoid the pattern 321 (i.e., are the union of two increasing sequences, and in one line notation contain no nontrivial block of values which form an interval). Let $\{r_n\}_{n\geq 0}$ be the sequence of Riordan numbers. The sequence $\{r_n\}_{n\geq 0}$ is sequence A005043 in the OEIS [9] and the value of r_n is the number of Dyck paths of semilength n with no peaks at odd level. There is an identity between $\{T_n\}_{n\geq 0}$ and $\{r_n\}_{n\geq 0}$, which is

$$T_n = r_{n+1} + (-1)^n. \tag{3}$$

For more properties of $\{T_n\}_{n\geq 0}$, see sequence A187306 in the OEIS [9]. In this paper, we investigate some properties of $\{T_n\}_{n\geq 0}$.

Now we recall some definitions. A sequence $\{z_n\}_{n\geq 0}$ of positive numbers is called *log-convex* (*log-concave*) if $z_n^2 \leq z_{n-1}z_{n+1}$ ($z_n^2 \geq z_{n-1}z_{n+1}$) for each $n \geq 1$. A log-convex sequence $\{z_n\}_{n\geq 0}$ is called *log-balanced* if $\{\frac{z_n}{n!}\}_{n\geq 0}$ is log-concave (Došlić [3] gave this definition). A sequence $\{z_n\}_{n\geq 0}$ of positive numbers is *tempered* (see Došlić [7]) if its quotient sequence $x_n = \frac{z_{n+1}}{z_n}$ satisfies

$$\frac{n}{n+1}x_{n-1} \leq x_n \leq \frac{n+1}{n}x_{n-1}.$$

It is known that a sequence $\{z_n\}_{n\geq 0}$ is log-convex (log-concave) if and only if its quotient sequence $\{\frac{z_{n+1}}{z_n}\}_{n\geq 0}$ is nondecreasing (nonincreasing) and a log-convex sequence $\{z_n\}_{n\geq 0}$ is log-balanced if and only if $\{\frac{z_{n+1}}{(n+1)z_n}\}_{n\geq 0}$ is nonincreasing. It is clear that a log-balanced sequence is tempered. Log-convexity (Log-concavity) is not only instrumental in obtaining the growth rate and asymptotic behavior of a sequence, but also a fertile source of combinatorial inequalities. In addition, log-convexity (log-concavity) plays an important role in many subjects (see for instance [1, 2, 4, 5, 6, 8, 10]). Log-balancedness can help us find more inequalities. Many combinatorial sequences, including the Motzkin numbers, the large Schröder numbers, the central Delannoy numbers, and the Bell numbers are log-balanced. One can find more log-balanced sequences in Došlić [3]. Log-balanced sequences play an important role in other subjects. For instance, Bell numbers provide important examples in white noise distribution theory (for more details, see Asai et al. [1]). This paper is concerned with the log-balancedness of $\{T_n\}_{n\geq 0}$. In addition, we also discuss the log-behavior of some sequences related to $\{T_n\}_{n\geq 0}$.

2. Main Results

Došlić [3] proved that the Motzkin sequence $\{M_n\}_{n\geq 0}$ is log-balanced. Now we discuss the log-balancedness of the sequence $\{T_n\}_{n\geq 0}$ defined by (2).

Theorem 1. *For the sequence $\{T_n\}_{n\geq 0}$ defined by (2), $\{T_n\}_{n\geq 8}$ is log-balanced and $\{T_n\}_{n\geq 5}$ is tempered.*

Proof. It follows from (2) that the sequence $\{T_n\}_{n\geq 0}$ satisfies the recurrence relation

$$T_{n+1} = \left(\frac{M_{n+1}}{M_n} - 1\right)T_n + \frac{M_{n+1}}{M_n}T_{n-1}. \tag{4}$$

For $n \geq 0$, let $x_n = \frac{M_{n+1}}{M_n}$ and $y_n = \frac{T_{n+1}}{T_n}$ ($n \geq 2$). By (1), we have that the sequence $\{x_n\}_{n\geq 0}$ satisfies the recurrence relation

$$x_n = \frac{2n+3}{n+3} + \frac{3n}{(n+3)x_{n-1}}. \tag{5}$$

It follows from (4) that

$$y_n = x_n - 1 + \frac{x_n}{y_{n-1}} \quad \text{for} \quad n \geq 3. \tag{6}$$

In order to prove that the sequence $\{T_n\}_{n\geq 8}$ is log-balanced, we need to show that $\{y_n\}_{n\geq 8}$ is increasing and $\{\frac{y_n}{n+1}\}_{n\geq 8}$ is decreasing. Now we prove by induction that

$$x_n \leq y_n \leq x_{n+1} \quad \text{for} \quad n \geq 9. \tag{7}$$

We observe that $x_k < y_k < x_{k+1}$ for $k = 9, 10$. Assume that $x_k \leq y_k \leq x_{k+1}$ for $k \geq 10$. By applying (6), we have

$$y_{k+1} - x_{k+1} = -1 + \frac{x_{k+1}}{y_k}$$

and

$$y_{k+1} - x_{k+2} = x_{k+1} - 1 + \frac{x_{k+1}}{y_k} - x_{k+2}.$$

For $k \geq 10$, noting that $x_k \leq y_k \leq x_{k+1}$, we obtain $y_{k+1} - x_{k+1} \geq 0$ and

$$y_{k+1} - x_{k+2} \leq x_{k+1} - 1 + \frac{x_{k+1}}{x_k} - x_{k+2}.$$

By means of (5), we get

$$\begin{aligned} x_{k+1} - 1 - x_{k+2} &= x_{k+1} - 1 - \frac{2k+7}{k+5} - \frac{3(k+2)}{(k+5)x_{k+1}} \\ &= \frac{2k+5}{k+4} + \frac{3(k+1)}{(k+4)x_k} - 1 - \frac{2k+7}{k+5} - \frac{3(k+2)(k+4)x_k}{(k+5)[(2k+5)x_k + 3(k+1)]} \\ &= -\frac{k^2+9k+23}{(k+4)(k+5)} + \frac{3(k+1)}{(k+4)x_k} - \frac{3(k+2)(k+4)x_k}{(k+5)[(2k+5)x_k + 3(k+1)]} \end{aligned}$$

and

$$\frac{x_{k+1}}{x_k} = \frac{2k+5}{(k+4)x_k} + \frac{3(k+1)}{(k+4)x_k^2}.$$

Došlić [6] proved that

$$b_n \le x_n \le b_{n+1} \quad \text{for} \quad n \ge 2, \tag{8}$$

where $b_n = \frac{6(n+1)}{2n+5}$. By using (8), we obtain

$$\begin{aligned} x_{k+1} - 1 - x_{k+2} &\le -\frac{k^2+9k+23}{(k+4)(k+5)} + \frac{3(k+1)}{(k+4)b_k} - \frac{3(k+2)(k+4)b_k}{(k+5)[(2k+5)b_k+3(k+1)]} \\ &= -2 - \frac{3}{(k+4)(k+5)} + \frac{2k+5}{2(k+4)} + \frac{3(k+3)}{(k+5)(2k+5)} \\ &= -\frac{2k+11}{2(k+4)} - \frac{3}{(k+4)(k+5)} + \frac{3(k+3)}{(k+5)(2k+5)} \end{aligned}$$

and

$$\begin{aligned} \frac{x_{k+1}}{x_k} &\le \frac{2k+5}{(k+4)b_k} + \frac{3(k+1)}{(k+4)b_k^2} \\ &= 1 + \frac{9}{4(k+1)(k+4)}. \end{aligned}$$

Then we have

$$\begin{aligned} y_{k+1} - x_{k+2} &\le -\frac{3}{2(k+4)} - \frac{3}{(k+4)(k+5)} + \frac{9}{4(k+1)(k+4)} + \frac{3(k+3)}{(k+5)(2k+5)} \\ &= -\frac{3(k+1)}{2(k+4)(k+5)(2k+5)} - \frac{3(k-11)}{4(k+1)(k+4)(k+5)} \\ &= -\frac{3(4k^2-13k-53)}{4(k+1)(k+4)(k+5)(2k+5)} < 0 \quad \text{for} \quad k \ge 10. \end{aligned}$$

This implies that the sequence $\{y_n\}_{n\ge 9}$ is increasing. On the other hand, we observe that $y_8 < y_9$. Thus, the sequence $\{T_n\}_{n\ge 8}$ is log-convex. Now we prove that the sequence $\{\frac{y_n}{n+1}\}_{n\ge 9}$ is decreasing. By using (7) and (8), we have

$$\begin{aligned} (n+2)y_n - (n+1)y_{n+1} &\ge (n+2)x_n - (n+1)x_{n+2} \\ &\ge (n+2)b_n - (n+1)b_{n+3} \\ &= \frac{6(n+1)(n+2)}{2n+5} - \frac{6(n+1)(n+4)}{2n+11} \\ &= \frac{12(n+1)^2}{(2n+5)(2n+11)} > 0 \quad \text{for} \quad n \ge 9. \end{aligned}$$

Thus, the sequence $\{\frac{y_n}{n+1}\}_{n\ge 9}$ is decreasing. On the other hand, it is clear that $y_8 < y_9$ and $\frac{y_8}{9} > \frac{y_9}{10}$. Then $\{y_n\}_{n\ge 8}$ is increasing and $\{\frac{y_n}{n+1}\}_{n\ge 8}$ is decreasing.

Because the sequence $\{T_n\}_{n\ge 8}$ is log-balanced, $\{T_n\}_{n\ge 8}$ is tempered. On the other hand, we observe that $\{T_k\}_{5\le k\le 9}$ is tempered. Then $\{T_n\}_{n\ge 5}$ is tempered. $\square$

Corollary 1. *For the sequence of Riordan numbers* $\{r_n\}_{n\geq 0}$, $\{r_{2n}\}_{n\geq 0}$ *is log-convex.*

Proof. We have that $r_{2n+2} = T_{2n+1} + 1$ from (3). It follows from Theorem 1 that the sequence $\{T_{2n+1}\}_{n\geq 4}$ is log-convex. Then $\{r_{2n+2}\}_{n\geq 4}$ is log-convex. On the other hand, we find that $\{r_{2k}\}_{0\leq k\leq 5}$ is log-convex. Thus, the sequence $\{r_{2n}\}_{n\geq 0}$ is log-convex.

□

Theorem 2. *For the sequence* $\{T_n\}_{n\geq 0}$ *defined by* (2), $\{T_nT_{n+1}\}_{n\geq 6}$ *is log-convex and* $\{T_nT_{n+1}\}_{n\geq 9}$ *is log-balanced.*

Proof. It follows from the log-convexity of the sequence $\{T_n\}_{n\geq 8}$ that $\{T_nT_{n+1}\}_{n\geq 8}$ is log-convex. We observe that $\{T_jT_{j+1}\}_{6\leq j\leq 9}$ is log-convex. Thus, the sequence $\{T_nT_{n+1}\}_{n\geq 6}$ is log-convex. For $n \geq 9$, let $x_n = \frac{M_{n+1}}{M_n}$ and $y_n = \frac{T_{n+1}}{T_n}$. It is evident that $\{y_ny_{n+1}\}_{n\geq 9}$ is the quotient sequence of $\{T_nT_{n+1}\}_{n\geq 9}$. We find that

$$\frac{y_ny_{n+1}}{n+1} - \frac{y_{n+1}y_{n+2}}{n+2} = \frac{(n+2)y_n - (n+1)y_{n+2}}{(n+1)(n+2)}y_{n+1}.$$

It follows from (7) and (8) that

$$\begin{aligned}(n+2)y_n - (n+1)y_{n+2} &\geq (n+2)x_n - (n+1)x_{n+3}\\ &\geq (n+2)b_n - (n+1)b_{n+4}\\ &= \frac{6(n+1)(2n+1)}{(2n+5)(2n+13)} > 0 \quad (n \geq 9).\end{aligned}$$

This means that the sequence $\{\frac{y_ny_{n+1}}{n+1}\}_{n\geq 9}$ is decreasing. Thus, $\{T_nT_{n+1}\}_{n\geq 9}$ is log-balanced. □

Theorem 3. *For the Motzkin sequence* $\{M_n\}_{n\geq 0}$ *and the sequence* $\{T_n\}_{n\geq 0}$ *defined by* (2), $\{M_n + T_n\}_{n\geq 6}$ *is log-balanced.*

Proof. It follows from the log-convexity of the sequences $\{M_n\}_{n\geq 0}$ and $\{T_n\}_{n\geq 9}$ that $\{M_n + T_n\}_{n\geq 8}$ is log-convex. Now we prove that the sequence $\{\frac{M_n+T_n}{n!}\}_{n\geq 8}$ is log-concave. For $n \geq 9$, let $x_n = \frac{M_{n+1}}{M_n}$ and $y_n = \frac{T_{n+1}}{T_n}$. For $n \geq 10$, put

$$\Psi_n = (n+1)(M_n + T_n)^2 - n(M_{n-1} + T_{n-1})(M_{n+1} + T_{n+1}).$$

In order to prove that the sequence $\{\frac{M_n+T_n}{n!}\}_{n\geq 9}$ is log-concave, we only need to show that $\Psi_n \geq 0$ for $n \geq 10$. It is obvious that

$$\begin{aligned}\Psi_n = {}& (n+1)M_n^2 - nM_{n-1}M_{n+1} + (n+1)T_n^2 - nT_{n-1}T_{n+1}\\ &+2(n+1)M_nT_n - nM_{n-1}T_{n+1} - nM_{n+1}T_{n-1}.\end{aligned}$$

It follows from the log-balancedness of the sequences $\{M_n\}_{n\geq 0}$ and $\{T_n\}_{n\geq 8}$ that

$$(n+1)M_n^2 - nM_{n-1}M_{n+1} \geq 0 \quad \text{and} \quad (n+1)T_n^2 - nT_{n-1}T_{n+1} \geq 0.$$

Then we have

$$\Psi_n \geq 2(n+1)M_nT_n - nM_{n-1}T_{n+1} - nM_{n+1}T_{n-1}.$$

We note that

$$\Psi_n \geq M_{n-1}T_{n-1}[2(n+1)x_{n-1}y_{n-1} - ny_{n-1}y_n - nx_{n-1}x_n].$$

By applying (7) and (8), we have

$$\begin{aligned}
\Psi_n &\geq M_{n-1}T_{n-1}[2(n+1)x_{n-1}^2 - nx_nx_{n+1} - nx_{n-1}x_n] \\
&\geq M_{n-1}T_{n-1}[2(n+1)b_{n-1}^2 - nb_{n+1}b_{n+2} - nb_nb_{n+1}] \\
&\geq 2M_{n-1}T_{n-1}[(n+1)b_{n-1}^2 - nb_{n+2}^2] \\
&= 2M_{n-1}T_{n-1}\left[\frac{36(n+1)n^2}{(2n+3)^2} - \frac{36n(n+3)^2}{(2n+9)^2}\right] \\
&= \frac{72nM_{n-1}T_{n-1}(4n^3 - 81n - 81)}{(2n+3)^2(2n+9)^2} > 0 \quad \text{for} \quad n \geq 10.
\end{aligned}$$

Thus, the sequence $\{M_n + T_n\}_{n\geq 9}$ is log-balanced. On the other hand, we observe that $\{M_k + T_k\}_{6\leq k\leq 10}$ is log-balanced. Then the sequence $\{M_n + T_n\}_{n\geq 6}$ is log-balanced. □

For the sequence $\{T_n\}_{n\geq 0}$ of the alternating sums of Motzkin numbers, we proved that $\{T_n\}_{n\geq 5}$ is tempered. In the final of this section, we consider the alternating sums of Fibonacci numbers F_n. For $n \geq 0$, let

$$W_n = \sum_{k=0}^{n} (-1)^{n-k} F_k. \tag{9}$$

The sequence $\{(-1)^n W_n\}_{n\geq 0}$ is sequence A119282 in the OEIS [9]. For $n \geq 1$, the closed form of $(-1)^n W_n$ is (see sequence A119282 in the OEIS [9])

$$(-1)^n W_n = (-1)^n F_{n-1} - 1.$$

For $n \geq 1$, it is evident that

$$W_n = F_{n-1} - (-1)^n. \tag{10}$$

Some values of $\{W_n\}_{n\geq 0}$ are given in Table 3.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
W_n	0	1	0	2	1	4	4	9	12	22	33	56	88	145	232

Table 3: Some values of $\{W_n\}_{n\geq 0}$

Now we prove that the sequence $\{W_n\}_{n\geq 11}$ is tempered.

Theorem 4. *For the sequence $\{W_n\}_{n\geq 0}$ defined by (9), $\{W_n\}_{n\geq 11}$ is tempered.*

Proof. In order to prove that the sequence $\{W_n\}_{n\geq 11}$ is tempered, we only need to show that $\{\frac{W_n}{n!}\}_{n\geq 11}$ is log-concave and $\{n!W_n\}_{n\geq 11}$ is log-convex. The sequence $\{\frac{W_n}{n!}\}_{n\geq 11}$ is log-concave if and only if $(n+1)W_n^2 - nW_{n-1}W_{n+1} \geq 0$ for $n \geq 12$, and the sequence $\{n!W_n\}_{n\geq 11}$ is log-convex if and only if $nW_n^2 - (n+1)W_{n-1}W_{n+1} \leq 0$ for $n \geq 12$. For convenience, set

$$\Sigma_n = (n+1)W_n^2 - nW_{n-1}W_{n+1} \quad \text{and} \quad \Omega_n = nW_n^2 - (n+1)W_{n-1}W_{n+1}.$$

It follows from (10) that

$$\begin{aligned}\Sigma_n &= (n+1)[F_{n-1} - (-1)^n]^2 - n[F_{n-2} - (-1)^{n-1}][F_n - (-1)^{n+1}] \\ &= n(F_{n-1}^2 - F_{n-2}F_n) + F_{n-1}^2 - 2(-1)^n F_{n-1} - (-1)^n n(2F_{n-1} + F_{n-2} + F_n) + 1\end{aligned}$$

and

$$\begin{aligned}\Omega_n &= n[F_{n-1} - (-1)^n]^2 - (n+1)[F_{n-2} - (-1)^{n-1}][F_n - (-1)^{n+1}] \\ &= n(F_{n-1}^2 - F_{n-2}F_n) - F_{n-2}F_n - (-1)^n(F_{n-2} + F_n) \\ &\quad -(-1)^n n(2F_{n-1} + F_{n-2} + F_n) - 1.\end{aligned}$$

Applying $F_{n-1}^2 - F_{n-2}F_n = (-1)^n$ and

$$F_{n+1} = F_n + F_{n-1}, \tag{11}$$

we obtain

$$\Sigma_n = (-1)^n n + F_{n-1}^2 - 2(-1)^n F_{n-1} - (-1)^n nF_{n+2} + 1$$

and

$$\Omega_n = (-1)^n n - F_{n-2}F_n - (-1)^n(F_{n-2} + F_n) - (-1)^n nF_{n+2} - 1.$$

It is obvious that

$$\Sigma_n \geq F_{n-1}^2 - 2F_{n-1} - nF_{n+2} - n + 1$$

and

$$\Omega_n \leq n - F_{n-2}F_n + 2F_n + nF_{n+2} - 1.$$

We observe that $\Sigma_{12} > 0$ and $\Omega_{12} < 0$. For $n \geq 13$, one can prove by induction that

$$F_{n-1} - 2 > 7n \tag{12}$$

and

$$F_{n-2} - 2 \geq 6n \tag{13}$$

By using (11)–(13), we have

$$\begin{aligned} \Sigma_n &> 7nF_{n-1} - nF_{n+2} - n + 1 = 6nF_{n-1} - 2nF_n - n + 1 \\ &= 2nF_{n-1} + 2nF_{n-3} - n + 1 > 3n + 1 > 0 \end{aligned}$$

and

$$\begin{aligned} \Omega_n &\leq n - 6nF_n + nF_{n+2} - 1 = n - 5nF_n + nF_{n+1} - 1 \\ &= n - 4nF_n + nF_{n-1} - 1 < n - 3nF_n - 1 < -2n - 1 < 0. \end{aligned}$$

Then the sequence $\{W_n\}_{n\geq 11}$ is tempered. □

3. Concluding Remarks

For the sequence $\{T_n\}_{n\geq 0}$ defined by (2), we mainly proved that $\{T_n\}_{n\geq 8}$ is log-balanced. In addition, we also discussed the log-behavior of some sequences involving T_n. For $\{T_n\}_{n\geq 0}$, we now give a conjecture.

Conjecture 1. For the sequence $\{T_n\}_{n\geq 0}$ defined by (2), $\{\sqrt[n]{T_n}\}_{n\geq 9}$ is log-concave.

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