A SQUARE–SPLITTING A-CONVOLUTION OF NARKIEWICZ TYPE

Champak Talukdar
Department of Mathematics, Gauhati University, Assam, India
Department of Mathematics, Behali Degree College, Assam, India
champak.nlb.2012@gmail.com

Helen K. Saikia
Department of Mathematics, Gauhati University, Assam, India
hsaikia@yahoo.com

Received: 2/4/26, Revised: 3/11/26, Accepted: 4/14/26, Published: 5/1/26

Abstract

Motivated by Narkiewicz's divisor systems and the induced A-convolutions, we introduce an explicit divisor system arising from the canonical decomposition $n = u(n)v(n)^2$ with $u(n)$ squarefree. The resulting A-convolution $\star$ splits the divisor structure into squarefree and square parts and yields a convolution algebra canonically isomorphic to a two-variable Dirichlet convolution. We develop the associated multiplicative theory: a closed formula for the $\star$-Möbius function, a sharp inversion principle, and Euler-product formulae for generalized divisor sums. We also define an Euler-type totient $\varphi_\star$ by $\sum_{d\in A(n)} \varphi_\star(d) = n$ and obtain an explicit prime-power description. Moreover, we establish an effective asymptotic formula for $\sum_{n\le x} \varphi_\star(n)$, whose leading constant is given by an absolutely convergent Euler product. We further derive a sharp asymptotic formula for $\sum_{n\le x} \tau_\star(n)$ and determine the natural density of the support of $\mu_\star$. Numerical examples illustrate the identities and the predicted constants.

1. Introduction

Let $\mathcal{A}$ denote the complex vector space of arithmetic functions $f : \mathbb{N} \to \mathbb{C}$. The classical Dirichlet convolution is

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d),$$

DOI: 10.5281/zenodo.19949682

with identity $\varepsilon(1) = 1$ and $\varepsilon(n) = 0$ for $n > 1$. Narkiewicz introduced a far-reaching generalization by restricting the divisor sum to a divisor system $A(n) \subseteq \{d : d \mid n\}$ and studying when the induced operation

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)g(n/d),$$

behaves regularly, namely in the sense of associativity, preservation of multiplicativity, and an inversion theory [5]. This initiated a broad literature on regular convolutions and related Euler-type functions [5, 4, 7, 1].

In this paper, we propose a computable A-divisor system built from the canonical decomposition $n = u(n)v(n)^2$ [4], where $u(n)$ is squarefree and $v(n)$ records the square part. This yields a regular convolution $\star$ which is canonically isomorphic to a two-variable Dirichlet convolution via the transform $Tf(u, v) = f(uv^2)$. The bivariate model gives an explicit inversion theory. In particular, it provides a closed formula for the $\star$-Möbius function $\mu_\star$, and leads to new Euler-type functions, including a totient $\varphi_\star$ characterized by $\sum_{d \in A(n)} \varphi_\star(d) = n$, with a computable mean-value constant and summatory asymptotics.

2. A Square–Splitting Divisor System

Definition 1. For each $n \in \mathbb{N}$, define $u(n), v(n) \in \mathbb{N}$ by the unique factorization

$$n = u(n)v(n)^2,$$

where $u(n)$ is squarefree. Equivalently, if $n = \prod_p p^{\nu_p(n)}$, then

$$u(n) = \prod_{p:\, \nu_p(n) \text{ odd}} p, \qquad v(n) = \prod_p p^{\lfloor \nu_p(n)/2 \rfloor}.$$

Remark 1. Write every $n \geq 1$ uniquely as

$$n = u(n)v(n)^2,$$

where $u(n)$ is squarefree. Then the divisor system

$$A(n) = \{u_1 v_1^2 :\ u_1 \mid u(n),\ v_1 \mid v(n)\}$$

may be viewed as the set of lattice points in the product poset

$$\mathcal{L}(n) := \mathrm{Div}(u(n)) \times \mathrm{Div}(v(n)),$$

ordered by $(u_1, v_1) \preceq (u_2, v_2)$ if and only if $u_1 \mid u_2$ and $v_1 \mid v_2$. Equivalently, the map

$$(u_1, v_1) \longmapsto u_1 v_1^2$$

is a bijection from $\mathcal{L}(n)$ onto $A(n)$. In this sense, the $\star$-convolution is the incidence convolution on the two-coordinate divisor lattice $\mathcal{L}(n)$: the factorization $n = a \star b$ corresponds to choosing a decomposition

$$u(n) = u(a)u(b), \qquad v(n) = v(a)v(b),$$

where $\gcd(u(a), u(b)) = 1$. Equivalently, it provides a partition of the prime support of $u(n)$ between $u(a)$ and $u(b)$, while the v-coordinate corresponds to splitting prime exponents in $v(n)$. In particular, $\gcd(v(a), v(b))$ need not be 1 in general. This viewpoint yields an immediate counting interpretation of $\tau_\star$. Indeed,

$$\tau_\star(n) = \#A(n) = \#\mathcal{L}(n) = d(u(n))\, d(v(n)),$$

so $\tau_\star(n)$ counts the number of pairs (u_1, v_1) with $u_1 \mid u(n)$ and $v_1 \mid v(n)$, that is, the number of divisor choices in the two independent coordinates [4]. Moreover, $\tau_\star(n)$ also counts the number of ordered $\star$-factorizations $n = a \star b$, equivalently, the number of ordered decompositions of the pair $(u(n), v(n))$ into two coordinatewise factors. Thus, the arithmetic of the $\star$-convolution is governed by a natural product-poset combinatorics coming from the squarefree-square splitting.

Theorem 1. *Define, for each n, the set*

$$A(n) := \{d = u_1 v_1^2 : \ u_1 \mid u(n),\ v_1 \mid v(n)\}.$$

Then the following statements hold:

1. *$A(n)$ is a divisor system: $1, n \in A(n)$ and every $d \in A(n)$ divides n,*
2. *A is multiplicative in the strong sense: if $\gcd(m, n) = 1$, then*

$$A(mn) = \{ab : \ a \in A(m),\ b \in A(n)\},$$

3. *The counting function $\tau_\star(n) := |A(n)|$ satisfies*

$$\tau_\star(n) = d(u(n))\, d(v(n)) = 2^{\omega(u(n))} d(v(n)),$$

where d is the classical divisor function and ω counts distinct prime factors.

Proof. If $d = u_1 v_1^2$ with $u_1 \mid u(n)$ and $v_1 \mid v(n)$, then $u_1 \mid u(n)$ and $v_1^2 \mid v(n)^2$, and hence, $d \mid u(n)v(n)^2 = n$. The choices $u_1 = 1$, $v_1 = 1$ and $u_1 = u(n)$, $v_1 = v(n)$ give $1, n \in A(n)$.

If $\gcd(m, n) = 1$, then $u(mn) = u(m)u(n)$ and $v(mn) = v(m)v(n)$, and divisors split uniquely across coprime components. Thus, $u_1 \mid u(mn)$ if and only if $u_1 = u_{1m}u_{1n}$ with $u_{1m} \mid u(m)$ and $u_{1n} \mid u(n)$, and similarly, $v_1 = v_{1m}v_{1n}$ with $v_{1m} \mid v(m)$ and $v_{1n} \mid v(n)$. Hence, $A(mn) = \{(u_{1m}v_{1m}^2)(u_{1n}v_{1n}^2)\}$, as claimed.

Since $u(n)$ is squarefree, $d(u(n)) = 2^{\omega(u(n))}$. Also, the map $(u_1, v_1) \mapsto u_1 v_1^2$ is injective because u_1 captures the parity of exponents and v_1 captures the square part. Therefore, $|A(n)| = d(u(n))d(v(n))$. □

Definition 2. For arithmetic functions f and g, define

$$(f \star g)(n) := \sum_{d \in A(n)} f(d)g(n/d),$$

where $A(n)$ is given in Theorem 1.

Theorem 2. *Let $\mathcal{A}$ be the space of arithmetic functions. Then $(\mathcal{A}, \star)$ is a commutative ring with identity ε. Moreover, the bivariate function*

$$\mathcal{T}f(u,v) := f(uv^2),$$

where u is squarefree and $v \in \mathbb{N}$, is a ring isomorphism between $(\mathcal{A}, \star)$ and the bivariate Dirichlet convolution ring

$$(F \odot G)(u,v) := \sum_{u_1 \mid u} \sum_{v_1 \mid v} F(u_1, v_1) G(u/u_1, v/v_1),$$

defined on the domain $\{(u,v) : u \text{ is squarefree}\}$.

Proof. Commutativity is immediate from the symmetric summand $f(d)g(n/d)$. The identity ε satisfies $(f \star \varepsilon)(n) = f(n)$ because only the term $d = n$ contributes.

For associativity, use the bivariate model. Fix n and write $n = uv^2$ with u squarefree. By Theorem 1, the elements of $A(n)$ are in bijection with pairs (u_1, v_1) with $u_1 \mid u$ and $v_1 \mid v$. Thus,

$$(f \star g)(uv^2) = \sum_{u_1 \mid u} \sum_{v_1 \mid v} f(u_1 v_1^2)\, g\left(\frac{u}{u_1}\left(\frac{v}{v_1}\right)^2\right) = (\mathcal{T}f \odot \mathcal{T}g)(u,v).$$

Hence, $\mathcal{T}(f \star g) = \mathcal{T}f \odot \mathcal{T}g$. Associativity and distributivity for $\star$ follow from the corresponding properties of $\odot$, which is the standard Dirichlet convolution on the product lattice of divisors in two coordinates. Finally, $\mathcal{T}$ is bijective since every n is uniquely of the form uv^2 with u squarefree. □

Corollary 1. *If f and g are multiplicative, then $f \star g$ is multiplicative.*

Proof. Under $\mathcal{T}$, multiplicativity corresponds to Euler factorization over primes in each coordinate, and the product form in the second statement of Theorem 1 yields the standard factorization argument over coprime components. □

3. Möbius Theory and Inversion

Definition 3. Let $\mathbf{1}$ denote the constant function $\mathbf{1}(n) \equiv 1$. The $\star$-Möbius function $\mu_\star$ is the $\star$-inverse of $\mathbf{1}$. That is,

$$\mu_\star \star \mathbf{1} = \varepsilon.$$

Theorem 3. *For all n,*

$$\mu_\star(n) = \mu(u(n))\mu(v(n)),$$

where μ is the classical Möbius function and $n = u(n)v(n)^2$. In particular, for prime powers p^k,

$$\mu_\star(p^k) = \begin{cases} -1, & k = 1 \text{ or } 2, \\ +1, & k = 3, \\ 0, & k \geq 4. \end{cases}$$

Moreover, the Dirichlet generating series admits the Euler product

$$\sum_{n\geq 1} \frac{\mu_\star(n)}{n^s} = \prod_p \left(1 - p^{-s} - p^{-2s} + p^{-3s}\right) = \frac{1}{\zeta(s)\zeta(2s)},$$

where $\Re(s) > 1$.

Proof. Work in the bivariate model of Theorem 2. The inverse of the constant function $\mathbf{1}(uv^2) = 1$ under $\odot$ is $(u, v) \mapsto \mu(u)\mu(v)$, since

$$\sum_{u_1|u} \sum_{v_1|v} \mu(u_1)\mu(v_1) = \varepsilon(u)\varepsilon(v),$$

by the usual one-variable identity $\sum_{d|n} \mu(d) = \varepsilon(n)$ [4] applied in each coordinate. Transporting back via $\mathcal{T}^{-1}$ yields $\mu_\star(n) = \mu(u(n))\mu(v(n))$.

For prime powers, if $k = 2b$ is even, then $u(p^k) = 1$ and $v(p^k) = p^b$. Hence, $\mu_\star(p^{2b}) = \mu(p^b)$, which equals -1 for $b = 1$ and 0 for $b \geq 2$. If $k = 2b + 1$ is odd, then $u(p^k) = p$ and $v(p^k) = p^b$, giving $\mu_\star(p^{2b+1}) = \mu(p)\mu(p^b) = (-1)\mu(p^b)$. Hence, -1 for $b = 0$, $+1$ for $b = 1$, and 0 for $b \geq 2$.

The Euler product follows from multiplicativity and the prime-power values:

$$\sum_{k\geq 0} \frac{\mu_\star(p^k)}{p^{ks}} = 1 - \frac{1}{p^s} - \frac{1}{p^{2s}} + \frac{1}{p^{3s}} = (1 - p^{-s})(1 - p^{-2s}).$$

Multiplying over primes yields

$$\prod_p (1 - p^{-s}) \prod_p (1 - p^{-2s}) = \frac{1}{\zeta(s)\zeta(2s)}.$$

□

Theorem 4. *For arithmetic functions f and F, the following are equivalent:*

$$F(n) = \sum_{d\in A(n)} f(d) \quad \textit{for all } n \quad \textit{if and only if} \quad f(n) = \sum_{d\in A(n)} \mu_\star(d)F(n/d) \quad \textit{for all } n.$$

Equivalently, $F = \mathbf{1} \star f$ if and only if $f = \mu_\star \star F$.

Proof. This is the standard convolution inversion. If $F = \mathbf{1} \star f$, then convolving both sides by $\mu_\star$ gives $\mu_\star \star F = (\mu_\star \star \mathbf{1}) \star f = \varepsilon \star f = f$. The converse follows in the same way. □

Theorem 5. *Let f be multiplicative with $f(1) \neq 0$. Then f is $\star$-invertible. That is, there exists $f^{(-1)_\star}$ such that $f \star f^{(-1)_\star} = \varepsilon$, and the $\star$-inverse is multiplicative. On prime powers, writing $g = f^{(-1)_\star}$, one has the recursion*

$$\sum_{j=0}^{\lfloor k/2 \rfloor} f(p^{2j})g(p^{k-2j}) + \mathbf{1}_{k \text{ odd}} \sum_{j=0}^{\lfloor k/2 \rfloor} f(p^{2j+1})g(p^{k-(2j+1)}) = \varepsilon(p^k),$$

where $\mathbf{1}_{k \text{ odd}}$ is 1 if k is odd and 0 otherwise. In particular,

$$\begin{aligned} g(1) &= 1/f(1), \\ g(p) &= -\frac{f(p)}{f(1)^2}, \\ g(p^2) &= -\frac{f(p^2)}{f(1)^2}, \\ g(p^3) &= \frac{2f(p)f(p^2) - f(p^3)f(1)}{f(1)^3}, \end{aligned}$$

and for $k \geq 4$ the values are determined uniquely by the recursion.

Proof. Transport to the bivariate ring. Under $\mathcal{T}$, invertibility in $\star$ is equivalent to invertibility in $\odot$. For multiplicative f with $f(1) \neq 0$, invertibility in the Dirichlet-type convolution follows prime by prime by a standard triangular recursion on the divisor lattice, and the inverse is multiplicative. The displayed recursion is exactly the prime-power specialization of $(f \star g)(p^k) = \varepsilon(p^k)$ using the description of $A(p^k)$: if $k = 2b$ is even then $A(p^{2b}) = \{p^0, p^2, \ldots, p^{2b}\}$, while if $k = 2b + 1$ is odd then $A(p^{2b+1})$ contains all $p^0, \ldots, p^{2b+1}$. The explicit formulas for small k follow by solving the first few triangular equations. □

4. Generalized Divisor Sums and an Euler-Type Totient

4.1. $\star$-Divisor Sums

For $\alpha \in \mathbb{C}$ define

$$\sigma_{\star,\alpha}(n) := \sum_{d \in A(n)} d^\alpha, \qquad \tau_\star(n) = \sigma_{\star,0}(n) = |A(n)|.$$

Theorem 6. *Let $n = u(n)v(n)^2$. Then for every $\alpha \in \mathbb{C}$,*

$$\sigma_{\star,\alpha}(n) = \sigma_\alpha(u(n))\, \sigma_{2\alpha}(v(n)),$$

where σ_β is the classical divisor sum $\sigma_\beta(m) = \sum_{d|m} d^\beta$. In particular, $\sigma_{\star,\alpha}$ is multiplicative and for prime powers we have

$$\sigma_{\star,\alpha}(p^{2b}) = 1 + p^{2\alpha} + \cdots + p^{2b\alpha}, \qquad \sigma_{\star,\alpha}(p^{2b+1}) = (1+p^\alpha)(1 + p^{2\alpha} + \cdots + p^{2b\alpha}).$$

Moreover, the Dirichlet series of $\sigma_{\star,\alpha}$ has Euler factor

$$\sum_{k\geq 0} \frac{\sigma_{\star,\alpha}(p^k)}{p^{ks}} = \frac{1 + p^{-s} + p^{-(s-\alpha)}}{(1-p^{-2s})(1-p^{-(2s-2\alpha)})},$$

and hence

$$\sum_{n\geq 1} \frac{\sigma_{\star,\alpha}(n)}{n^s} = \zeta(2s)\zeta(2s-2\alpha)\prod_p \left(1 + p^{-s} + p^{-(s-\alpha)}\right),$$

with absolute convergence in the right half-plane. In particular, it is absolutely convergent for $\Re(s) > \max\{1, 1 + \Re(\alpha)\}$.

Proof. By Theorem 1, each $d \in A(n)$ is uniquely of the form $d = u_1 v_1^2$ with $u_1 \mid u(n)$ and $v_1 \mid v(n)$. Therefore,

$$\sigma_{\star,\alpha}(n) = \sum_{u_1|u(n)} \sum_{v_1|v(n)} (u_1 v_1^2)^\alpha \tag{1}$$

$$= \left(\sum_{u_1|u(n)} u_1^\alpha\right)\left(\sum_{v_1|v(n)} v_1^{2\alpha}\right) \tag{2}$$

$$= \sigma_\alpha(u(n))\,\sigma_{2\alpha}(v(n)). \tag{3}$$

The prime-power formulas follow by substituting $u(p^{2b}) = 1$, $v(p^{2b}) = p^b$ and $u(p^{2b+1}) = p$, $v(p^{2b+1}) = p^b$. Summing the resulting geometric series gives the rational Euler factor, and the Euler product follows from multiplicativity. □

Definition 4. Define $\varphi_\star$ as the unique arithmetic function satisfying

$$\sum_{d\in A(n)} \varphi_\star(d) = n,$$

for all $n \in \mathbb{N}$. Equivalently, $\varphi_\star \star \mathbf{1} = \mathrm{id}$, where $\mathrm{id}(n) = n$.

Theorem 7. *The function $\varphi_\star$ is multiplicative. For a prime p and $b \geq 1$,*

$$\varphi_\star(p^{2b}) = (p^2-1)p^{2b-2}, \qquad \varphi_\star(p^{2b+1}) = (p-1)(p^2-1)p^{2b-2},$$

$\varphi_\star(p) = p - 1$ and $\varphi_\star(1) = 1$. Equivalently, for $n = u(n)v(n)^2$,

$$\varphi_\star(n) = \varphi(u(n)) \cdot \left(v(n)^2 \prod_{p|v(n)} \left(1 - \frac{1}{p^2}\right)\right),$$

where φ is the classical Euler totient.

Proof. Uniqueness follows since $\mathbf{1}$ is $\star$-invertible and $\varphi_\star = \mathrm{id} \star \mu_\star$. Multiplicativity follows from Corollary 1. For prime powers, use the defining relation. If $k = 2b$ is even, then $A(p^{2b}) = \{p^0, p^2, \ldots, p^{2b}\}$, so

$$\varphi_\star(1) + \varphi_\star(p^2) + \cdots + \varphi_\star(p^{2b}) = p^{2b}.$$

Subtracting the same identity for $b-1$ gives

$$\varphi_\star(p^{2b}) = p^{2b} - p^{2b-2} = (p^2-1)p^{2b-2}.$$

If $k = 2b+1$ is odd, then $A(p^{2b+1}) = \{p^0, p^1, \ldots, p^{2b+1}\}$, from which it follows that

$$\sum_{j=0}^{2b+1} \varphi_\star(p^j) = p^{2b+1}.$$

Splitting even and odd exponents, we obtain

$$\sum_{j=0}^{b} \varphi_\star(p^{2j}) + \sum_{j=0}^{b} \varphi_\star(p^{2j+1}) = p^{2b+1}.$$

But the even sum equals p^{2b} by the even-case identity, so

$$\sum_{j=0}^{b} \varphi_\star(p^{2j+1}) = p^{2b}(p-1).$$

Subtracting the same identity for $b-1$ yields, for $b \geq 1$,

$$\varphi_\star(p^{2b+1}) = p^{2b}(p-1) - p^{2b-2}(p-1) = (p-1)(p^2-1)p^{2b-2}.$$

The factorized form in terms of $u(n)$ and $v(n)$ follows by multiplicativity and the prime-power formulas. $\square$

Theorem 8. *Let $\varphi_\star$ be the Euler-type function defined by*

$$\sum_{d \in A(n)} \varphi_\star(d) = n,$$

for all $n \in \mathbb{N}$. Then the constant

$$M := \prod_p \left(1 - \frac{1}{p^2} + \frac{(p-1)^2}{p^5}\right)$$

converges absolutely, and for every $\varepsilon \in (0, \frac{1}{2})$ we have

$$\sum_{n \leq x} \varphi_\star(n) = \frac{M}{2} x^2 + O_\varepsilon\left(x^{1+\varepsilon}\right),$$

as $x \to \infty$.

Proof. Set

$$L(s,\varphi_\star) := \sum_{n\geq 1} \frac{\varphi_\star(n)}{n^s},$$

where $\Re(s) > 2$. A computation of the Euler factors, using the prime-power values from Theorem 7, gives the factorization

$$L(s,\varphi_\star) = \frac{\zeta(s-1)}{\zeta(s)}\, G(s), \qquad G(s) := \prod_p \left(1 + \frac{p-1}{p^s(p^s+p)}\right), \tag{4}$$

where $G(s)$ converges absolutely for $\Re(s) > 1$. Write

$$G(s) = \sum_{n\geq 1} \frac{g(n)}{n^s},$$

where $\Re(s) > 1$. Since $\zeta(s-1)/\zeta(s)$ is the Dirichlet series of Euler's totient function φ, Equation (4) implies that

$$\varphi_\star = \varphi * g$$

under the ordinary Dirichlet convolution. Hence, for $x \geq 2$,

$$\sum_{n\leq x} \varphi_\star(n) = \sum_{d\leq x} g(d) \sum_{k\leq x/d} \varphi(k).$$

Since the Dirichlet series for $G(s)$ converges absolutely for $\Re(s) > 1$, standard coefficient bounds imply that for every $\varepsilon \in (0, \frac{1}{2})$,

$$\sum_{n\leq t} |g(n)| \ll_\varepsilon t^{1+\varepsilon}.$$

By partial summation, this yields

$$\sum_{n\leq t} \frac{|g(n)|}{n} \ll_\varepsilon t^{\varepsilon}, \qquad \sum_{n>t} \frac{|g(n)|}{n^2} \ll_\varepsilon t^{-1+\varepsilon}.$$

We use the effective asymptotic for the summatory totient [3]:

$$\sum_{k\leq y} \varphi(k) = \frac{1}{2\zeta(2)} y^2 + O\Big(y(\log y)^{2/3} \max\left\{1, (\log\log y)^{1/3}\right\}\Big),$$

for $y \geq 2$. Split the sum over d into the ranges $d \leq x/2$ and $x/2 < d \leq x$.
If $d \leq x/2$, then $x/d \geq 2$, so applying the above asymptotic with $y = x/d$ gives

$$\sum_{k\leq x/d} \varphi(k) = \frac{1}{2\zeta(2)} \left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\left(\log\frac{x}{d}\right)^{2/3} \max\left\{1, \left(\log\log\frac{x}{d}\right)^{1/3}\right\}\right).$$

Therefore,

$$\sum_{d\le x/2} g(d) \sum_{k\le x/d} \varphi(k) = \frac{x^2}{2\zeta(2)} \sum_{d\le x/2} \frac{g(d)}{d^2} + O\left(x(\log x)^{2/3} \max\{1, (\log\log x)^{1/3}\} \sum_{d\le x/2} \frac{|g(d)|}{d} \right),$$

and hence,

$$\sum_{d\le x/2} g(d) \sum_{k\le x/d} \varphi(k) = \frac{x^2}{2\zeta(2)} \sum_{d\le x/2} \frac{g(d)}{d^2} + O_\varepsilon\left(x^{1+\varepsilon}\right).$$

If $x/2 < d \le x$, then $1 \le x/d < 2$, so $\sum_{k\le x/d} \varphi(k) \ll 1$. Thus,

$$\sum_{x/2<d\le x} g(d) \sum_{k\le x/d} \varphi(k) \ll \sum_{x/2<d\le x} |g(d)| \ll_\varepsilon x^{1+\varepsilon}.$$

Combining the two ranges, we obtain

$$\sum_{n\le x} \varphi_\star(n) = \frac{x^2}{2\zeta(2)} \sum_{d\le x/2} \frac{g(d)}{d^2} + O_\varepsilon\left(x^{1+\varepsilon}\right).$$

Now,

$$\sum_{d\le x/2} \frac{g(d)}{d^2} = \sum_{d\ge 1} \frac{g(d)}{d^2} - \sum_{d>x/2} \frac{g(d)}{d^2} = G(2) + O_\varepsilon\left(x^{-1+\varepsilon}\right),$$

since

$$\sum_{d>x/2} \frac{|g(d)|}{d^2} \ll_\varepsilon x^{-1+\varepsilon}.$$

Therefore,

$$\sum_{n\le x} \varphi_\star(n) = \frac{G(2)}{2\zeta(2)} x^2 + O_\varepsilon\left(x^{1+\varepsilon}\right).$$

Finally,

$$\frac{G(2)}{\zeta(2)} = \frac{1}{\zeta(2)} \prod_p \left(1 + \frac{p-1}{p^2(p^2+p)}\right) = \prod_p \left(1 - \frac{1}{p^2} + \frac{(p-1)^2}{p^5}\right) = M,$$

which gives the stated asymptotic formula. □

4.2. Summatory Order of $\tau_\star$ and Density Results for $\mu_\star$

Recall from Theorem 1 that $\tau_\star(n) = |A(n)| = d(u(n))d(v(n))$. This function is multiplicative, and its Dirichlet series admits a factorization strong enough to yield an explicit main term with a logarithmic factor.

Theorem 9. *For $\Re(s) > 1$,*

$$\sum_{n\geq 1} \frac{\tau_\star(n)}{n^s} = \prod_p \left(\sum_{k\geq 0} \frac{\tau_\star(p^k)}{p^{ks}} \right) = \prod_p \frac{1+2p^{-s}}{(1-p^{-2s})^2},$$

where p denotes a prime. Moreover, one has the analytic factorization

$$\sum_{n\geq 1} \frac{\tau_\star(n)}{n^s} = \zeta(s)^2 \zeta(2s)^2 H(s),$$

where

$$H(s) := \prod_p \left(1 - 3p^{-2s} + 2p^{-3s}\right)$$

converges absolutely and defines a holomorphic function for $\Re(s) > \frac{1}{2}$. In particular, $H(s)$ is holomorphic in a neighborhood of $s = 1$ and $H(1) \neq 0$.

Proof. For p^k, where p is a prime, and k is a positive integer, write $k = 2b$ or $k = 2b+1$. Then $u(p^{2b}) = 1$ and $v(p^{2b}) = p^b$, so $\tau_\star(p^{2b}) = d(1)d(p^b) = b+1$. Also, $u(p^{2b+1}) = p$ and $v(p^{2b+1}) = p^b$, so $\tau_\star(p^{2b+1}) = d(p)d(p^b) = 2(b+1)$. Hence,

$$\sum_{k\geq 0} \frac{\tau_\star(p^k)}{p^{ks}} = \sum_{b\geq 0} \frac{b+1}{p^{2bs}} + \sum_{b\geq 0} \frac{2(b+1)}{p^{(2b+1)s}} \tag{5}$$

$$= \frac{1}{(1-p^{-2s})^2} + \frac{2p^{-s}}{(1-p^{-2s})^2} \tag{6}$$

$$= \frac{1+2p^{-s}}{(1-p^{-2s})^2}. \tag{7}$$

This gives the Euler product. For the factorization, write $x = p^{-s}$. Then

$$1 + 2x = (1-x)^{-2}(1 - 3x^2 + 2x^3),$$

which is verified by multiplying the series $(1-x)^{-2} = \sum_{n\geq 0}(n+1)x^n$ by $1-3x^2+2x^3$ and observing that all coefficients beyond x^1 cancel. Thus,

$$\prod_p (1+2p^{-s}) = \prod_p (1-p^{-s})^{-2} \prod_p (1 - 3p^{-2s} + 2p^{-3s}) = \zeta(s)^2 H(s),$$

and since $\prod_p (1-p^{-2s})^{-2} = \zeta(2s)^2$, the claimed factorization follows. Absolute convergence of $H(s)$ for $\Re(s) > \frac{1}{2}$ follows from $\log(1 - 3p^{-2s} + 2p^{-3s}) = O(p^{-2\Re(s)})$. □

4.3. A Sharp Asymptotic Formula for the Summatory Function of $\tau_\star$

Lemma 1. *Let*

$$F(s) = \sum_{n\geq 1} \frac{f(n)}{n^s} = \frac{\zeta(s)^2}{\zeta(2s)}\, G(s),$$

where $G(s)$ is holomorphic and absolutely convergent for $\Re(s) > \sigma_0$, with $\sigma_0 < \frac{1}{2}$. Then, as $x \to \infty$,

$$\sum_{n\leq x} f(n) = \frac{G(1)}{\zeta(2)} x \log x + \frac{G(1)}{\zeta(2)} \left(2\gamma - 1 + \frac{G'(1)}{G(1)} - 2\frac{\zeta'(2)}{\zeta(2)}\right) x + O\Big(x^{1/2} (\log x)^3\Big).$$

Proof. This follows from the main theorem of Zhai [8] by specializing to $L(s; \chi, q) = \zeta(s)$, $M(s) = 1$, and $H(s) = 1/\zeta(s)$, together with the residue computation at $s = 1$. □

Theorem 10. *Let $\tau_\star(n) = |A(n)|$. Define*

$$A_\tau := \prod_p \left(1 - \frac{1}{(p+1)^2}\right) \quad \text{and} \quad B_\tau := 6 \sum_p \frac{\log p}{(p-1)(p+1)(p+2)}.$$

Then, as $x \to \infty$,

$$\sum_{n\leq x} \tau_\star(n) = A_\tau x \log x + A_\tau \left(2\gamma - 1 - B_\tau - 2\frac{\zeta'}{\zeta}(2)\right) x + O\Big(x^{1/2} (\log x)^3\Big).$$

Proof. We may use the alternative factorization

$$\sum_{n\geq 1} \frac{\tau_\star(n)}{n^s} = \frac{\zeta(s)^2}{\zeta(2s)}\, G(s),$$

where

$$G(s) := \prod_p \left(1 + \frac{2p^s + 1}{(p^s - 1)(p^s + 1)^3}\right),$$

and $G(s)$ converges absolutely in the half-plane $\Re(s) > \frac{1}{3}$. Applying Lemma 1, we obtain

$$\sum_{n\leq x} \tau_\star(n) = \frac{G(1)}{\zeta(2)} x \log x + \frac{G(1)}{\zeta(2)} \left(2\gamma - 1 + \frac{G'(1)}{G(1)} - 2\frac{\zeta'(2)}{\zeta(2)}\right) x + O\Big(x^{1/2} (\log x)^3\Big).$$

Now,

$$\frac{G(1)}{\zeta(2)} = \prod_p \left(1 - \frac{1}{(p+1)^2}\right) = A_\tau.$$

Moreover, for each prime p the Euler factor of $G(s)$ is

$$G_p(s) = 1 + \frac{2p^s + 1}{(p^s - 1)(p^s + 1)^3},$$

so a direct differentiation gives

$$-\frac{G_p'(1)}{G_p(1)} = \frac{6 \log p}{(p-1)(p+1)(p+2)}.$$

Summing over primes yields

$$-\frac{G'(1)}{G(1)} = 6 \sum_p \frac{\log p}{(p-1)(p+1)(p+2)} = B_\tau.$$

Substituting these identities completes the proof. □

Theorem 11. *One has $\mu_\star(n) \neq 0$ if and only if $\nu_p(n) \in \{0, 1, 2, 3\}$ for every prime p. In particular, the set*

$$\mathcal{S} := \{n \in \mathbb{N} : \mu_\star(n) \neq 0\}$$

has natural density

$$\delta(\mathcal{S}) = \prod_p \left(1 - \frac{1}{p^4}\right) = \frac{1}{\zeta(4)}.$$

Equivalently,

$$\sum_{n \leq x} \mu_\star(n)^2 \sim \frac{1}{\zeta(4)} x.$$

Proof. By Theorem 3, $\mu_\star(n) = \mu(u(n))\mu(v(n))$. Since $u(n)$ is squarefree, $\mu(u(n)) \neq 0$ always. Thus, $\mu_\star(n) \neq 0$ if and only if $\mu(v(n)) \neq 0$. That is, $\mu_\star(n) \neq 0$ if and only if $v(n)$ is squarefree. Now

$$v(n) = \prod_p p^{\lfloor \nu_p(n)/2 \rfloor}$$

is squarefree if and only if $\lfloor \nu_p(n)/2 \rfloor \in \{0, 1\}$ for all p. Equivalently, it is squarefree if and only if $\nu_p(n) \in \{0, 1, 2, 3\}$ for all p.

For the density, the local condition at a prime p is $\nu_p(n) \leq 3$. The probability that a random integer has $\nu_p(n) = k$ is $(1 - 1/p)p^{-k}$. Hence, the local contribution equals

$$\sum_{k=0}^{3} (1 - 1/p)p^{-k} = (1 - 1/p)\frac{1 - p^{-4}}{1 - p^{-1}} = 1 - p^{-4}.$$

Independence over primes gives $\delta(\mathcal{S}) = \prod_p (1 - p^{-4}) = 1/\zeta(4)$. Finally, $\mu_\star(n)^2$ is the indicator function of $\mathcal{S}$, hence the asserted asymptotic follows. □

5. Numerical Examples

In Table 1, we list $u(n)$, $v(n)$, the A-divisor count $\tau_\star(n)$, $\mu_\star(n)$, and $\varphi_\star(n)$ for $1 \le n \le 12$.

n	$(u(n), v(n))$	$\tau_\star(n)$	$\mu_\star(n)$	$\varphi_\star(n)$
1	$(1,1)$	1	1	1
2	$(2,1)$	2	-1	1
3	$(3,1)$	2	-1	2
4	$(1,2)$	2	-1	3
5	$(5,1)$	2	-1	4
6	$(6,1)$	4	$+1$	2
7	$(7,1)$	2	-1	6
8	$(2,2)$	4	$+1$	3
9	$(1,3)$	2	-1	8
10	$(10,1)$	4	$+1$	4
11	$(11,1)$	2	-1	10
12	$(3,2)$	4	$+1$	6

Table 1: Values of $u(n)$, $v(n)$, $\tau_\star(n)$, $\mu_\star(n)$, and $\varphi_\star(n)$ for $1 \le n \le 12$.

As a consistency check, for $n = 12$ we have $A(12) = \{1, 4, 3, 12\}$, and indeed,

$$\varphi_\star(1) + \varphi_\star(4) + \varphi_\star(3) + \varphi_\star(12) = 1 + 3 + 2 + 6 = 12.$$

To illustrate Theorem 8, we compare the ratio

$$\frac{1}{x^2} \sum_{n \le x} \varphi_\star(n)$$

with the predicted constant $M/2 \approx 0.3254359793$. Empirically,

$$\frac{1}{50^2} \sum_{n \le 50} \varphi_\star(n) = 0.3336,$$

$$\frac{1}{200^2} \sum_{n \le 200} \varphi_\star(n) = 0.3275,$$

$$\frac{1}{20000^2} \sum_{n \le 20000} \varphi_\star(n) = 0.3254425,$$

which is consistent with convergence to $M/2$.

6. Conclusion

We introduced a concrete square–splitting divisor system $A(n)$ and its induced A-convolution $\star$. The key structural result is the ring isomorphism with a two-variable Dirichlet convolution, which yields explicit and computable arithmetic: a closed formula for $\mu_\star$, a sharp inversion theorem, Euler products for generalized divisor sums, and a new Euler-type totient $\varphi_\star$ with exact prime-power values. This fits into the general divisor-system convolution framework initiated by Narkiewicz and developed further in related settings such as unitary and cross-convolutions [5, 2, 7].

We also established an effective asymptotic formula for $\sum_{n \le x} \varphi_\star(n)$, whose leading constant is given by a simple absolutely convergent Euler product. In addition, we obtained a sharp asymptotic formula for $\sum_{n \le x} \tau_\star(n)$ and determined the natural density of the support of $\mu_\star$. These results show that the square–splitting convolution admits a tractable analytic theory alongside its explicit algebraic structure.

Further questions include possible refinements and extensions of the asymptotic theory for $\sum_{n \le x} \tau_\star(n)$, as well as related distributional problems for $\mu_\star(n)$ and other arithmetic functions associated with this convolution; see also the analytic tools used in [6, 1].

Acknowledgements. The authors sincerely thank the editor for handling the manuscript and the anonymous referee for constructive comments and suggestions, which significantly improved both the presentation and the analytic part of the paper.

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