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# CERTAIN ALGEBRAIC FORMAL LAURENT SERIES WITH BOUNDED PARTIAL QUOTIENTS

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## Abstract

Two classes of algebraic formal Laurent series over a finite field with bounded partial quotients in their continued fraction expansions are constructed. The first class is based on and extends a result of Ayadi et al. in 2016. The second class is exhibited by generalizing a result of Lasjaunias in 1999; a complementary set of the second class with unbounded partial quotients is also derived.

## 1. Introduction

The problem of classifying elements in the field of formal Laurent series over a finite field $\mathbb{F}$ that have bounded partial quotients in their continued fractions is an interesting, yet still unsolved problem. Baum and Sweet [2] proved the following three main results.

**Theorem 1** ([2, Corollary 3])**.** *Let $f \in \mathbb{F}_2((x^{-1}))$, $\mathbb{F}_2$ the finite field with two elements, be a root of the equation $f^3 + p^{-1}f + 1 = 0$, where $p(x) \in \mathbb{F}_2[x]$ and $\deg p \geq 1$. Then $f$ has bounded partial quotients.*

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**Theorem 2** ([2, Theorem 5]). *If* $f \in \mathbb{F}_2((x^{-1}))$ *satisfies* $f^{2^n+1} + g^{-1}f + 1 = 0$, $(n > 1)$*, where* $g \in \mathbb{F}_2[x]$, $\deg g \geq 1$*, then* $f$ *has unbounded partial quotients.*

**Theorem 3** ([2]). *Let* $n \geq 0$*, and let* $P, Q \in \mathbb{F}_2[x]$ *be such that* $P + Q^{2^n} \neq 0, 1$*. Then* $f^{2^n+1} + Qf^{2^n} + Pf + QP + 1 = 0$ *has a unique root* $f$ *in* $\mathbb{F}_2((x^{-1}))$ *and its continued fraction is* $f = \left[Q; P + Q^{2^n}, P^{2^n} + Q^{2^{2n}}, P^{2^{2n}} + Q^{2^{3n}}, \cdots\right]$.

As mentioned in [1], Mills and Robbins [5] showed that irrational formal Laurent series that are solutions of equations of the form

$$x = \frac{Ax^{p^r} + B}{Cx^{p^r} + D}, \quad r \geq 0, \quad A, B, C, D \in \mathcal{F}[X],\ \mathcal{F} \text{ a finite field of characteristic } p, \tag{1.1}$$

tend to have regular patterns in their continued fraction expansions. In 1999, Lasjaunias [4] generalized the work of [2] and obtained a root of the cubic equation

$$xT^3 + DT + x^\ell = 0, \quad \ell \geq 1,\ D \in \mathbb{F}_2[x],\ D(0) = 1$$

with bounded partial quotients. Recently, in 2016, among other things, Ayadi et al. [1] proved: if $\alpha$ over $\mathbb{F}_2$ is the unique formal Laurent series of positive degree that is a root of the equation

$$x^\ell\alpha^3 + A\alpha^2 + 1 = 0, \quad \ell \geq 2,\ A = x^{2\ell} + \sum_{i=2}^{2\ell-1} e_i x^i + x + 1,\ e_i \in \mathbb{F}_2, \tag{1.2}$$

then $\alpha$ has bounded partial quotients. It is also mentioned that more examples of elements with bounded and/or unbounded partial quotients ought to be explored for the classification problem.

We exhibit here two further classes of algebraic formal Laurent series with bounded partial quotients. The first class extends the one in (1.2) to the field of any characteristic. The elements in the second class are derived using a result of Lasjaunias in [4]; they are algebraic formal Laurent series closely related to (1.1). In fact, in this second class, it is possible to completely classify whether they have bounded partial quotients using their valuations.

Throughout the rest of the paper, let $\mathbb{F}_q$ be the finite field of $q$ elements, $\mathbb{F}_q[x]$ the ring of polynomials over $\mathbb{F}_q$, and let $\mathbb{F} := \mathbb{F}_q((x^{-1}))$ be the field of formal Laurent series over $\mathbb{F}_q$, complete with respect to the degree valuation $|\cdot|$, and set $\mathbb{N} = \{1, 2, 3, \ldots\}$. For $P, Q \in \mathbb{F}_q[x]\backslash\{0\}$, by $\gcd(P, Q)$ we mean the monic polynomial $d \in \mathbb{F}_q[x]$ satisfying $d \mid P, d \mid Q$, and if $d' \in \mathbb{F}_q[x]$ such that $d' \mid P, d' \mid Q$, then $d' \mid d$. Moreover, $P$ and $Q$ are said to be associates if there exists $c \in \mathbb{F}_q\backslash\{0\}$ such that $P = cQ$.

For a basic introduction to continued fractions in the field of formal Laurent series, we refer to the works [6] and [1, Section 2]. Recall that for each nonzero element

$$\alpha = c_m x^m + \ldots + c_1 x + c_0 + \frac{c_{-1}}{x} + \frac{c_{-2}}{x^2} + \ldots \in \mathbb{F},$$

where $m \in \mathbb{Z}, c_i \in \mathbb{F}_q$ $(i \leq m)$, and $c_m \neq 0$, the degree valuation is defined by $|\alpha| = |x|^m$ and $|0| = 0$. Every element $\alpha \in \mathbb{F}$ can be represented uniquely as a finite or infinite expression of the form

$$\alpha = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}} := [b_0, b_1, b_2, \ldots], \tag{1.3}$$

where $b_0 \in \mathbb{F}_q[x]$ and $b_i \in \mathbb{F}_q[x] \backslash \mathbb{F}_q$ $(i \geq 1)$. The polynomials $b_n$ are called the *partial quotients* of $\alpha$ and $\alpha_n := [b_n, b_{n+1}, \ldots]$ is called the $n^{th}$ *complete quotient* of $\alpha$. We say that $\alpha \in \mathbb{F}$ has *bounded partial quotients* if $\sup_{n \geq 0} |b_n| < \infty$, and say that $\alpha$ has unbounded partial quotients otherwise. Equivalently, $\alpha$ has bounded partial quotients if the degrees of all polynomials $b_n$ are bounded. Define

$$P_{-1} = 1,\ P_0 = b_0,\ P_{n+1} = b_{n+1}P_n + P_{n-1} \quad (n \geq 0),$$
$$Q_{-1} = 0,\ Q_0 = 1,\ Q_{n+1} = b_{n+1}Q_n + Q_{n-1} \quad (n \geq 0).$$

The quotient $P_n/Q_n$ is called the $n^{th}$ *convergent* of $\alpha$. For convenience, the $(n+1)^{th}$ partial quotient $b_{n+1}$ in (1.3) is denoted by $b_\alpha(P_n, Q_n)$. The following properties are well-known.

**Lemma 1.** *Keeping the above notation, for $n \geq 0$ and $\beta \in \mathbb{F} \backslash \{0\}$, we have*

*i)* $\frac{\beta P_n + P_{n-1}}{\beta Q_n + Q_{n-1}} = [b_0, b_1, b_2, \ldots, b_n, \beta]$,

*ii)* $P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1}$,

*iii)* $|Q_n| > |Q_{n-1}| > 0$,

*iv)* $|Q_n| = |b_1 b_2 \ldots b_n| \qquad (n \geq 1)$,

*v)* $\left|\alpha - \frac{P_n}{Q_n}\right| = \frac{1}{|b_{n+1}||Q_n|^2} \qquad (n \geq 1)$,

*vi)* $|Q_{n+1}\alpha - P_{n+1}| < |Q_n \alpha - P_n| \qquad (n \geq 1)$.

## 2. The First Class

To construct our first class of algebraic formal Laurent series with bounded partial quotients, we need the following lemmas.

**Lemma 2.** *If the continued fraction expansion of $\alpha$ is $[b_0, b_1, b_2, \ldots]$, then the continued fraction expansion of $\alpha^q$ is $[b_0^q, b_1^q, b_2^q, \ldots]$ whose $n^{th}$ convergent is*

$$\frac{\widetilde{P}_n}{\widetilde{Q}_n} = [b_0^q, b_1^q, b_2^q, \ldots, b_n^q],$$

*where $\widetilde{P}_n = P_n^q$ and $\widetilde{Q}_n = Q_n^q$, with $P_n/Q_n$ being the $n^{th}$ convergent of $\alpha$.*

*Proof.* By induction on $n \geq 0$, it is easily verified that

$$\frac{P_n^q}{Q_n^q} = [b_0^q, b_1^q, b_2^q, \ldots, b_n^q].$$

The desired result now follows by noting that, by Lemma 1 (v), we have

$$\left|\alpha^q - \frac{\widetilde{P}_n}{\widetilde{Q}_n}\right| = \left|\alpha - \frac{P_n}{Q_n}\right|^q < \frac{1}{|b_{n+1}|^q |Q_n|^{2q}} \longrightarrow 0 \quad (n \to \infty).$$

□

The following lemma generalizes corresponding results of Baum and Sweet in [2].

**Lemma 3** ([2, Lemma 1])**.** *Let* $P, Q \in \mathbb{F}_q[x]$ *with* $\gcd(P, Q) = 1$.

*i) If* $|Q\alpha - P| = |x|^{-s}|Q|^{-1}$ *for some* $s \in \mathbb{N}$*, then there exists* $n \in \mathbb{N} \cup \{0\}$ *such that* $P$ *and* $Q$ *are associates of* $P_n$ *and* $Q_n$*, respectively. Moreover, for such* $n$*, the* $(n+1)^{th}$ *partial quotient* $b_{n+1}$ *satisfies* $|b_{n+1}| = |x|^s$.

*ii) If* $|Q\alpha - P| = |Q|^{-1}$*, then there exist* $n \in \mathbb{N} \cup \{0\}$ *and* $c_n^*, d_n^* \in \mathbb{F}_q \backslash \{0\}$ *such that*

$$P = c_n^* P_n + d_n^* P_{n-1}, \ Q = c_n^* Q_n + d_n^* Q_{n-1}.$$

*Proof.* (i) Assume $|Q\alpha - P| = |x|^{-s}|Q|^{-1}$ for some $s \in \mathbb{N}$. Since $s > 0$, we have $|\alpha - P/Q| < 1/|Q|^2$. Choosing $n \in \mathbb{N} \cup \{0\}$ so that $|Q_n| \leq |Q| < |Q_{n+1}|$, we get

$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{|Q_n||Q|}.$$

Since $P_n/Q_n$ is a best approximation to $\alpha$, we must have

$$|Q_n\alpha - P_n| \leq |Q\alpha - P| < \frac{1}{|Q|}.$$

Thus,

$$\left|\frac{P}{Q} - \frac{P_n}{Q_n}\right| \leq \max\left\{\left|\frac{P}{Q} - \alpha\right|, \left|\alpha - \frac{P_n}{Q_n}\right|\right\} < \frac{1}{|Q_n||Q|}. \tag{2.1}$$

To show that $P/Q = P_n/Q_n$, we assume to the contrary that $P/Q \neq P_n/Q_n$. Then

$$\left|\frac{P}{Q} - \frac{P_n}{Q_n}\right| = \frac{|PQ_n - QP_n|}{|Q_n||Q|} \geq \frac{1}{|Q_n||Q|},$$

which contradicts (2.1), and so $P/Q = P_n/Q_n$. Making use of the fact that $\gcd(P, Q) = 1 = \gcd(P_n, Q_n)$, we conclude that $P$ and $P_n$ are associates and so are $Q$ and $Q_n$. Therefore, by Lemma 1, parts (iv) and (v), we have

$$|Q\alpha - P| = |Q_n\alpha - P_n| = \frac{1}{|b_{n+1}||Q_n|} = \frac{1}{|b_{n+1}||Q|}.$$

Combining with the main hypothesis, we deduce at once that $|b_{n+1}| = |x|^s$.

(ii) As in the proof of (i), choosing $n \in \mathbb{N} \cup \{0\}$ such that $|Q_n| \le |Q| < |Q_{n+1}|$, we get

$$\left|\alpha - \frac{P}{Q}\right| = \frac{1}{|Q|^2} \le \frac{1}{|Q_n||Q|}.$$

Again using the fact that $P_n/Q_n$ is a best approximation of $\alpha$, we see that

$$|Q_n\alpha - P_n| \le |Q\alpha - P| = \frac{1}{|Q|},$$

and this leads to

$$\left|\frac{P}{Q} - \frac{P_n}{Q_n}\right| \le \max\left\{\left|\frac{P}{Q} - \alpha\right|, \left|\alpha - \frac{P_n}{Q_n}\right|\right\} \le \frac{1}{|Q_n||Q|}. \tag{2.2}$$

We note now that $P/Q$ cannot be a convergent to $\alpha$; for otherwise, by Lemma 1 we would have $|Q\alpha - P| < |Q|^{-1}$, a contradiction. Since $P/Q \ne P_n/Q_n$ for any $n$, we have

$$\left|\frac{P}{Q} - \frac{P_n}{Q_n}\right| = \frac{|PQ_n - QP_n|}{|Q_n||Q|} \ge \frac{1}{|Q_n||Q|},$$

which together with (2.2) implies that

$$|PQ_n - QP_n| = 1.$$

Then $PQ_n - QP_n := d_n \in \mathbb{F}_q\backslash\{0\}$ and so the matrix $\begin{bmatrix} P & P_n \\ Q & Q_n \end{bmatrix}$ is nonsingular, with inverse $\frac{1}{d_n}\begin{bmatrix} Q_n & -P_n \\ -Q & P \end{bmatrix}$. Observe that

$$\frac{1}{d_n}\begin{bmatrix} Q_n & -P_n \\ -Q & P \end{bmatrix}\begin{bmatrix} P_{n-1} \\ Q_{n-1} \end{bmatrix} = \frac{1}{d_n}\begin{bmatrix} Q_nP_{n-1} - P_nQ_{n-1} \\ -QP_{n-1} + PQ_{n-1} \end{bmatrix} = \begin{bmatrix} (-1)^n/d_n \\ c_n/d_n \end{bmatrix}, \tag{2.3}$$

where $c_n = -QP_{n-1} + PQ_{n-1}$. The relation (2.3) assures us that the system of two equations

$$\frac{(-1)^n}{d_n}P + \frac{c_n}{d_n}P_n = P_{n-1}, \quad \frac{(-1)^n}{d_n}Q + \frac{c_n}{d_n}Q_n = Q_{n-1},$$

has a unique solution with $\frac{(-1)^n}{d_n}, \frac{c_n}{d_n} \in \mathbb{F}_q[x]$. Equivalently, we have

$$P = (-1)^n(d_nP_{n-1} - c_nP_n), \quad Q = (-1)^n(d_nQ_{n-1} - c_nQ_n).$$

Note that $c_n \ne 0$, for if $c_n = 0$, then $P/Q = P_{n-1}/Q_{n-1}$ is a convergent of $\alpha$, a contradiction. Thus,

$$\begin{aligned} Q\alpha - P &= (-1)^n(d_nQ_{n-1} - c_nQ_n)\alpha - (-1)^n(d_nP_{n-1} - c_nP_n) \\ &= (-1)^n(d_n(Q_{n-1}\alpha - P_{n-1}) - c_n(Q_n\alpha - P_n)). \end{aligned} \tag{2.4}$$

By the strong triangle inequality, we get

$$|Q| = |Q_{n-1} - c_n Q_n| = |-c_n Q_n| = |c_n||Q_n|. \tag{2.5}$$

By Lemma 1, (2.5), and the choice of $n$, we see that

$$|c_n(Q_n\alpha - P_n)| = \frac{|c_n|}{|Q_{n+1}|} \cdot \frac{|Q_n|}{|Q_n|} = \frac{|Q|}{|Q_{n+1}Q_n|} < \frac{1}{|Q_n|} = |d_n(Q_{n-1}\alpha - P_{n-1})|.$$

Using this last estimate, the relation (2.4) yields

$$|Q\alpha - P| = |d_n(Q_{n-1}\alpha - P_{n-1})| = |Q_{n-1}\alpha - P_{n-1}|.$$

Since $|Q\alpha - P| = 1/|Q|$ and $|Q\alpha - P| = |Q_{n-1}\alpha - P_{n-1}| = 1/|Q_n|$, we have $|Q| = |Q_n|$ and so $|c_n| = 1$. Therefore, $c_n \in \mathbb{F}_q \setminus \{0\}$.

Hence, there exist $n \in \mathbb{N} \cup \{0\}$ and $c_n^*, d_n^* \in \mathbb{F}_q \setminus \{0\}$ such that $P = c_n^* P_n + d_n^* P_{n-1}$ and $Q = c_n^* Q_n + d_n^* Q_{n-1}$ as desired. □

Recall that each element $\alpha \in \mathbb{F}$ can be written uniquely as

$$\alpha = [\alpha] + (\alpha),$$

where

$$[\alpha] := c_m x^m + \ldots + c_1 x + c_0, \quad (\alpha) := \frac{c_{-1}}{x} + \frac{c_{-2}}{x^2} + \ldots.$$

The element $\alpha$ is said to be *of positive degree* if $[\alpha] \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$.

Our first main result is stated as follows.

**Theorem 4.** *If $\alpha \in \mathbb{F}$ has positive degree and satisfies the equation*

$$CT^{q+1} + AT^q + 1 = 0, \tag{2.6}$$

*where $C = x^\ell$, for some $\ell \geq 2$, and*

$$A = e_{2\ell}x^{2\ell} + \sum_{i=2}^{2\ell-1} e_i x^i + e_1 x + e_0 \in \mathbb{F}_q[x]$$

*with nonzero $e_{2\ell}, e_1$, and $e_0$, then $\alpha$ has bounded partial quotients.*

*Proof.* Assume that $\alpha \in \mathbb{F}$ has positive degree and satisfies Equation (2.6). Then

$$\alpha = -\frac{A}{C} - \frac{1}{C\alpha^q}.$$

Since $\alpha$ is of positive degree, we have $[\alpha] = -[A/C]$ and $|\alpha| = |C|$. If $\alpha$ is rational, then we are done. Otherwise, adopting the same notation as in Section 1, let $\alpha = [b_0, b_1, b_2, \ldots]$ and $b_\alpha(P_n, Q_n) = b_{n+1}$. Here, $|b_0| = |\alpha| = |x|^\ell$.

We must show that $b_n$ $(n \geq 1)$ are bounded. Let $n \in \mathbb{N} \cup \{0\}$, $P := P_n$, and $Q := Q_n$. By Lemma 1 (ii) and (v), we have $\gcd(P, Q) = 1$ and

$$|Q\alpha - P| = \frac{1}{|b_\alpha(P, Q)||Q|}. \tag{2.7}$$

Since $\alpha = (-A\alpha^q - 1)/C\alpha^q$, we have

$$|C\alpha - (-A)| = \left|\frac{-A\alpha^q - 1}{\alpha^q} + A\right| = |x|^{-(q-1)\ell}|C|^{-1}.$$

Notice that $\gcd(A, C) = 1$. By Lemma 3, we know that $-A/C$ is the $m^{th}$ convergent of $\alpha$ with $|b_{m+1}| = |b_\alpha(-A, C)| = |x|^{(q-1)\ell}$. Again, by Lemma 1 (v), we obtain

$$|C\alpha + A| = \frac{1}{|b_\alpha(-A, C)||C|}. \tag{2.8}$$

If $P/Q$ is a convergent of $\alpha$ preceding $-A/C$, then, by Lemma 1 (vi), we get $|C\alpha + A| < |Q\alpha - P|$. By (2.7) and (2.8), we have

$$|b_\alpha(P, Q)| < \frac{|b_\alpha(-A, C)||C|}{|Q|} < |b_\alpha(-A, C)||C| = |x|^{q\ell}.$$

We have thus shown that the partial quotients $b_{n+1}$ are bounded by $|x|^{q\ell}$ for all $0 \leq n \leq m - 1$.

We proceed next to show that the partial quotients $b_n$ are bounded by $|x|^{(q^2-1)\ell-(q-2)}$ for all $n > m$. Assume now that $P/Q$ is the convergent of $\alpha$ right after $-A/C$. By Lemma 1 (v), we get $|C| < |Q|$, and so $|C||Q\alpha - P| < |Q||C\alpha + A|$. Note that

$$\begin{aligned} |QA + PC| &= |Q(C\alpha + A) - C(Q\alpha - P)| \\ &= \max\{|Q||C\alpha + A|, |C||Q\alpha - P|\} = |Q||C\alpha + A| = \frac{|Q|}{|C|^q}. \end{aligned} \tag{2.9}$$

We have

$$|Q\alpha - P| = \left|Q(\frac{-A\alpha^q - 1}{C\alpha^q}) - P\right| = \frac{|(QA + PC)\alpha^q + Q|}{|C|^{q+1}}.$$

Hence,

$$|(QA + PC)\alpha^q + Q| = \frac{|C|^{q+1}}{|b_\alpha(P, Q)||Q|}. \tag{2.10}$$

Now, from (2.9) and (2.10), we get

$$|(QA + PC)\alpha^q + Q| = \frac{|C|}{|b_\alpha(P, Q)||QA + PC|}.$$

Let

$$D = \gcd(Q, QA + PC) = \gcd(Q, PC) = \gcd(Q, C).$$

Write $Q = P'D$ and $QA + PC = Q'D$ where $P', Q' \in \mathbb{F}_q[x]$ and $\gcd(P', D') = 1$. Thus,

$$|Q'\alpha^q + P'| = \frac{|C|}{|b_\alpha(P, Q)||Q'||D|^2}. \tag{2.11}$$

We claim that if $P'$ or $Q'$ is not a $q^{th}$ power of an element in $\mathbb{F}_q[x]$, then $|b_\alpha(P, Q)| < |C|/|D|^2$. To see this, suppose that $|b_\alpha(P, Q)| = |C|/|D|^2$. Then Equation (2.11) becomes

$$|Q'\alpha^q - (-P')| = \frac{1}{|Q'|}. \tag{2.12}$$

By (2.12) and Lemma 3, there exist an integer $n \geq 0$ and $c_n^*, d_n^* \in \mathbb{F}_q\backslash\{0\}$ such that

$$-P' = c_n^*\widetilde{P}_n + d_n^*\widetilde{P}_{n-1}, \quad Q' = c_n^*\widetilde{Q}_n + d_n^*\widetilde{Q}_{n-1},$$

where $\widetilde{P}_{n-1}/\widetilde{Q}_{n-1}, \widetilde{P}_n/\widetilde{Q}_n$ are consecutive convergents of $\alpha^q$. Since all convergents of $\alpha^q$ are $q^{th}$ powers of polynomials in $\mathbb{F}_q[x]$, so are $\widetilde{P}_{n-1}, \widetilde{P}_n, \widetilde{Q}_{n-1}$, and $\widetilde{Q}_n$, and we are done. If $|b_\alpha(P, Q)| > |C|/|D|^2$, then, by (2.11) and Lemma 3, $-P'/Q'$ is a convergent of $\alpha^q$. Similarly, $P'$ and $Q'$ are both $q^{th}$ powers in $\mathbb{F}_q[x]$. This proves the claim.

Since $D$ divides $C = x^\ell$, the element $D$ must be of the form $D = bx^i$, where $0 \leq i \leq \ell$. We consider three cases.

`Case 1.` $D = bx^i$ where $0 \leq i \leq \ell - 2$. Observe that the polynomial $P'$ has a constant term (for otherwise write $P' = \ldots + a_{k+1}x^{k+1} + a_kx^k \in \mathbb{F}_q[x]$ where $k \geq 1$ is the smallest degree of $x$. Therefore, $Q = P'D = (\ldots + a_kb)x^{k+i}$ and so $x^{i+1} \mid Q$. Now, we have $x^{i+1} \mid \gcd(Q, C) = bx^i$, a contradiction.) If $P'$ is not a $q^{th}$ power of an element in $\mathbb{F}_q[x]$, then, by the claim, the partial quotient $b_\alpha(P, Q)$ is bounded as required. Now, we consider the case that $P'$ is a $q^{th}$ power, say $P' = \ldots + c_2x^{2q} + c_1x^q + c_0$ with $c_0 \in \mathbb{F}_q \setminus \{0\}$. Then, since $q$ and $\ell - i \geq 2$, we have

$$Q' = \frac{QA + PC}{D} = AP' + b^{-1}x^{\ell-i}P = \ldots + e_1c_0x + e_0c_0,$$

where $e_1c_0 \in \mathbb{F}_q\backslash\{0\}$. Consequently, there is a linear term $e_1c_0x$ in the expansion, and so $Q'$ is not a $q^{th}$ power. By the claim, we have

$$|b_\alpha(P, Q)| < \frac{|C|}{|D|^2} = |x|^{\ell-2i} \leq |x|^\ell.$$

Therefore, in this case the partial quotients $b_n$ are bounded by $|x|^{\ell-1}$ for all $n > m$.

`Case 2.` $D = bx^{\ell-1}$. By the same argument as in `Case 1`, $P'$ must have a constant term. Therefore, $Q' = AP' + b^{-1}xP$ also has a constant term. If $P'$ or $Q'$ is not a $q^{th}$

power, then by the claim, we have $|b_\alpha(P,Q)| \leq |x|^{\ell-1}$. Assume $P'$ and $Q'$ are both $q^{th}$ powers. Write $P' = U^q$ and $Q' = V^q$ where $U, V \in \mathbb{F}_q[x]$. Since $\gcd(P', Q') = 1$, we obtain $\gcd(U, V) = 1$. Then Equation (2.11) becomes

$$|V^q\alpha^q - (-U^q)| = \frac{1}{|b_\alpha(P,Q)||x|^{\ell-2}} \cdot \frac{1}{|V|^q}.$$

Since $|b_\alpha(P,Q)| > 1$ and $\ell \geq 2$, by Lemma 3, we have $-U^q/V^q$ is a convergent of $\alpha^q$ with $|b_{\alpha^q}(-U^q, V^q)| = |b_\alpha(P,Q)||x|^{\ell-2}$. Since $|b_{\alpha^q}(-U^q, V^q)|$ is a $q^{th}$ power of a positive integer, we have

$$|V\alpha - (-U)| = \frac{1}{\sqrt[q]{|b_\alpha(P,Q)||x|^{\ell-2}}|V|} \tag{2.13}$$

satisfies the condition of Lemma 3. This implies that $-U/V$ is a convergent of $\alpha$ with

$$|b_\alpha(-U,V)| = \sqrt[q]{|b_\alpha(P,Q)||x|^{\ell-2}}. \tag{2.14}$$

If $-U/V$ is a convergent of $\alpha$ preceding $-A/C$, then by Lemma 1 (vi), $|C\alpha + A| < |V\alpha - (-U)|$. By (2.8) and (2.13), we have

$$|b_\alpha(-U,V)| < |b_\alpha(-A,C)||C| = |x|^{(q-1)\ell}|x|^\ell = |x|^{q\ell}.$$

Hence, we get $|b_\alpha(-U,V)| \leq |x|^{q\ell-1}$, implying that

$$|b_\alpha(P,Q)| = |b_\alpha(-U,V)|^q|x|^{2-\ell} \leq |x|^{(q^2-1)\ell-(q-2)}.$$

If $-U/V$ is a convergent of $\alpha$ which is equal to $-A/C$, then $CU = AV$. Since $P' = U^q$ and $Q' = V^q$ both contain constant terms, so do $U$ and $V$. Write $U = \ldots + u_0$ and $V = \ldots + v_0$ where $u_0, v_0 \in \mathbb{F}_q \setminus \{0\}$. Now we have

$$\ldots + u_0x^\ell = CU = AV = \ldots + e_0v_0,$$

where $\ell \geq 2$ and $e_0v_0 \neq 0$, and this is impossible. If $-U/V$ is a convergent of $\alpha$ that comes after $-A/C$, then $|C||V\alpha + U| < |V||C\alpha + A|$ and so

$$|AV - CU| = \max\{|C||V\alpha + U|, |V||C\alpha + A|\} = |V||A + C\alpha| = \frac{|V|}{|C|^q}. \tag{2.15}$$

Since $|V\alpha + U| = 1/(|b_\alpha(-U,V)||V|)$ and $|V\alpha + U| = |(AV - CU)\alpha^q + V|/|C|^{q+1}$, we have

$$|(AV - CU)\alpha^q + V| = \frac{|C|^{q+1}}{|b_\alpha(-U,V)||V|}. \tag{2.16}$$

From (2.15) and (2.16), we conclude that

$$|(AV - CU)\alpha^q + V| = \frac{|C|}{|b_\alpha(-U,V)||AV - CU|}.$$

By the same argument as before, we let

$$D' = \gcd(V, AV - CU) = \gcd(V, -CU) = \gcd(V, C).$$

Then $V = U'D'$ and $AV - CU = V'D'$ where $U', V' \in \mathbb{F}_q[x]$. Since $V$ has a constant term, we have $\gcd(V, C) = 1$ and so $D' = 1$. Then $U' = V$ and $V' = AV - CU$. Now, we get

$$|V'\alpha^q + U'| = \frac{|C|}{|b_\alpha(-U, V)|} \cdot \frac{1}{|V'|}.$$

If $U'$ is not a $q^{th}$ power, then by the claim, we get $|b_\alpha(-U, V)| \le |x|^{\ell-1}$. By (2.14), we have

$$|b_\alpha(P, Q)| \le |x|^{(q-1)\ell-(q-2)}.$$

Otherwise, if $U'$ is a $q^{th}$ power, then

$$V' = AV - CU = (\ldots + e_1v_0x + e_0v_0) - x^\ell U.$$

Since $\ell \ge 2$ and $e_1v_0 \ne 0$, there is a linear term $e_1v_0x$ in the expansion, and so $V'$ is not a $q^{th}$ power. By the claim again, we have $|b_\alpha(-U, V)| \le |x|^{\ell-1}$. Also, by (2.14), we have

$$|b_\alpha(P, Q)| \le |x|^{(q-1)\ell-(q-2)}.$$

Therefore, in `Case 2`, the partial quotients $b_n$ for all $n > m$ are bounded by $\max\{|x|^{\ell-1}, |x|^{(q^2-1)\ell-(q-2)}, |x|^{(q-1)\ell-(q-2)}\} = |x|^{(q^2-1)\ell-(q-2)}$.

`Case 3.` $D = bx^\ell$. We shall show that

$$|b_\alpha(P, Q)| \le |x|^{\lfloor \frac{\ell}{q-1} \rfloor},$$

where $\lfloor \ell/(q-1) \rfloor$ is the greatest integer less than or equal to $\ell/(q-1)$. Suppose, to the contrary, that there exist a convergent $P/Q$ such that $|b_\alpha(P, Q)| > |x|^{\lfloor \ell/(q-1) \rfloor}$. We may assume that $b_\alpha(P, Q)$ has the smallest degree greater than $\lfloor \ell/(q-1) \rfloor$. If $P'$ or $Q'$ is not a $q^{th}$ power, then $|b_\alpha(P, Q)| < |C|/|D|^2 = |x|^{-\ell}$, a contradiction. Thus, $P'$ and $Q'$ must be $q^{th}$ powers, say $P' = U^q, Q' = V^q$ where $U, V \in \mathbb{F}_q[x]$. Since $\gcd(U^q, V^q) = 1 = \gcd(P', Q')$, we have $\gcd(U, V) = 1$. Substituting $P' = U^q$, $Q' = V^q$, and $D = bx^\ell$ in (2.11), we get

$$|V^q\alpha^q + U^q| = \frac{1}{|b_\alpha(P, Q)||x|^\ell} \cdot \frac{1}{|V|^q} = \left(\frac{1}{|x|^{t'/q}|V|}\right)^q,$$

where $t'/q \in \mathbb{N}$. Since $|b_\alpha(P, Q)| > 1$ and $\ell \ge 2$, we have $|b_\alpha(P, Q)||x|^\ell = |x|^{t'}$ for some integer $t' > 0$. By Lemma 3, we have $-U^q/V^q$ is a convergent of $\alpha^q$ with $|b_{\alpha^q}(-U^q, V^q)| = |b_\alpha(P, Q)||x|^\ell$. Since $b_{\alpha^q}(-U^q, V^q)$ is a $q^{th}$ power and $q \mid t'$, we deduce that

$$|V\alpha + U| = \frac{1}{\sqrt[q]{|b_\alpha(P, Q)||x|^\ell}|V|}.$$

By Lemma 3 again, $-U/V$ is a convergent of $\alpha$ such that

$$|b_\alpha(-U,V)| = \sqrt[q]{|b_\alpha(P,Q)||x|^\ell} > |x|^{\left\lfloor \frac{\ell}{q-1} \right\rfloor}.$$

Since $\deg b_\alpha(P,Q) > \lfloor \ell/(q-1) \rfloor$, we get $\deg b_\alpha(-U,V) < \deg b_\alpha(P,Q)$ which contradicts the assumption that the convergent $P/Q$ has $b_\alpha(P,Q)$ of smallest degree greater than $\lfloor \ell/(q-1) \rfloor$. In this case $b_n$ is bounded by $|x|^{\lfloor \ell/(q-1) \rfloor}$ for all $n > m$.

From the above three cases, for all $n > m$, we have $b_n$ is bounded by

$$\max\{|x|^{\ell-1}, |x|^{(q^2-1)\ell-(q-2)}, |x|^{\lfloor \ell/(q-1) \rfloor}\} = |x|^{(q^2-1)\ell-(q-2)}.$$

Summing up, for all $n \in \mathbb{N} \cup \{0\}$, the partial quotient $b_n$ is bounded by

$$\max\{|x|^{\ell}, |x|^{q\ell}, |x|^{(q-1)\ell}, |x|^{(q^2-1)\ell-(q-2)}\} = |x|^{(q^2-1)\ell-(q-2)}.$$

Hence, $\alpha$ has bounded partial quotients as desired. □

Taking $q = 2$, we get the following result of Ayadi, Beldi, and Lee.

**Corollary 1** ([1])**.** *If $\alpha \in \mathbb{F}_2\left((x^{-1})\right)$ has positive degree and satisfies the equation*

$$CT^3 + AT^2 + 1 = 0,$$

*where $C = x^\ell$, for some $\ell \geq 2$, and $A = x^{2l} + \sum_{i=2}^{2l-1} e_i x^i + x + 1 \in \mathbb{F}_2[x]$, then $\alpha$ has bounded partial quotients.*

As another application, the following result in [3] is an example of a quartic formal Laurent series over $\mathbb{F}_3$ with bounded partial quotients.

**Example 1.** Consider the equation

$$x^2T^4 + (x^4 + x^3 + x^2 + x + 1)T^3 + 1 = 0,$$

all of whose coefficients are in $\mathbb{F}_3(x)$. If $\alpha \in \mathbb{F}_3((x^{-1}))$ is its irrational root, by Theorem 4, all partial quotients of $\alpha$ are bounded by

$$\max\{|x|^2, |x|^6, |x|^4, |x|^{15}\} = |x|^{15}.$$

## 3. The Second Class

In this section, we deal with equations of the form

$$xT^{q+1} + DT + x^\ell = 0,$$

where $\ell \geq 1$ and $D \in \mathbb{F}_q[x]$ with $D(0) \neq 0$. The existence of an irrational root of this equation depends on the choice of $q, D$, and $\ell$. For example, if $q = 2, D = 1$, and

$\ell = 1$, the root is given by Baum and Sweet in [2]. For $m = \deg D$ and if $1 \le \ell \le m$ with $(\ell, m) \neq (1,1)$, the equation has a unique root $\alpha$ with $|\alpha| = |x|^{\ell - m}$, [4].

Throughout this section, let

$$E = \left\{ \frac{P}{Q} \,:\, \frac{P}{Q} \text{ is a convergent of } \alpha \text{ and } \gcd(P,Q) = 1 \right\}.$$

We begin with two lemmas.

**Lemma 4.** *Let* $\alpha = [b_0, b_1, b_2, \ldots] \in \mathbb{F}_q((x^{-1}))$ *be irrational and let* $P_n/Q_n$ *be its* $n^{th}$ *convergent.*

- *i) If* $|\alpha| \ge 1$, *then* $|\alpha - P_n/Q_n| < |\alpha|$ *for all* $n \ge 0$.
- *ii) If* $|\alpha| < 1$, *then* $|\alpha - P_0/Q_0| = |\alpha|$ *and* $|\alpha - P_n/Q_n| < |\alpha|$ *for all* $n \ge 1$.

*Proof.* (i) Assume that $|\alpha| \ge 1$. If $n = 0$, then

$$\left|\alpha - \frac{P_0}{Q_0}\right| = |\alpha - b_0| = |\alpha - [\alpha]| < 1 \le |\alpha|.$$

For $n \ge 1$, by Lemma 1 (iv) and (v), we have

$$\left|\alpha - \frac{P_n}{Q_n}\right| = \frac{1}{|b_{n+1}||Q_n|^2} = \frac{1}{|b_{n+1}||b_n \ldots b_2 b_1|^2} < 1 \le |\alpha|.$$

Therefore, $|\alpha - P_n/Q_n| < |\alpha|$ for all $n \ge 0$.

(ii) Assume that $|\alpha| < 1$. Then $b_0 = [\alpha] = 0$ and $\alpha = [0, b_1, b_2, \ldots]$ yielding

$$\left|\alpha - \frac{P_0}{Q_0}\right| = |\alpha - 0| = |\alpha|.$$

Since $\alpha = [0, b_1, b_2, \ldots]$, write $1/\alpha = b_1 + \beta$ where $\beta \in \mathbb{F}_q((x^{-1}))$ with $|\beta| < 1$. Then $|1 - \alpha b_1| = |\alpha\beta| < |\alpha| < 1$. Therefore,

$$\left|\alpha - \frac{P_1}{Q_1}\right| = \left|\alpha - \frac{1}{b_1}\right| = \frac{|\alpha b_1 - 1|}{|b_1|} < \frac{1}{|b_1|} = |\alpha|.$$

By Lemma 1 (vi), we have $|\alpha - P_{n+1}/Q_{n+1}| < |\alpha - P_n/Q_n|$ for all $n \ge 1$, and so $|\alpha - P_n/Q_n| < |\alpha|$ for all $n \ge 1$. □

Observe that, by Lemma 4, if $|\alpha| \ge 1$, then $|\alpha - P/Q| < |\alpha|$ for all $P/Q \in E$. Otherwise, we have $b_0 = 0 = P_0/Q_0$, $|b_1| = 1/|\alpha|$, and $|\alpha - P/Q| < |\alpha|$ for all $P/Q \in E \backslash \{0\}$. Consequently, we have

$$\left|\alpha - \frac{P}{Q}\right| < |\alpha| \quad \text{for all } \frac{P}{Q} \in E \backslash \{0\}.$$

We shall also need the following result of Lasjaunias in [4].

**Lemma 5** ([4, Lemma 1, p. 50])**.** *Let $\alpha$ be an irrational element in $\mathbb{F}_q((x^{-1}))$ satisfying $\alpha = \frac{A\alpha^q+B}{C\alpha^q+D}$ where $\Delta = AD - BC \neq 0$. If $P/Q \in E$ is such that*

$$\left|\alpha - \frac{P}{Q}\right| < \left|\alpha - \frac{A}{C}\right|$$

*and if $R_1'$ or $S_1'$ is not a $q^{th}$ power of an element in $\mathbb{F}_q[x]$, then*

$$|b_\alpha(P,Q)| < |\Delta||\delta_1(P,Q)|^{-2}.$$

We now state our second main theorem.

**Theorem 5.** *Assume $\alpha \in \mathbb{F}_q\left((x^{-1})\right)$ is an irrational root of the equation*

$$xT^{q+1} + DT + x^\ell = 0, \tag{3.1}$$

*where $\ell \geq 1$, and $D \in \mathbb{F}_q[x]$ is such that $D(0) \neq 0$. If $|\alpha| \geq |x|^{-((q-1)\ell+1)}$, then $\alpha$ has bounded partial quotients.*

To prove the theorem, we need some more lemmas.

**Lemma 6.** *Let $A, B, C, D \in \mathbb{F}_q[x]$ be such that $\gcd(A,B,C,D) = 1$ and $AD - BC \neq 0$. Set $\Delta := AD - BC$. Suppose that there exists an irrational element $\alpha$ in $\mathbb{F}_q((x^{-1}))$ satisfying*

$$\alpha = \frac{A\alpha^q + B}{C\alpha^q + D}.$$

*Let $f$ be the linear fractional transformation defined by*

$$f(T) = \frac{AT + B}{CT + D}.$$

*For $P/Q \in E$ such that $P/Q \neq A/C$ and $(P/Q)^q \neq -D/C$, let*

$$R_1 := DP - BQ, \quad S_1 := -CP + AQ \tag{3.2}$$

$$R_2 := AP^q + BQ^q, \quad S_2 := CP^q + DQ^q. \tag{3.3}$$

*Then*

$$\alpha^q - \frac{R_1}{S_1} = f^{-1}(\alpha) - f^{-1}\left(\frac{P}{Q}\right) = \frac{\Delta(Q\alpha - P)}{(A - C\alpha)S_1} \tag{3.4}$$

*and*

$$\alpha - \frac{R_2}{S_2} = f(\alpha^q) - f\left(\left(\frac{P}{Q}\right)^q\right) = \frac{\Delta(Q\alpha - P)^q}{(C\alpha^q + D)S_2}. \tag{3.5}$$

*Proof.* Since $\Delta \neq 0$, we have $A/C \neq B/D$. Note that $\alpha \neq A/C$ since $\alpha$ is irrational. It is known that if $T \neq A/C$, then $f$ is invertible and $f^{-1}(T) = \frac{DT-B}{-CT+A}$. Note that

$$f(\alpha^q) = \frac{A\alpha^q + B}{C\alpha^q + D} = \alpha, \quad f^{-1}(\alpha) = \frac{D\alpha - B}{-C\alpha + A} = \alpha^q.$$

Recall that

$$|Q\alpha - P| = \frac{1}{|b_\alpha(P,Q)||Q|}. \tag{3.6}$$

Since $S_1 \neq 0$ and $S_2 \neq 0$, it is easily checked that

$$\frac{R_1}{S_1} = f^{-1}\left(\frac{P}{Q}\right), \quad \frac{R_2}{S_2} = f\left(\left(\frac{P}{Q}\right)^q\right).$$

For $T_1, T_2 \in \mathbb{F}_q((x^{-1}))\setminus\{-D/C, A/C\}$, we have

$$f(T_1)-f(T_2) = \frac{\Delta(T_1-T_2)}{(CT_1+D)(CT_2+D)}, \; f^{-1}(T_1)-f^{-1}(T_2) = \frac{\Delta(T_1-T_2)}{(A-CT_1)(A-CT_2)},$$

and the desired results follow. □

The next lemma yields more information when the convergent $P/Q$ is subject to certain restrictions.

**Lemma 7.** *Keeping the notation of Lemma 6, let* $P/Q \in E$.

*i) If* $|\alpha - P/Q| < |\alpha - A/C|$, *then*

$$|S_1\alpha^q - R_1| = |\Delta||b_\alpha(P,Q)|^{-1}|S_1|^{-1}. \tag{3.7}$$

*ii) If* $|\alpha - P/Q| < |\alpha^q + D/C|^{1/q}$, *then*

$$|S_2\alpha - R_2| = |\Delta||b_\alpha(P,Q)|^{-q}|S_2|^{-1}. \tag{3.8}$$

*Proof.* (i) Assume that $|\alpha - P/Q| < |\alpha - A/C|$. Then $|C(P/Q) - C\alpha| < |C\alpha - A|$. Therefore,

$$\left|C\left(\frac{P}{Q}\right) - A\right| = \left|C\left(\frac{P}{Q}\right) - C\alpha + C\alpha - A\right| = |C\alpha - A|. \tag{3.9}$$

By (3.2), we have $S_1 = -CP + AQ$ and thus

$$|S_1| = |Q|\left|C\left(\frac{P}{Q}\right) - A\right| = |Q|\left|C\alpha - A\right|.$$

Using (3.4), (3.6), and (3.9), the first assertion follows.

(ii) Assume that $|\alpha - P/Q| < |\alpha^q + D/C|^{1/q}$. Then $|C(P/Q)^q - C\alpha^q| < |C\alpha^q + D|$. Therefore,

$$\left|C\left(\frac{P}{Q}\right)^q + D\right| = \left|C\left(\frac{P}{Q}\right)^q - C\alpha^q + C\alpha^q + D\right| = |C\alpha^q + D|. \tag{3.10}$$

By (3.3), we have $S_2 = CP^q + DQ^q$ and thus

$$|S_2| = |Q|^q\left|C\left(\frac{P}{Q}\right)^q + D\right| = |Q|^q\left|C\alpha^q + D\right|.$$

Relations (3.5), (3.6), and (3.10) imply the second assertion. □

Observe that the relations (3.7) and (3.8) can be further refined by setting

$$\delta_1(P,Q) := \gcd(R_1,S_1) \quad \text{and} \quad \delta_2(P,Q) := \gcd(R_2,S_2).$$

From the two relations in (3.2), we see that $P$ and $Q$ satisfy

$$AR_1 + BS_1 = \Delta P, \quad CR_1 + DS_1 = \Delta Q.$$

Similarly, the two relations in (3.3) give

$$DR_2 - BS_2 = \Delta P^q, \quad -CR_2 + AS_2 = \Delta Q^q.$$

From $\delta_1(P,Q) \mid \Delta P$ and $\delta_1(P,Q) \mid \Delta Q$, since $\gcd(P,Q) = 1$, we have $\delta_1(P,Q) \mid \Delta$. Similarly, we have $\delta_2(P,Q) \mid \Delta P^q$ and $\delta_2(P,Q) \mid \Delta Q^q$. Since $\gcd(P^q,Q^q) = 1$, we have $\delta_2(P,Q) \mid \Delta$. Putting

$$R_1 = R_1'\delta_1(P,Q), \quad S_1 = S_1'\delta_1(P,Q),$$

and

$$R_2 = R_2'\delta_2(P,Q), \quad S_2 = S_2'\delta_2(P,Q),$$

the relations (3.7) and (3.8) become

$$|S_1'\alpha^q - R_1'| = |\Delta||b_\alpha(P,Q)|^{-1}|S_1'|^{-1}|\delta_1(P,Q)|^{-2} \tag{3.11}$$

and

$$|S_2'\alpha - R_2'| = |\Delta||b_\alpha(P,Q)|^{-q}|S_2'|^{-1}|\delta_2(P,Q)|^{-2},$$

respectively. We are now ready to prove Theorem 5.

*Proof of Theorem 5.* If $\alpha$ is an irrational root of the equation (3.1), then $\alpha$ satisfies $\alpha(x\alpha^q + D) = -x^\ell$, so we can write

$$\alpha = \frac{0 \cdot \alpha^q + (-x^\ell)}{x \cdot \alpha^q + D}.$$

Here, $A = 0, B = -x^\ell, C = x$ yielding $\Delta = AD - BC = x^{\ell+1}$ and $|\Delta| = |x|^{\ell+1}$. Assume $|\alpha| \geq |x|^{-((q-1)\ell+1)}$. Let $P/Q \in E$. By Lemma 4, we have

$$\left|\alpha - \frac{P}{Q}\right| < |\alpha| = \left|\alpha - \frac{A}{C}\right|$$

for all $P/Q \in E \setminus \{0\}$. Then, in this case, we have

$$R_1 = DP - BQ = DP + x^\ell Q \qquad \text{and} \quad S_1 = -CP + AQ = -xP.$$

Using the above setting, since $\delta_1(P,Q) \mid \Delta$, we can write $\delta_1(P,Q) = x^j$ where $j \in \{0, 1, \ldots, \ell+1\}$.

We observe that $x \nmid D$. Otherwise, there exists $D' \in \mathbb{F}_q[x]$ such that $D = xD'$. Then $D(0) = 0 \cdot D'(0) = 0$, which is a contradiction. Therefore, $\gcd(x, D) = 1$.

We now show that

$$\delta_1(P, Q) = 1 \text{ if and only if } x \nmid P.$$

To see this, assume $x \mid P$. Then $P = xP'$ for some $P' \in \mathbb{F}_q[x]$. Since $R_1 = DP + x^\ell Q = x(DP' + x^{\ell-1}Q)$, we get $x \mid R_1$. Likewise, $S_1 = -xP$, so $x \mid S_1$, and hence $x \mid \gcd(R_1, S_1)$, i.e., $\delta_1(P, Q) \neq 1$. To prove the converse, assume $\delta_1(P, Q) \neq 1$. Since $\delta_1(P, Q) \mid \Delta$, we write $\delta_1(P, Q) = x^j$ for some $j \in \{1, 2, \ldots, \ell + 1\}$. Then $R_1 = x^j R_1'$ and $S_1 = x^j S_1'$. Since $S_1 = -xP$, we obtain $x^j S_1' = -xP$, and hence $P = -x^{j-1}Q_1'$, i.e., $x^{j-1} \mid P$. For $j \in \{2, 3, \ldots, \ell+1\}$, we have $x \mid x^{j-1}$ so that $x \mid P$. For $j = 1$, we have $\delta_1(P, Q) = x$. Then $x \mid R_1$, i.e., $x \mid DP + x^\ell Q$. Since $x \mid x^\ell Q$, we get $x \mid DP$. As $\gcd(x, D) = 1$, it follows that $x \mid P$. Therefore, $\delta_1(P, Q) = 1$ if and only if $x \nmid P$.

Next, we show that for all $1 \leq j \leq \ell - 1$, we have

$$\delta_1(P, Q) = x^j \text{ if and only if } x^j \mid P,\ x^{j+1} \nmid P.$$

To prove this, let $1 \leq j \leq \ell - 1$. Assume $\delta_1(P, Q) = x^j$. Then $x^j \mid R_1$, i.e., $x^j \mid DP + x^\ell Q$. Since $x^j \mid x^\ell$, we have $x^j \mid x^\ell Q$, so $x^j \mid DP$. Since $\gcd(x^j, D) = 1$, we see that $x^j \mid P$. Next, we suppose that $x^{j+1} \mid P$. Since $x^{j+1} \mid x^\ell$ and $R_1 = DP + x^\ell Q$, it follows that $x^{j+1} \mid R_1$. Likewise, since $S_1 = -xP$ and $x^j \mid P$, we have $x^{j+1} \mid S_1$. Hence, $x^{j+1} \mid \gcd(R_1, S_1) = \delta_1(P, Q)$, contradicting $\delta_1(P, Q) = x^j$. Therefore, $x^{j+1} \nmid P$. Conversely, assume that $x^j \mid P$ and $x^{j+1} \nmid P$. Since $R_1 = DP + x^\ell Q$ and $S_1 = -xP$, we get $x^j \mid R_1$ and $x^j \mid S_1$. Then $x^j \mid \delta_1(P, Q)$. Next, we suppose that $x^{j+1} \mid \delta_1(P, Q)$. Then $x^{j+1} \mid R_1$, i.e., $x^{j+1} \mid DP + x^\ell Q$. Since $x^{j+1} \mid x^\ell$, we have $x^{j+1} \mid DP$, but $\gcd(x, D) = 1$ yielding $x^{j+1} \mid P$, a contradiction. Therefore, $x^{j+1} \nmid \delta_1(P, Q)$. Since $\delta_1(P, Q)$ is a power of $x$, we conclude that $\delta_1(P, Q) = x^j$.

To finish the proof, consider three possible cases.

`Case 1.` $j = 0$, i.e., $\delta_1(P, Q) = 1$. We have $x \nmid P$. So the constant term of the polynomial $P$ is nonzero, say $c_0$. Moreover, since $S_1' = S_1$ and $S_1 = -xP$, it follows that $S_1'$ is of the form $S_1' = \ldots - xc_0$. Therefore, $S_1'$ is not a $q^{th}$ power, so by Lemma 5, we have

$$|b_\alpha(P, Q)| < |x|^{\ell+1}.$$

`Case 2.` $\delta_1(P, Q) = x^j$ where $j \in \{1, 2, \ldots, \ell - 1\}$. We have $x^j \mid P$ and $x^{j+1} \nmid P$. Since $S_1 = S_1' x^j$ and $S_1 = -xP$,

$$S_1' = -x^{1-j} P = -x \left( \frac{P}{x^j} \right).$$

Since $x$ is irreducible and $P/x^j$ is not divisible by $x$, we see that $S_1'$ cannot be a $q^{th}$

power. Hence, by Lemma 5, we have

$$|b_\alpha(P,Q)| < |x|^{\ell+1}.$$

`Case 3.` $\delta_1(P,Q) = x^j$ where $j = \ell$ or $\ell+1$. Write $S_1 = x^j S_1'$ and $R_1 = x^j R_1'$ such that $x \nmid R_1'$ and $x \nmid S_1'$. The equation (3.11) becomes

$$|S_1'\alpha^q - R_1'| = |x|^{\ell-2j+1}|b_\alpha(P,Q)|^{-1}|S_1'|^{-1}.$$

Since $|x|^{\ell-2j+1}|b_\alpha(P,Q)|^{-1} < 1$, we see that $R_1'/S_1'$ is a convergent of $\alpha^q$ by Lemma 3. Hence, by Lemma 2, there exist $U, V \in \mathbb{F}_q[x]$ such that $U^q = R_1'$, $V^q = S_1'$, and

$$|V\alpha - U| = \sqrt[q]{|x|^{\ell-2j+1}|b_\alpha(P,Q)|^{-1}}|V|^{-1}.$$

Again, since $\sqrt[q]{|x|^{\ell-2j+1}|b_\alpha(P,Q)|^{-1}} < 1$, by Lemma 3, $U/V$ is a convergent of $\alpha$ . Since $|V\alpha - U| = 1/|b_\alpha(U,V)||V|$, we have

$$|b_\alpha(U,V)| = \sqrt[q]{|x|^{2j-\ell-1}|b_\alpha(P,Q)|}. \tag{3.12}$$

Let $U_1 := DU + x^\ell V$ and $V_1 := -xU$. Set $U_1 = U_1'\delta_1(U,V)$ and $V_1 = V_1'\delta_1(U,V)$, where $\delta_1(U,V) = \gcd(U_1, V_1)$. Since $x \nmid R_1'$, we get $x \nmid U$. Then $\delta_1(U,V) = 1$. By the same proof as in `Case 1`, we have $V_1'$ is not a $q^{th}$ power. Also, by Lemma 5, we have that

$$|b_\alpha(U,V)| < |\Delta||\delta_1(U,V)|^{-2}.$$

Since $\delta_1(U,V) = 1$, we see that

$$|b_\alpha(U,V)| \le |x|^\ell.$$

By Equation (3.12), we can conclude that $\sqrt[q]{|x|^{2j-\ell-1}|b_\alpha(P,Q)|} \le |x|^\ell$, i.e.,

$$|b_\alpha(P,Q)| \le |x|^{\ell-2j+1}|x|^{q\ell} = |x|^{(q+1)\ell-2j+1},$$

and so

$$|b_\alpha(P,Q)| \le |x|^{(q-1)\ell+1}.$$

The three cases show that $\alpha$ has bounded partial quotients. □

As a corollary of Theorem 5, we re-obtain a result of Baum and Sweet by taking $q = 2$ and $D = 1$.

**Corollary 2** ([2, Theorem 1, p. 598]). *Assume $\alpha \in \mathbb{F}_2((x^{-1}))$ is a root of the irreducible equation*

$$xT^3 + T + x = 0.$$

*Then $\alpha$ has bounded partial quotients with bounded degree at most 2.*

The method developed at the beginning of this section also gives a criterion for formal Laurent series to have unbounded partial quotients. Let

$$F = \left\{ \frac{P}{Q} \in E : \left| \alpha - \frac{P}{Q} \right| < \left| \alpha^q + \frac{D}{C} \right|^{1/q} \right\}.$$

Recall the following lemma of Lasjaunias [4].

**Lemma 8** ([4]). *Let $\alpha$ be an irrational element in $\mathbb{F}_q((x^{-1}))$ satisfying*

$$\alpha = \frac{A\alpha^q + B}{C\alpha^q + D}$$

*where $\Delta = AD - BC \neq 0$. Suppose that there exists $P/Q \in F$ such that*

$$\left| \alpha - \frac{P}{Q} \right| < |\Delta|^{\frac{1}{q-1}} |C\alpha^q + D|^{\frac{2}{q-1}} \quad \text{and} \quad |b_\alpha(P, Q)| > |\Delta|^{\frac{1}{q-1}}.$$

*Then $\alpha$ has unbounded partial quotients.*

We now state and prove our last main result.

**Theorem 6.** *Assume $\alpha \in \mathbb{F}_q\left((x^{-1})\right)$ is an irrational root of the equation*

$$xT^{q+1} + DT + x^\ell = 0,$$

*where $\ell \geq 1$, and $D \in \mathbb{F}_q[x]$ is such that $D(0) \neq 0$. If $|\alpha| < |x|^{-((q-1)\ell+1)}$, then $\alpha$ has unbounded partial quotients.*

*Proof.* Assume $|\alpha| < |x|^{-((q-1)\ell+1)} < 1$. By Lemma 4, we have

$$|\alpha| = \left| \alpha - \frac{P_0}{Q_0} \right| = \frac{1}{|b_\alpha(P_0, Q_0)||Q_0|^2}.$$

Therefore,

$$|b_\alpha(0,1)| = |b_\alpha(P_0, Q_0)| = |\alpha|^{-1} > |x|^{(q-1)\ell+1}.$$

Clearly, $|b_\alpha(0,1)|$ satisfies the second condition in Lemma 8 because

$$|x|^{(q-1)\ell+1} > (|x|^{\ell+1})^{\frac{1}{q-1}} = |\Delta|^{\frac{1}{q-1}}.$$

It remains to show that $P_0/Q_0$ is in $F$ and satisfies the first condition in Lemma 8. Since $|\alpha| < |x|^{-((q-1)\ell+1)} < 1$, we have

$$|x\alpha^q| < |x|^{-(q(q-1)\ell+q)+1} < 1.$$

Since $|D| \geq 1$, we have $|x\alpha^q| < |D|$, so that

$$|x\alpha^q + D| = |D|. \tag{3.13}$$

It follows that $|\alpha^q + D/x| = |D|/|x| \geq |x|^{-1}$, and thus

$$\left|\alpha^q + \frac{D}{x}\right|^{\frac{1}{q}} \geq |x|^{-\frac{1}{q}} > |x|^{-((q-1)\ell+1)} > \left|\alpha - \frac{P_0}{Q_0}\right|.$$

Hence, $P_0/Q_0 \in F$. From (3.13), we have $|x\alpha^q + D|^{2/(q-1)} = |D|^{2/(q-1)}$ and so

$$|\Delta|^{\frac{1}{q-1}} |x\alpha^q + D|^{\frac{2}{q-1}} > |x|^{-((q-1)\ell+1)} > |\alpha|,$$

i.e., $P_0/Q_0 \in F$ satisfies the first condition in Lemma 8. Therefore, the partial quotients of $\alpha$ are unbounded. $\square$

Applying Theorems 5 and 6 to the case $q = 2$, we obtain the following result of Lasjaunias.

**Corollary 3** ([4]). *Let $\ell \in \mathbb{N}$ and let $D \in \mathbb{F}_2[x]$ be such that $D(0) = 1$. If $\alpha \in \mathbb{F}_2\left((x^{-1})\right)$ is an irrational root of the equation*

$$xT^3 + DT + x^\ell = 0,$$

*then*

*i) for $|\alpha| \geq |x|^{-(\ell+1)}$, the partial quotients of $\alpha$ are bounded by $|x|^{\ell+1}$;*

*ii) for $|\alpha| < |x|^{-(\ell+1)}$, the partial quotients of $\alpha$ are unbounded.*

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## References

[1] K. Ayadi, S. Beldi, and K. Lee, Bounded partial quotients of some cubic power series with binary coefficients, *Bull. Korean Math. Soc.* **53** (4) (2016), 1005–1015.

[2] L. E. Baum and M. M. Sweet, Continued fractions of algebraic power series in characteristic 2, *Ann. of Math.* **103** (3) (1976), 593–610.

[3] N. Chitmung and T. Chaichana, Bounded partial quotients of quartic Laurent series over the finite field $\mathbb{F}_3$, *AMM2019 Proc.* (2019), 485–491.

[4] A. Lasjaunias, Continued fractions for algebraic formal power series over a finite base field, *Finite Fields Appl.* **5** (1) (1999), 46–56.

[5] W. H. Mills and D. P. Robbins, Continued fractions for certain algebraic power series, *J. Number Theory* **23** (3) (1986), 388–404.

[6] W. M. Schmidt, On continued fractions and diophantine approximation in power series fields, *Acta Arith.* **95** (2000), 139-166.