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THE UNION OF TWO ARITHMETIC PROGRESSIONS WITH DIFFERENT COMMON DIFFERENCES IS NOT AN MSTD SET

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Abstract

Let A be a nonempty finite set of integers. The *sumset* and the *difference set* of A are defined as $A+A=\{s+t : s,t\in A\}$ and $A-A=\{s-t : s,t\in A\}$, respectively. A set A is said to be *sum-dominant* or a *more-sum-than-difference* (MSTD) set, if $|A+A|>|A-A|$. Let $A=X_1\cup X_2$ with $X_1=\{a+id_1 : 0\le i\le n-1\}$ and $X_2=\{b+jd_2 : 0\le j\le m-1\}$, where a,b,i,j,d_1,d_2,n, and m are integers. Recently, Chu (J. Integer Seq., Art. 19.3.7, 2019) made a conjecture that the set $A=X_1\cup X_2$ is not an MSTD set. Soon after, Chu (Integers, #A87, 2020) confirmed the conjecture for the case $d_1=d_2$. In this article, we prove the conjecture for the case $d_1\ne d_2$ and $\max X_1<\min X_2$. We also confirm Chu's conjecture in the case $\max X_1>\min X_2$ with the exception of two unsolved cases.

1. Notation

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ be the set of positive integers, integers, rational numbers, and real numbers, respectively. Let $|A|$ denote the cardinality of the set A. For real

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numbers α and β, and a nonempty set of real numbers A, define

$$\alpha * A := \{\alpha a : a \in A\},$$
$$A + \beta := \{a + \beta : a \in A\}.$$

The greatest common divisor of the integers $x_1, \ldots, x_k$ is denoted by $d\,(x_1, \ldots, x_k)$. For $k \geq 2$, let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a finite set of integers with $a_0 < a_1 < \cdots < a_{k-1}$,

$$d(A - a_0) = d(a_1 - a_0, a_2 - a_0, \ldots, a_{k-1} - a_0),$$

and

$$A^{(N)} = \left\{ a_i' = \frac{a_i - a_0}{d(A - a_0)} : a_i \in A \right\}.$$

The set $A^{(N)}$ is called the *normal form* of A. Here $d(A^{(N)}) = 1$ and $\min(A^{(N)}) = 0$. In fact, every finite set of integers can be converted into a normal form. We also use the following notation in this article.

For a set A of real numbers, we use A^+ for the set $\{a \in A : a > 0\}$. For integers m and n with $m \leq n$, we use $[m, n]$ for the set $\{m, m+1, \ldots, n\}$. In particular, $I_n = \{0, 1, 2, \ldots, n-1\} = [0, n-1]$. A $(k+1)$-term arithmetic progression of real numbers whose first term is a and the common difference is d is denoted by $J(a, k, d)$, i.e., $J(a, k, d) = \{a + id : 0 \leq i \leq k\}$.

2. Introduction

Let A be a nonempty finite set of real numbers. Then the *sumset* and the *difference set* of A are defined as

$$A + A := \{a + b : a, b \in A\}$$

and

$$A - A := \{a - b : a, b \in A\},$$

respectively. A set A is said to be

- *sum-dominant* or *more sum than difference* (MSTD, in short), if $|A + A| > |A - A|$,
- *balanced*, if $|A + A| = |A - A|$,
- *difference-dominated*, if $|A + A| < |A - A|$.

In the group of integers, the fact that the addition of two elements a and b is commutative, while the subtraction may not be unless $a = b$, suggests that sum-dominant sets are rare. Roesler [14] proved that for integers n and k with $1 \leq k \leq n$, the average value of the quotient $|A - A|/|A + A|$, where A is a subset of $\{0, 1, \ldots, n\}$

with k elements, lies in the interval $[1, 2)$. However, a surprising result of Martin and O'Bryant [10] states that the proportion of MSTD subsets of $\{0, 1, \ldots, n-1\}$ is bounded below by a positive constant (about $2 \cdot 10^{-7}$) as $n \to \infty$. Later, Zhao improved the lower bound [17] to about $4 \cdot 10^{-4}$.

The existence of MSTD sets was an intriguing and unanswered question that dates back to 1960. It is believed that Conway gave the first example; $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$ of an MSTD set in 1969, but other initial examples were found by Marica [9] and Freiman and Pigarev [6]. Since then, MSTD sets have been an active research area (see [4, 7, 8, 9, 11, 12, 13, 15, 16] and references therein).

Recently, Chu [1, 2, 3, 5] gave some interesting families of sets that are not MSTD. In [1], he proposed the following conjecture.

Conjecture 1. The union of two arithmetic progressions of integers is not a sum-dominant set.

Let $x_1, x_2 \in \mathbb{Z}$ and $d_1, d_2, m, n \in \mathbb{N}$, and let $A = J(x_1, n, d_1) \cup J(x_2, m, d_2)$. Then Chu's conjecture says that A is not a sum-dominant set. Chu [3] settled the conjecture for the case $d_1 = d_2$. Hence, in this article, we assume that $d_1 \neq d_2$.

Let $A = X_1 \cup X_2$ be the union of two arithmetic progressions. It is easy to see that

$$((\alpha * A) + \beta) + ((\alpha * A) + \beta) = \alpha * (A + A) + 2\beta$$

and

$$((\alpha * A) + \beta) - ((\alpha * A) + \beta) = \alpha * (A - A)$$

for $\alpha, \beta \in \mathbb{R}$. Therefore, $|A + A|$ and $|A - A|$ are translation and dilation invariants. So, it is enough to solve the conjecture for the sets A of real numbers with $\min X_1 = 0$ and $d(X_1) = 1$. In particular, $A = X_1 \cup X_2$, where $X_1 = I_n$ and $X_2 = J(x, m, d)$ with $x, d \in \mathbb{R}$ and $m \in \mathbb{N}$. In Section 3, we prove Conjecture 1 when $\max X_1 < \min X_2$. In Section 4, we confirm the conjecture when $\max X_1 > \min X_2$ except for two unsolved cases mentioned in the next paragraph.

If

$$\frac{x_2 - x_1}{d_1} = s_1 + \frac{p_1}{q_1} \quad \text{and} \quad \frac{d_2}{d_1} = s_2 + \frac{p_2}{q_2},$$

where $s_i \in \mathbb{N} \cup \{0\}$, $p_i, q_i \in \mathbb{N}$, and $\gcd(p_i, q_i) = 1$ for $i \in \{1, 2\}$, then $X_1 \cup X_2$ is not an MSTD set except in the following two unsolved cases.

(i) $\gcd(q_1, q_2) = l > 1$ and $0 < d_2 < (n-1)d_1$,

(ii) $q_1 = 2$, $\gcd(2, q_2) = 1$, and $0 < d_2 < (n-1)d_1$.

3. When $\max X_1 < \min X_2$

Theorem 1. *The union of two finite arithmetic progressions* X_1 *and* X_2 *with* $\max X_1 < \min X_2$ *is not an MSTD set.*

Let $A = X_1 \cup X_2$. Without loss of generality, we can take $X_1 = I_n$ and $X_2 = J(x, m, d)$ with $x > n - 1$. Then

$$A = \{0, 1, \ldots, n-1\} \cup \{x + id : i \in [0, m]\}.$$

The cases $d = 1$, $1 \leq n \leq 2$, or $0 \leq m \leq 1$ were already proved in [1, 3]. So, we consider $n \geq 3$, $d > 1$, and $m \geq 2$. The proof of Theorem 1 is divided into four Subsections 3.1, 3.2, 3.3, and 3.4 with the help of a series of lemmas.

Since $A = \{0, 1, \ldots, n-1\} \cup \{x + id : i \in [0, m]\}$, we have

$$A + A \subseteq [0, 2n-2] \cup \Big(\bigcup_{i=0}^{m}(x + id + I_n)\Big) \cup \{2x + id : i \in [0, 2m]\} \tag{1}$$

and

$$[1, n-1] \cup \bigcup_{i=0}^{m}(x + id - I_n) \subseteq (A - A)^+. \tag{2}$$

If $x \in [n+1, 2n-2]$, then

$$A + A \subseteq [0, x+n-1] \cup \Big(\bigcup_{i=1}^{m}(x + id + I_n)\Big) \cup \{2x + id : i \in [0, 2m]\} \tag{3}$$

and

$$[1, x] \cup \bigcup_{i=1}^{m}(x + id - I_n) \subseteq (A - A)^+. \tag{4}$$

3.1. $A = I_n \cup J(x, m, d)$ with x, $d \in \mathbb{N}$

Lemma 1. *Let* $n \geq 3$, $m \geq 2$, *and* $d \geq 2$ *be integers. Let* $A = I_n \cup J(x, m, d)$ *with* $x \in [n+1, 2n-2]$. *Then* A *is not an MSTD set.*

Proof. We prove this in two cases.

`Case 1.` $d \leq n-1$. We have $A + A \subseteq [0, 2x + 2md]$ and so $|A + A| \leq 2x + 2md + 1$. Since $x + (i+1)d - (n-1) \leq x + id$ for $i \in [0, m-1]$, we have

$$(x + id - I_n) \cup (x + (i+1)d - I_n) = x + id + [-(n-1), d]$$

and

$$\bigcup_{i=0}^{m}(x + id - I_n) = [x - (n-1), x + md].$$

Therefore, using Relation (2), we have

$$(A - A)^+ = [1, n-1] \cup [x - (n-1), x + md] = [1, x + md]$$

and

$$|A - A| = 2x + 2md + 1 \geq |A + A|.$$

`Case 2.` $d \geq n$. Since $x + (i+1)d - (n-1) > x + id$ for $i \in [0, m-1]$, we have

$$(x + id - I_n) \cap (x + jd - I_n) = \emptyset$$

for $1 \leq i < j \leq m$, and $[1, x] \cap (x + id - I_n) = \emptyset$ for $i \in [1, m]$. Now, using Relations (3) and (4), we get

$$\begin{aligned} |A - A| - |A + A| &\geq (2x + 2mn + 1) - (x + n + mn + 2m + 1) \\ &= (x - n) + m(n - 2) > 0. \end{aligned}$$

This proves that A is not an MSTD set. $\square$

Lemma 2. *Let $n \geq 3$, $m \geq 2$, and $d \geq 2$ be integers. Let $A = I_n \cup J(x, m, d)$ with $x \geq 2n - 1$. Then A is not an MSTD set.*

Proof. We prove this in two cases.

`Case 1.` $2 \leq d \leq n - 1$. Since $x + (i+1)d - (n-1) \leq x + id$ for $i \in [0, m-1]$, we have

$$\bigcup_{i=0}^{m} (x + id + I_n) = [x, x + md + n - 1]$$

and

$$\bigcup_{i=0}^{m} (x + id - I_n) = [x - (n-1), x + md].$$

Also, $x \geq 2n - 1$ implies that $[1, n-1] \cap [x - (n-1), x + md] = \emptyset$. Now, using Relation (2), we get

$$[1, n-1] \cup [x - (n-1), x + md] \subseteq (A - A)^+,$$

where the sets on the left hand side are disjoint. Thus $|A - A| \geq 4n + 2md - 1$. Next, we find an upper bound for $|A + A|$. Using Relation (1), we get

$$A + A \subseteq [0, 2n-2] \cup [x, x + md + n - 1] \cup \{2x + id : i \in [0, 2m]\},$$

so $|A + A| \leq 3n + md + 2m$. Thus

$$\begin{aligned} |A - A| - |A + A| &\geq (4n + 2md - 1) - (3n + md + 2m) \\ &= (n - 1) + m(d - 2) > 0. \end{aligned}$$

`Case 2.` $d \geq n$. Since $(x + id - I_n) \cap (x + jd - I_n) = \emptyset$ for $0 \leq i < j \leq m$ and $I_n \cap (x + id - I_n) = \emptyset$ for $i \in [0, m]$, using Relations (1) and (2), we have

$$\begin{aligned} |A - A| - |A + A| &\geq (4n + 2mn - 1) - (3n + mn + 2m) \\ &= (n - 1) + m(n - 2) > 0. \end{aligned}$$

This proves that A is not an MSTD set. □

3.2. $A = I_n \cup J(x, m, d)$ with $x \notin \mathbb{N}$ and $d \in \mathbb{N}$

Lemma 3. *Let $n \geq 3$, $m \geq 2$, and $d \geq 2$ be integers. Let $A = I_n \cup J(x, m, d)$ with $x > n - 1$. Then A is not an MSTD set.*

Proof. Since $x \notin \mathbb{N}$, we have

$$[1, n - 1] \cap \bigcup_{i=0}^{m} (x + id - I_n) = \emptyset$$

and

$$[0, 2n - 2] \cap \bigcup_{i=0}^{m} (x + id + I_n) = \emptyset.$$

Now, by arguments similar to those used in Lemma 2, we can prove that A is not an MSTD set. □

3.3. $A = I_n \cup J(x, m, d)$ with $x \in \mathbb{N}$ and $d \notin \mathbb{N}$

Lemma 4. *Let $n \geq 3$ and $m \geq 2$ be integers. Let $d \in \mathbb{R} \setminus \mathbb{Q}$ be such that $d > 0$. Let $A = I_n \cup J(x, m, d)$ with $x \geq n + 1$. Then A is not an MSTD set.*

Proof. We have $(x + id - I_n) \cap (x + jd - I_n) = \emptyset$ for $1 \leq i < j \leq m$. Now, the rest of the proof is similar to the proof of Case 2 of both Lemma 1 and Lemma 2. So, A is not an MSTD set. □

Lemma 5. *Let $n \geq 3$ be an integer. Let $d \in \mathbb{Q} \setminus \mathbb{N}$ be such that $d > 0$ and $d = s + \frac{p}{q}$, where $\gcd(p, q) = 1$, $p < q$, $s \in \mathbb{N} \cup \{0\}$, and p and q are positive integers. Let $A = I_n \cup J(x, m, d)$ with $x \geq n + 1$. If either $m \in [2, q - 1]$ or $m \geq q$ and $qd > n - 1$, then A is not an MSTD set.*

Proof. We prove this in two cases.

`Case 1.` $n + 1 \leq x \leq 2n - 2$. We have $[1, n - 1] \cup (x - I_n) = [1, x]$. Since $d \notin \mathbb{N}$ and $2 \leq m \leq q - 1$, or $m \geq q$ and $qd > n - 1$, we have

$$[1, x] \cap \Big(\bigcup_{i=1}^{m} (x + id - I_n) \Big) = \emptyset.$$

Claim. For $0 \leq i < j \leq m$, we have $(x + id - I_n) \cap (x + jd - I_n) = \emptyset$.

Proof of the claim. If $(x+id-I_n) \cap (x+jd-I_n) \neq \emptyset$, then there exist $0 \leq i_0 < j_0 \leq m$ and $s_1, s_2 \in I_n$ such that $x + i_0 d - s_1 = x + j_0 d - s_2$. This gives

$$(j_0 - i_0)d = s_2 - s_1 \in \mathbb{Z}. \tag{5}$$

Now, consider the following two situations.
(i) If $m \in [2, q-1]$, then $j_0 - i_0 < m < q$. This contradicts (5) because q is the smallest positive integer such that $qd \in \mathbb{Z}$.
(ii) If $m \geq q$ and $qd > n - 1$, then $(s_2 - s_1) \leq n - 1 < qd \leq (j_0 - i_0)d$, which contradicts (5). This completes the proof of the claim.

Using Relations (3) and (4), we get

$$\begin{aligned}|A - A| - |A + A| &\geq (2x + 2mn + 1) - (2m + mn + x + n + 1)\\ &= (x - n) + m(n - 2) > 0.\end{aligned}$$

`Case 2.` $x \geq 2n - 1$. Since $d \notin \mathbb{N}$ and $2 \leq m \leq q - 1$, or $m \geq q$ and $qd > n - 1$, we have

$$[1, n-1] \cap \Big(\bigcup_{i=0}^{m} (x + id - I_n) \Big) = \emptyset.$$

Similar to the previous case, we have $(x+id-I_n) \cap (x+jd-I_n) = \emptyset$ for $0 \leq i < j \leq m$. Therefore, using Relations (1) and (2), we get

$$\begin{aligned}|A - A| - |A + A| &\geq (4n + 2mn - 1) - (3n + mn + 2m)\\ &= (n - 1) + m(n - 2) > 0.\end{aligned}$$

This proves that A is not an MSTD set. $\square$

Lemma 6. *Let $n \geq 3$ be an integer. Let $d \in \mathbb{Q} \setminus \mathbb{N}$ be such that $d = s + \frac{p}{q}$, where $\gcd(p, q) = 1$, $p < q$, $s \in \mathbb{N} \cup \{0\}$, and p and q are positive integers with $qd \leq n - 1$. Let $A = I_n \cup J(x, m, d)$ with $x \geq n + 1$ and $m \geq q$. Then A is not an MSTD set.*

Proof. Let $m = bq + c$, where $b \in \mathbb{N}$ and $0 \leq c < q$.

Claim 1. If $(x+id-I_n) \cap (x+jd-I_n) \neq \emptyset$ for $0 \leq i < j \leq m$, then $i \equiv j \pmod{q}$.

Proof of Claim 1. Assume that $(x + id - I_n) \cap (x + jd - I_n) \neq \emptyset$. Then there exist $0 \leq i_0 < j_0 \leq m$ and $s_1, s_2 \in I_n$ such that

$$x + i_0 d - s_1 = x + j_0 d - s_2.$$

This gives $(j_0 - i_0)d = s_2 - s_1 \in \mathbb{Z}$. Therefore, $j_0 - i_0$ is a multiple of q, i.e., $j_0 \equiv i_0 \pmod{q}$. So, for a fixed $i \in [0, c]$, the set $(x + id - I_n)$ can intersect only with

$$(x + (i+q)d - I_n), (x + (i+2q)d - I_n), \ldots, \text{ or } (x + (i+bq)d - I_n).$$

This completes the proof of the claim.

Let $A^* = I_n \cup \{x+id : i \in [0, q-1]\}$. By Lemma 5, we have $|A^*-A^*|-|A^*+A^*| > 0$. Consider

$$A_j = A^* \cup \{x+qd, x+(q+1)d, \dots, x+(q+j)d\}$$

for $j \in [0, m-q]$. Then $A_{m-q} = A$.

Claim 2. For $j \in [0, m-q]$, we have

$$|A_j + A_j| \le |A^* + A^*| + (j+1)(qd+2). \tag{6}$$

Proof of Claim 2. For the base case, we have

$$[x, x+n-1] \cup \{2x, 2x+d, \dots, 2x+(2q-2)d\} \subset A^* + A^*.$$

On the other hand,

$$\begin{aligned} x+qd+A^* &= (x+qd+I_n) \cup (x+qd+\{x+id : i \in [0,q-1]\}) \\ &= [x+qd, x+qd+n-1] \\ &\quad \cup \{2x+qd, 2x+(q+1)d, \dots, 2x+(2q-1)d\}. \end{aligned}$$

Since $[x, x+n-1] \subset A^*+A^*$, $[x+qd, x+qd+n-1] \subset x+qd+A^*$, and $qd \le n-1$, we have

$$[x+qd, x+qd+n-1] \setminus (A^*+A^*) \subset [x+n, x+qd+n-1].$$

Consequently, we have

$$(x+qd+A^*) \setminus (A^*+A^*) \subset [x+n, x+qd+n-1] \cup \{2x+(2q-1)d\}.$$

Therefore,

$$|A_0 + A_0| \le |A^* + A^*| + (qd+2).$$

Assume the claim holds for $j = k-1 < m-q$, i.e.,

$$|A_{k-1} + A_{k-1}| \le |A^* + A^*| + k(qd+2).$$

For $j = k$, we have

$$[x+kd, x+kd+n-1] \cup \{2x, 2x+d, \dots, 2x+(2q+2k-2)d\} \subset A_{k-1} + A_{k-1}.$$

On the other hand,

$$\begin{aligned} x+(q+k)d+A_{k-1} &= (x+(q+k)d+I_n) \cup (x+(q+k)d \\ &\qquad + \{x+id : i \in [0, q+k-1]\}) \\ &= [x+(q+k)d, x+(q+k)d+n-1] \\ &\quad \cup \{2x+(q+k)d, 2x+(q+k+1)d, \dots, 2x+(2q+2k-1)d\}. \end{aligned}$$

Since

$$\begin{aligned}[x+kd, x+kd+n-1] &\subset A_{k-1}+A_{k-1},\\ [x+(q+k)d, x+(q+k)d+n-1] &\subset x+(q+k)d+A_{k-1},\end{aligned}$$

and $qd \le n-1$, we have

$$x+(q+k)d, x+(q+k)d+n-1] \setminus (A_{k-1}+A_{k-1}) \subset [x+kd+n, x+(q+k)d+n-1].$$

Consequently, we have

$$\begin{aligned}(x+(q+k)d+A_{k-1}) \setminus (A_{k-1}+A_{k-1}) \subset\ & [x+kd+n, x+(q+k)d+n-1]\\ & \cup \{2x+(2q+2k-1)d\}.\end{aligned}$$

Therefore,

$$|A_k+A_k| \le |A_{k-1}+A_{k-1}| + (qd+2) \le |A^*+A^*| + (k+1)(qd+2).$$

This completes the proof of the claim.

Claim 3. For $j \in [0, m-q]$, we have

$$|A_j - A_j| \ge |A^* - A^*| + 2(j+1)qd. \tag{7}$$

Proof of Claim 3. For $(A^*-A^*)^+$, there are three components; however, we consider only two: $(I_n - I_n)^+ = [1, n-1]$ and

$$\begin{aligned}\{x+id : i \in [0, q-1]\} - I_n =\ & [x-(n-1), x] \cup [x+d-(n-1), x+d] \cup \cdots\\ & \cup [x+(q-1)d-(n-1), x+(q-1)d].\end{aligned}$$

On the other hand,

$$[x+1, x+qd] \subset [x+qd-(n-1), x+qd] \subset x+qd-A^*.$$

Since by Claim 1, $[x+1, x+qd]$ is disjoint from each of the above components of (A^*-A^*), we obtain

$$|A_0 - A_0| \ge |A^* - A^*| + 2qd.$$

Assume the claim holds for $j = k-1 < m-q$, i.e.,

$$|A_{k-1} - A_{k-1}| \ge |A^* - A^*| + 2kqd.$$

For $(A_{k-1} - A_{k-1})^+$, there are three components; however, we consider only two: $(I_n - I_n)^+ = [1, n-1]$ and

$$\begin{aligned}\{x+id : i \in [0, q+k-1]\} - I_n =\ & [x-(n-1), x] \cup [x+d-(n-1), x+d] \cup \cdots\\ & \cup [x+(q+k-1)d-(n-1), x+(q+k-1)d].\end{aligned}$$

On the other hand,

$$[x+kd+1, x+(q+k)d] \subset [x+(q+k)d-(n-1), x+(q+k)d] \subset x+(q+k)d-A_{k-1}.$$

By Claim 1, $[x+kd+1, x+(q+k)d]$ is disjoint from each of the above components of $(A_{k-1}-A_{k-1})$. So, we obtain

$$|A_k - A_k| \geq |A_{k-1} - A_{k-1}| + 2qd \geq |A^* - A^*| + 2(k+1)qd.$$

This completes the proof of the claim.

By (6) and (7), we have

$$\begin{aligned}|A_j - A_j| - |A_j + A_j| &\geq (|A^* - A^*| + 2(j+1)qd) - (|A^* + A^*| + (j+1)(qd+2)) \\ &= (|A^* - A^*| - |A^* + A^*|) + (qd-2)(j+1) > 0\end{aligned}$$

for $j \in [0, m-q]$ when $qd \neq 1$. Let $qd = 1$.

If $x \in [n+1, 2n-2]$, then by Case 1 of Lemma 5, we have

$$\begin{aligned}|A^* - A^*| - |A^* + A^*| + (1-2)(j+1) &\geq (x-n) + m(n-2) - j - 1 \\ &\geq (x-n-1) + m(n-2) + q - m \\ &\geq (x-n-1) + m(n-3) + q > 0.\end{aligned}$$

If $x \geq 2n-1$, then by Case 2 of Lemma 5, we have

$$\begin{aligned}|A^* - A^*| - |A^* + A^*| + (1-2)(j+1) &\geq (n-1) + m(n-2) - j - 1 \\ &\geq (n-2) + m(n-2) + q - m \\ &\geq (n-2) + m(n-3) + q > 0.\end{aligned}$$

This proves that A is not an MSTD set. □

3.4. $A = I_n \cup J(x, m, d)$ with $x, d \notin \mathbb{N}$

Lemma 7. *Let $n \geq 3$ be an integer. Let $x \in \mathbb{R} \setminus \mathbb{N}$ and $d \in \mathbb{R} \setminus \mathbb{Q}$ be such that $d > 0$ and $A = I_n \cup J(x, m, d)$ with $x > n-1$. Moreover, x and d are such that $x + id \notin \mathbb{N}$ for $i \in [0, m]$. Then A is not an MSTD set.*

Proof. Since $x + id \notin \mathbb{N}$ for $i \in [0, m]$, we have

$$[1, n-1] \cap \Big(\bigcup_{i=0}^{m} (x + id - I_n) \Big) = \emptyset.$$

Also, $(x + id - I_n) \cap (x + jd - I_n) = \emptyset$ for $0 \leq i < j \leq m$. Assume that

$$(x + id - I_n) \cap (x + jd - I_n) \neq \emptyset.$$

This gives that $d \in \mathbb{Q}$, a contradiction. Now, using Relations (1) and (2), we get

$$\begin{aligned}|A-A|-|A+A| &\geq (4n+2mn-1)-(3n+mn+2m)\\ &= (n-1)+m(n-2) > 0.\end{aligned}$$

This proves that A is not an MSTD set. □

Lemma 8. *Let $x \in \mathbb{R} \setminus \mathbb{N}$ and $d \in \mathbb{R} \setminus \mathbb{Q}$ be such that x and d are positive. If $x + id \in \mathbb{N}$ for $i \in \mathbb{N}$, then i is unique.*

Proof. Let $x + id \in \mathbb{N}$ for $i \in \mathbb{N}$. If there exist $i \neq j \in \mathbb{N}$ such that $x + jd \in \mathbb{N}$, then $(i-j)d \in \mathbb{Z}$. This gives that $d \in \mathbb{Q}$, a contradiction. This completes the proof of the lemma. □

Lemma 9. *Let $n \geq 3$ be an integer. Let $x \in \mathbb{R} \setminus \mathbb{N}$ and $d \in \mathbb{R} \setminus \mathbb{Q}$ be such that $d > 0$. Let $A = I_n \cup J(x, m, d)$ with $x > n-1$ and $x + jd \in \mathbb{N}$ for some $j \in [1, m]$. Then A is not an MSTD set.*

Proof. Using arguments similar to those used in Lemma 7, we have

$$(x + i_1 d - I_n) \cap (x + i_2 d - I_n) = \emptyset$$

for $0 \leq i_1 < i_2 \leq m$. Since $x + jd \in \mathbb{N}$ for some $j \in [1, m]$, we have j is unique by Lemma 8. Now, consider the following two cases.

`Case 1.` $x + jd > 2n - 2$. We have

$$[1, n-1] \cap \Big(\bigcup_{i=0}^{m} (x + id - I_n) \Big) = \emptyset.$$

So, using Relations (1) and (2), we have

$$\begin{aligned}|A-A|-|A+A| &\geq (4n+2mn-1)-(3n+mn+2m)\\ &= (n-1)+m(n-2) > 0.\end{aligned}$$

`Case 2.` $x + jd \leq 2n - 2$. We have $[1, n-1] \cup (x + jd - I_n) = [1, x + jd]$ and

$$[1, x+jd] \cap \Big(\bigcup_{j \neq i=0}^{m} (x + id - I_n) \Big) = \emptyset.$$

Thus,

$$|A - A| \geq 2(x + jd + mn) + 1.$$

Also, $[0, 2n-2] \cup (x + jd + I_n) = [0, x + jd + n - 1]$. Now, using Relation (1), we have

$$A + A \subseteq [0, x + jd + n - 1] \cup \Big(\bigcup_{j \neq i=0}^{m} (x + id + I_n) \Big) \cup \{2x + id : i \in [0, 2m]\}$$

and

$$|A+A| \leq x + jd + (m+1)n + 2m + 1.$$

Therefore,

$$\begin{aligned}|A-A| - |A+A| &\geq (2(x+jd+mn)+1) - (x+jd+(m+1)n+2m+1)\\ &= (x+jd-n) + m(n-2) > 0.\end{aligned}$$

This proves that A is not an MSTD set. ☐

Lemma 10. *Let $n \geq 3$ be an integer. Let $x \in \mathbb{R} \setminus \mathbb{N}$ and $d \in \mathbb{Q} \setminus \mathbb{N}$ be such that $d = s + \frac{p}{q}$, where $\gcd(p,q) = 1$, $p < q$, $s \in \mathbb{N} \cup \{0\}$, and p and q are positive integers. Let $A = I_n \cup J(x,m,d)$ with $x > n-1$. Moreover, x and d are such that $x + id \notin \mathbb{N}$ for $i \in [0,m]$. If either $m \in [2, q-1]$ or $m \geq q$ and $qd > n-1$, then A is not an MSTD set.*

Proof. Since $x + id \notin \mathbb{N}$ for $i \in [0,m]$, we have

$$[1, n-1] \cap \Big(\bigcup_{i=0}^{m} (x + id - I_n) \Big) = \emptyset.$$

Now, the rest of the proof is similar to the proof of Case 2 of Lemma 5. Therefore, A is not an MSTD set. ☐

Lemma 11. *Let $x \in \mathbb{R} \setminus \mathbb{N}$ and $d \in \mathbb{Q} \setminus \mathbb{N}$ be such that $d = s + \frac{p}{q}$, where $\gcd(p,q) = 1$, $p < q$, $s \in \mathbb{N} \cup \{0\}$, and p and q are positive integers. If $x + id \in \mathbb{N}$ for $i \in [1, q-1]$, then i is unique.*

Proof. Let $x + id \in \mathbb{N}$ for $i \in [1, q-1]$. If there exist $1 \leq j < i \leq q-1$ such that $x + jd \in \mathbb{N}$, then $(i-j)\frac{p}{q} \in \mathbb{N}$. This implies that q divides $(i-j)$, a contradiction. This completes the proof of the lemma. ☐

Lemma 12. *Let $n \geq 3$ be an integer. Let $x \in \mathbb{R} \setminus \mathbb{N}$ and $d \in \mathbb{Q} \setminus \mathbb{N}$ be such that $d = s + \frac{p}{q}$, where $\gcd(p,q) = 1$, $p < q$, $s \in \mathbb{N} \cup \{0\}$, and p and q are positive integers. Let $x + jd \in \mathbb{N}$ for some $j \in [1, q-1]$ and $A = I_n \cup J(x,m,d)$ with $x > n-1$. If either $m \in [2, q-1]$, or $m \geq q$ and $qd > n-1$, then A is not an MSTD set.*

Proof. Using arguments similar to those used in Lemma 5, we have

$$(x + i_1 d - I_n) \cap (x + i_2 d - I_n) = \emptyset$$

for $0 \leq i_1 < i_2 \leq m$. Also, $x + jd \in \mathbb{N}$, where $j \in [1, q-1]$ is unique by Lemma 11. This gives

$$(x + (j + j_1 q)d \pm I_n) \subseteq \mathbb{N}$$

for $j_1 \in \mathbb{N}$ and $1 \leq j + j_1 q \leq m$. Now, consider the following cases.

`Case 1.` $x + jd > 2n - 2$. The proof is similar to the proof of Case 1 of Lemma 9.

`Case 2.` $x + jd \leq 2n - 2$. We have $[1, n-1] \cup (x + jd - I_n) = [1, x + jd]$ and

$$[0, 2n-2] \cup (x + jd + I_n) = [0, x + jd + n - 1].$$

(i) Let $m \in [2, q-1]$. Since $[1, x + jd] \subseteq \mathbb{N}$,

$$[1, x + jd] \cap \Big(\bigcup_{j \neq i = 0}^{m} (x + id - I_n) \Big) = \emptyset.$$

So, using Relation (2), we have $|A - A| \geq 2(x + jd + mn) + 1$. Now, using Relation (1), we have

$$A + A \subseteq [0, x + jd + n - 1] \cup \Big(\bigcup_{j \neq i = 0}^{m} (x + id + I_n) \Big) \cup \{2x + id : i \in [0, 2m]\}.$$

Thus,

$$\begin{aligned} |A - A| - |A + A| &\geq (2(x + jd + mn) + 1) - ((x + jd + n) + mn + 2m + 1) \\ &= (x + jd - n) + m(n-2) > 0. \end{aligned}$$

(ii) Let $m \geq q$ and $qd > n - 1$. We have

$$[1, x + jd] \cap (x + (j + j_1 q)d - I_n) = \emptyset$$

for $j_1 \in \mathbb{N}$ and $1 \leq j + j_1 q \leq m$. Now, the remaining proof is similar to the proof of (i).

This proves that A is not an MSTD set. □

Lemma 13. *Let $n \geq 3$ be an integer. Let $x \in \mathbb{R} \setminus \mathbb{N}$ and $d \in \mathbb{Q} \setminus \mathbb{N}$ be such that $d = s + \frac{p}{q}$, where $\gcd(p, q) = 1$, $p < q$, $s \in \mathbb{N} \cup \{0\}$, and p and q are positive integers with $qd \leq n - 1$. Let $A = I_n \cup J(x, m, d)$ with $x > n - 1$ and $m \geq q$. Then A is not an MSTD set.*

Proof. The proof of the lemma is similar to the proof of Lemma 6. □

4. When $\min X_2 < \max X_1$

Theorem 2. *Let $X_1 = J(x_1, m_1, d_1)$ and $X_2 = J(x_2, m_2, d_2)$ with $\min X_2 < \max X_1$. If*

$$\frac{x_2 - x_1}{d_1} = s_1 + \frac{p_1}{q_1} \quad \text{and} \quad \frac{d_2}{d_1} = s_2 + \frac{p_2}{q_2},$$

where $s_i \in \mathbb{N} \cup \{0\}$, p_i and q_i are positive integers and $\gcd(p_i, q_i) = 1$ for $i \in \{1, 2\}$, then $X_1 \cup X_2$ is not an MSTD set except for the following two unsolved cases:

(i) $\gcd(q_1, q_2) = l > 1$ *and* $0 < d_2 < (m_1 - 1)d_1$,

(ii) $q_1 = 2$, $\gcd(2, q_2) = 1$, *and* $0 < d_2 < (m_1 - 1)d_1$.

Let $A = X_1 \cup X_2$. Without loss of generality, we can take $X_1 = I_n$ and $X_2 = J(x, m, d)$ with $x < n - 1$, i.e.,

$$A = \{0, 1, \ldots, n-1\} \cup \{x + id : i \in [0, m]\}.$$

The cases $d = 1$, $1 \leq n \leq 2$, or $0 \leq m \leq 1$ were already proved in [1, 3]. So, we consider $n \geq 3$, $d > 1$, and $m \geq 2$. The proof of Theorem 2 is divided into Subsections 4.1, 4.2, 4.3, and 4.4 with the help of a series of lemmas.

4.1. $A = I_n \cup J(x, m, d)$ with $x, d \in \mathbb{N}$

Lemma 14. *Let* $n \geq 3$, $m \geq 2$, *and* $d \geq 2$ *be integers. Let* $A = I_n \cup J(x, m, d)$ *with* $x \in [0, n-1]$. *Then* A *is not an MSTD set.*

Proof. If $J(x, m, d) \subseteq I_n$, then $A = I_n$, which is a balanced set. If there exists an $i \in [1, m]$ such that $x + id > n - 1$, then by Lemma 1 and Lemma 2, we see that A is not an MSTD set. This completes the proof of the lemma. ☐

4.2. $A = I_n \cup J(x, m, d)$ with $x \notin \mathbb{N}$ and $d \in \mathbb{N}$

Lemma 15. *Let* $n \geq 3$, $m \geq 2$, *and* $2 \leq d \leq n-1$ *be integers. Let* $A = I_n \cup J(x, m, d)$ *with* $x \in \mathbb{R} \setminus \mathbb{Q}$ *or* $x \in \mathbb{Q} \setminus \mathbb{N}$ *such that* $x = s + \frac{p}{q}$, $\gcd(p, q) = 1$, $p < q \neq 2$, $s \in \mathbb{N} \cup \{0\}$, *and* p *and* q *are positive integers with* $0 < x < n - 1$. *Then* A *is not an MSTD set.*

Proof. Let $c \in I_n$ be such that $c < x < c + 1$. For $2 \leq d \leq n - 1$, we have

$$\{x - c + i : 0 \leq i \leq md + c\} \subseteq \Big(\bigcup_{i=0}^{m} (x + id - I_n)\Big)^+ \tag{8}$$

and

$$\{i - x : c + 1 \leq i \leq n - 1\} \subseteq (x - I_n)^+.$$

We first prove that

$$\{x - c + i : 0 \leq i \leq md + c\} \cap \{i - x : c + 1 \leq i \leq n - 1\} = \emptyset.$$

Assume that for some $i \in [0, md + c]$ and $j \in [c + 1, n - 1]$, we have $x - c + i = j - x$. This gives that $2x \in \mathbb{Z}$, a contradiction.

Since $x \notin \mathbb{N}$ and $d \in \mathbb{N}$, we have $x + id \notin \mathbb{N}$ for $0 \leq i \leq m$. Therefore,

$$[1, n-1] \cap \Big(\bigcup_{i=0}^{m} (x + id - I_n)\Big) = \emptyset$$

and

$$|A-A| \geq 4n+2md-1.$$

Also, $d \leq n-1$, so, using Relation (1), we get $|A+A| \leq 3n+md+2m$. Therefore,

$$\begin{aligned}|A-A|-|A+A| &\geq (4n+2md-1)-(3n+md+2m)\\ &= (n-1)+m(d-2) > 0.\end{aligned}$$

This proves that A is not an MSTD set. □

Lemma 16. *Let $n \geq 3$, $m \geq 2$, and $2 \leq d \leq n-1$ be integers. Let $A = I_n \cup J(x,m,d)$ with $x = s+\frac{1}{2}$, $s \in \mathbb{N} \cup \{0\}$, and $0 < x < n-1$. Then A is not an MSTD set.*

Proof. Let $c, r \in I_n$ be such that $x = c+\frac{1}{2}$ and

$$x+rd < n-1 < x+(r+1)d.$$

Since $x+rd < n-1$, we have $c+rd \in I_n$. Now, consider the following cases.

`Case 1.` $c \geq \frac{n-1}{2}$. Using Inequality (8), which is also valid in this lemma, we get

$$[1,n-1] \cup \{x-c+i : 0 \leq i \leq md+c\} \subseteq (A-A)^+,$$

where the sets on the left-hand side are disjoint. Thus,

$$|A-A| \geq 2n+2md+2c+1.$$

Since $d \leq n-1$, using Relation (1), we get

$$\begin{aligned}|A-A|-|A+A| &\geq (2n+2md+2c+1)-(3n+md+2m)\\ &= (2c-n+1)+m(d-2) \geq 0.\end{aligned}$$

`Case 2.` $c < \frac{n-1}{2}$. Let

$$A^{(i)} = I_n \cup \{x+jd : 0 \leq j \leq i\},$$

where $i \in [0,r]$. Since $d \leq n-1$ and $x+id < n-1$, we have $2x+2id \leq 2n-3$. Therefore, using Relation (1), we get

$$A^{(i)}+A^{(i)} \subseteq [0,2n-2] \cup [x,x+id+n-1].$$

(i) Let $\max(A^{(i)}-A^{(i)}) = x+id$. Since $x+id \geq n-1-x$, Relation (8) gives

$$[1,n-1] \cup \{x-c+j : 0 \leq j \leq id+c\} \subseteq (A^{(i)}-A^{(i)})^+,$$

where the sets on the left-hand side are disjoint. Thus,

$$|A^{(i)} - A^{(i)}| \geq 2n + 2id + 2c + 1$$

and

$$\begin{aligned} |A^{(i)} - A^{(i)}| - |A^{(i)} + A^{(i)}| &\geq (2n + 2id + 2c + 1) - (3n + id - 1) \\ &\geq 2(c - x) + 1 = 0. \end{aligned}$$

If

$$A^{(r)} = I_n \cup \{x + id : 0 \leq i \leq r\},$$

then we have $|A^{(r)} - A^{(r)}| \geq |A^{(r)} + A^{(r)}|$. Consider

$$A_j = A^{(r)} \cup \{x + (r+1)d, x + (r+2)d, \ldots, x + (r+j)d\}$$

for $j \in [1, m - r]$. Then $A_{m-r} = A$. Using arguments similar to those used in Lemma 6, we can prove that

$$|A_j - A_j| \geq |A_j + A_j|$$

for $j \in [1, m - r]$.
(ii) Let $\max(A^{(i)} - A^{(i)}) = n - 1 - x$. Since $x + id \leq n - 1 - x$, we have

$$[1, n-1] \cup \{j - x : c + 1 \leq j \leq n - 1\} \subseteq (A^{(i)} - A^{(i)})^+,$$

where the sets on the left-hand side are disjoint. Thus,

$$|A^{(i)} - A^{(i)}| \geq 4n - 2c - 3$$

and

$$\begin{aligned} |A^{(i)} - A^{(i)}| - |A^{(i)} + A^{(i)}| &\geq (4n - 2c - 3) - (3n + id - 1) \\ &\geq 2(x - c) - 1 = 0. \end{aligned}$$

If $A^{(r+1)} = I_n \cup \{x + id : i \in [0, r+1]\}$, then $\max(A^{(r+1)} - A^{(r+1)}) = x + (r+1)d$.

Firstly, let $rd + d \leq n - 1$. Now, consider the following two situations.
(a) $2x + (2r+1)d \geq 2n - 1$. We have

$$[1, n-1] \cup \{x - c + i : 0 \leq i \leq (r+1)d + c\} \subseteq (A^{(r+1)} - A^{(r+1)})^+ \tag{9}$$

and

$$A^{(r+1)} + A^{(r+1)} \subseteq [0, 2n-2] \cup [x, x + (r+1)d + n - 1] \cup \{2x + (2r+1)d, 2x + (2r+2)d\}. \tag{10}$$

Since $x \notin \mathbb{N}$ and $d \in \mathbb{N}$, we have $x + id \notin \mathbb{N}$ for $0 \leq i \leq m$. Therefore,

$$[1, n-1] \cap \{x - c + i : 0 \leq i \leq (r+1)d + c\} = \emptyset.$$

Now, using Relations (9) and (10), we have

$$\begin{aligned} |A^{(r+1)} - A^{(r+1)}| - |A^{(r+1)} + A^{(r+1)}| &\geq (2n + (2r+2)d + 2c + 1) \\ &\quad - (3n + (r+1)d + 1) \\ &\geq n - rd - 2 \\ &\geq d - 1 \geq 0. \end{aligned}$$

(b) $2x + (2r+1)d \leq 2n - 2$. Since

$$A^{(r+1)} + A^{(r+1)} \subseteq [0, 2n-2] \cup [x, x + (r+1)d + n - 1] \cup \{2x + (2r+2)d\},$$

we have

$$\begin{aligned} |A^{(r+1)} - A^{(r+1)}| - |A^{(r+1)} + A^{(r+1)}| &\geq (2n + 2rd + 2d + 2c + 1) - (3n + rd + d) \\ &\geq n - rd - d - 1 \geq 0. \end{aligned}$$

Secondly, let $rd + d > n - 1$. Using Relation (10), we have

$$|A^{(r+1)} + A^{(r+1)}| \leq 3n + (r+1)d + 1.$$

Since $rd + d \in (A^{(r+1)} - A^{(r+1)})^{+}$, using Relation (9), we have

$$|A^{(r+1)} - A^{(r+1)}| \geq 2n + (2r+2)d + 2c + 3.$$

Therefore,

$$\begin{aligned} |A^{(r+1)} - A^{(r+1)}| - |A^{(r+1)} + A^{(r+1)}| &\geq (2n + (2r+2)d + 2c + 3) \\ &\quad - (3n + (r+1)d + 1) \\ &> 2c + 1 > 0. \end{aligned}$$

If

$$A^{(r+1)} = I_n \cup \{x + id : i \in [0, r+1]\},$$

then we have $|A^{(r+1)} - A^{(r+1)}| \geq |A^{(r+1)} + A^{(r+1)}|$. Consider

$$A_j = A^{(r+1)} \cup \{x + (r+2)d, x + (r+3)d, \ldots, x + (r+j+1)d\}$$

for $j \in [1, m-r-1]$. Then $A_{m-r-1} = A$. Using arguments similar to those used in Lemma 6, we can prove that

$$|A_j - A_j| \geq |A_j + A_j|$$

for $j \in [1, m-r-1]$. This proves that A is not an MSTD set. □

Lemma 17. *Let $n \geq 3$, $m \geq 2$, and $d \geq n$ be integers. Let $A = I_n \cup J(x, m, d)$ with $x \in \mathbb{R} \setminus \mathbb{N}$ such that $0 < x < n - 1$. Then A is not an MSTD set.*

Proof. Let $c \in I_n$ be such that $c < x < c + 1$. We have

$$[1, n-1] \cup \{x - c + i : 0 \leq i \leq c\} \cup \Big(\bigcup_{i=1}^{m} (x + id - I_n) \Big) \cup \{id : 1 \leq i \leq m\} \subseteq (A - A)^+.$$

Since $x \notin \mathbb{N}$, $d \in \mathbb{N}$, and $d \geq n$, the sets on the left-hand side are pairwise disjoint. Thus

$$|A - A| \geq 2n + 2c + 2mn + 2m + 1.$$

Now, using Relation (1), we get

$$\begin{aligned} |A - A| - |A + A| &\geq (2n + 2c + 2mn + 2m + 1) - (3n + mn + 2m) \\ &= n(m-1) + 2c + 1 > 0. \end{aligned}$$

This proves that A is not an MSTD set. □

4.3. $A = I_n \cup J(x, m, d)$ with $x \in \mathbb{N}$ and $d \notin \mathbb{N}$

This case is similar to Subsections 3.4 and 4.4. So, we omit the discussion.

4.4. $A = I_n \cup J(x, m, d)$ with $x, d \notin \mathbb{N}$

Lemma 18. *If any one of*

(i) *$x \in \mathbb{R} \setminus \mathbb{Q}$ and $d \in \mathbb{Q} \setminus \mathbb{N}$,*

(ii) *$x \in \mathbb{Q} \setminus \mathbb{N}$ and $d \in \mathbb{R} \setminus \mathbb{Q}$, or*

(iii) *$x \in \mathbb{Q} \setminus \mathbb{N}$ and $d \in \mathbb{Q} \setminus \mathbb{N}$ with $x = s_1 + \frac{p_1}{q_1}$, $d = s_2 + \frac{p_2}{q_2}$ with $s_i \in \mathbb{N} \cup \{0\}$, p_i and q_i are positive integers with $\gcd(p_i, q_i) = 1$ for $i \in \{1, 2\}$, and $\gcd(q_1, q_2) = 1$ with $q_1 \geq 3$,*

holds, then $x + id, 2x + jd \notin \mathbb{N}$ for $i, j \in \mathbb{N} \cup \{0\}$.

Proof. For (i) and (ii), if $x + id$ or $2x + jd \in \mathbb{N}$, then $x \in \mathbb{Q}$ or $d \in \mathbb{Q}$, respectively, which is a contradiction. For (iii), it is sufficient to prove the lemma only for $i, j \in [0, q_2]$. For $i = 0 = j$ or $i = q_2 = j$, this is trivial. If $x + id \in \mathbb{N}$ or $2x + jd \in \mathbb{N}$ for some $i, j \in [1, q_2 - 1]$, then

$$\frac{p_1}{q_1} + i\frac{p_2}{q_2} = n_1$$

or

$$\frac{2p_1}{q_1} + j\frac{p_2}{q_2} = n_2,$$

respectively, where $n_1, n_2 \in \mathbb{N}$. Thus, we have

$$p_1q_2 + ip_2q_1 = n_1q_1q_2$$

or

$$2p_1q_2 + jp_2q_1 = n_2q_1q_2,$$

respectively. Both of these cases imply q_2 divides q_1 and $\gcd(q_1, q_2) = q_2 \geq 2$, a contradiction. This completes the proof of the lemma. $\square$

Lemma 19. *Let $n \geq 3$ and $m \geq 2$ be integers. Let $A = I_n \cup J(x, m, d)$ with $0 < x < n - 1$. If any one of*

- (i) $x \in \mathbb{R} \setminus \mathbb{Q}$ *and* $d \in \mathbb{Q} \setminus \mathbb{N}$,
- (ii) $x \in \mathbb{Q} \setminus \mathbb{N}$ *and* $d \in \mathbb{R} \setminus \mathbb{Q}$, *or*
- (iii) $x \in \mathbb{Q}\setminus\mathbb{N}$ *and* $d \in \mathbb{Q}\setminus\mathbb{N}$ *with* $x = s_1 + \frac{p_1}{q_1}$, $d = s_2 + \frac{p_2}{q_2}$ *with* $s_i \in \mathbb{N} \cup \{0\}$, p_i *and* q_i *are positive integers with* $\gcd(p_i, q_i) = 1$ *for* $i \in \{1, 2\}$, *and* $\gcd(q_1, q_2) = 1$ *with* $q_1 \geq 3$,

holds, then A is not an MSTD set.

Proof. Let $c \in I_n$ be such that $c < x < c + 1$ and let $r \in [0, q_2 - 1]$ be the largest integer such that $x + rd < n - 1$. Let

$$A^* = I_n \cup \{x + id : 0 \leq i \leq r\}.$$

For $i \in [0, r]$, we have

$$\{x + id - \lfloor x + id\rfloor + j : j \in [0, \lfloor x + id\rfloor]\} \subseteq (x + id - I_n)^+ \tag{11}$$

and

$$\{\lfloor x + id\rfloor + j - (x + id) : j \in [1, n - \lfloor x + id\rfloor - 1]\} \subseteq (x + id - I_n)^+. \tag{12}$$

These subsets of $(x + id - I_n)^+$ are disjoint due to Lemma 18. Thus,

$$|A^* - A^*| \geq 4n + 2nr - 1.$$

Using Relation (1), we have $|A^* + A^*| \leq 3n + nr + 2r$. Therefore,

$$\begin{aligned} |A^* - A^*| - |A^* + A^*| &\geq (4n + 2nr - 1) - (3n + nr + 2r) \\ &= r(n - 2) + (n - 1) \geq 0. \end{aligned}$$

We first prove that $A^{'} = I_n \cup \{x + id : i \in [0, q_2 - 1]\}$ is not an MSTD set. If $r = q_2 - 1$, then we are done. If $r < q_2 - 1$, then $n - 1 < x + (r + 1)d$. This gives

$$(x + id - I_n) \subseteq (A' - A')^+$$

for $i \in [r+1, q_2-1]$. The subsets mentioned in Inequalities (11) and (12) and these sets of positive differences are disjoint due to Lemma 18. This gives

$$|A^{'} - A^{'}| \geq 2n + 2q_2n - 1.$$

Using Relation (1), we have

$$|A^{'} + A^{'}| \leq 2n + q_2n + 2q_2 - 2$$

and

$$\begin{aligned} |A^{'} - A^{'}| - |A^{'} + A^{'}| &\geq (2n + 2q_2n - 1) - (2n + q_2n + 2q_2 - 2) \\ &= q_2n - 2q_2 + 1 = q_2(n-2) + 1 \geq 0. \end{aligned}$$

Now, consider

$$A_i = A^{'} \cup \{x + q_2d, x + (q_2+1)d, \dots, x + (q_2 + i - 1)d\}$$

for $i \in [1, m - q_2 + 1]$. Then $A_{m-q_2+1} = A$. Using arguments similar to those used in Lemma 6, we can prove that $|A_i - A_i| \geq |A_i + A_i|$ for $i \in [1, m - q_2 + 1]$. This completes the proof of the lemma. ☐

Lemma 20. *Let* $x, d \in \mathbb{R} \backslash \mathbb{Q}$ *or* $x, d \in \mathbb{Q} \backslash \mathbb{N}$ *be such that* $x = s_1 + \frac{p_1}{q_1}$ *and* $d = s_2 + \frac{p_2}{q_2}$, *where* $s_i \in \mathbb{N} \cup \{0\}$, p_i *and* q_i *are positive integers, and* $\gcd(p_i, q_i) = 1$ *for* $i \in \{1, 2\}$. *Let* $\gcd(q_1, q_2) = l > 1$. *If* $x + id$ *or* $2x + jd \in \mathbb{N}$, *where* $i, j \in [1, q_2 - 1]$, *then* i *and* j *are unique with* $i \neq j$.

Proof. If $i = j$, then $x \in \mathbb{N}$, which is not possible. If $x + i_1d, x + i_2d \in \mathbb{N}$ for $i_1, i_2 \in [1, q_2 - 1]$ with $i_2 < i_1$, then

$$(x + i_1d) - (x + i_2d) \in \mathbb{N}$$

and so $d \in \mathbb{Q}$ or

$$(i_1 - i_2)\frac{p_2}{q_2} = n \in \mathbb{N}.$$

This implies that q_2 divides $(i_1 - i_2)$, which is not possible. The case $2x + jd$ can be dealt with the same way. This completes the proof of the lemma. ☐

Lemma 21. *Let* $n \geq 3$ *and* $m \geq 2$ *be integers. Let* $A = I_n \cup J(x, m, d)$ *with* $0 < x < n - 1$ *and* $d > n - 1$. *Let* $x, d \in \mathbb{R} \setminus \mathbb{Q}$ *or* $x, d \in \mathbb{Q} \setminus \mathbb{N}$ *be such that* $x = s_1 + \frac{p_1}{q_1}$ *and* $d = s_2 + \frac{p_2}{q_2}$, *where* $s_i \in \mathbb{N} \cup \{0\}$, p_i *and* q_i *are positive integers with* $\gcd(p_i, q_i) = 1$ *for* $i \in \{1, 2\}$. *Let* $\gcd(q_1, q_2) = l > 1$. *Then* A *is not an MSTD set.*

Proof. If $x + id$ and $2x + jd \notin \mathbb{N}$ for $i, j \in \mathbb{N}$, then the proof is the same as that of Lemma 19. Suppose that there exist i_1 and i_2 such that $x + i_1d, 2x + i_2d \in \mathbb{N}$ for

$i_1, i_2 \in [2, q_2-1]$. Also, $i_1 \neq i_2$ and i_1 and i_2 are unique due to Lemma 20. Since $2x+d \notin \mathbb{N}$ and $d > n-1$, we have

$$(x+id-I_n) \cap (x+jd-I_n) = \emptyset$$

for $0 \leq i < j \leq m$. Also, $x+d \notin \mathbb{N}$, therefore,

$$[1, n-1] \cap \Big(\bigcup_{i=0}^{m} (x+id-I_n) \Big) = \emptyset.$$

Thus, $|A-A| \geq 4n+2mn-1$. Using Relation (1), we get

$$\begin{aligned} |A-A| - |A+A| &\geq (4n+2mn-1) - (3n+mn+2m) \\ &= (n-1) + m(n-2) > 0. \end{aligned}$$

Now, consider the remaining cases as follows.

`Case 1.` $x+d \in \mathbb{N}$. Clearly, $2 \leq i_2 \leq q_2-1$. Consider $[1, n-1]$ and $(x+d-I_n)$. If $x+d > 2n-2$, then the proof is the same as the one given in the paragraph before Case 1. If $x+d \leq 2n-2$, then

$$[1, n-1] \cup (x+d-I_n) = [1, x+d]$$

and

$$[0, 2n-2] \cup (x+d+I_n) = [0, x+d+n-1].$$

Thus, $|A-A| \geq 2(x+d+mn)+1$ and

$$|A+A| \leq x+d+n+m(n+2)+1.$$

Now,

$$\begin{aligned} |A-A| - |A+A| &= (2x+2d+2mn+1) - (x+d+n+mn+2m+1) \\ &= (x+d-n) + m(n-2) > 0. \end{aligned}$$

`Case 2.` $2x+d \in \mathbb{N}$. Clearly, $2 \leq i_1 \leq q_2-1$. We have

$$[1, n-1] \bigcup_{i=0}^{m} (x+id-I_n) \bigcup \{id : 1 \leq i \leq m\} \subseteq A-A.$$

If the sets $\{i-x : c+1 \leq i \leq n-1\}$ and $(x+d-I_n)$ are disjoint, then the proof is the same as the one given in the paragraph before Case 1. If these sets intersect, then

$$\{i-x : c+1 \leq i \leq n-1\} \cup (x+d-I_n) = \{i-x : c+1 \leq i \leq 2x+d\}.$$

Now, consider the set $\{id : 1 \leq i \leq m\}$. Since $d > n-1$, we have $d > x$ and $d > (n-1) - x$. So $d \notin (x - I_n)$. Also,

$$\begin{gathered} d < x + d - (n-1) < \cdots < x + d < 2d \\ 2d < x + 2d - (n-1) < \cdots < x + 2d < 3d \\ \vdots \\ (m-1)d < x + (m-1)d - (n-1) < \cdots < x + (m-1)d < md. \end{gathered}$$

Thus,

$$|A - A| \geq 4x + 2d + 2m(n+1) + 1.$$

Now, using Relation (1), we get

$$\begin{aligned} |A - A| - |A + A| &\geq (4x + 2d + 2m(n+1) + 1) - (3n + mn + 2m) \\ &= 4x + n(m-3) + 2d + 1 > 0. \end{aligned}$$

This proves that A is not an MSTD set. □

Lemma 22. *Let $x = s_1 + \frac{1}{2}$ and $d = s_2 + \frac{p_2}{q_2}$, where $s_i \in \mathbb{N} \cup \{0\}$, and where p_i and q_i are positive integers for $i \in \{1, 2\}$. Let $\gcd(2, q_2) = 1$. Then $x + id \notin \mathbb{N}$ for $i \in \mathbb{N} \cup \{0\}$.*

The proof of Lemma 22 is similar to the proof of Lemma 18. So, we omit the proof.

Lemma 23. *Let $n \geq 3$ and $m \geq 2$ be integers. Let $A = I_n \cup J(x, m, d)$ with $0 < x < n-1$ and $d > n-1$. Let $x = s_1 + \frac{1}{2}$ and $d = s_2 + \frac{p_2}{q_2}$, where $s_i \in \mathbb{N} \cup \{0\}$, and where p_i and q_i are positive integers for $i \in \{1, 2\}$. Let $\gcd(2, q_2) = 1$. Then A is not an MSTD set.*

Proof. Let $c \in \mathbb{N}$ be such that $c < x < c+1$. Now, consider the following two cases.

`Case 1.` $x \geq \frac{n-1}{2}$. We have

$$\{x - c + i : 0 \leq i \leq c\} = (x - I_n)^+.$$

Since $d > n-1$, we have

$$(x + id - I_n) \cap (x + jd - I_n) = \emptyset$$

for $0 \leq i < j \leq m$. Also, due to Lemma 22, $x + id \notin \mathbb{N}$ for $0 \leq i \leq m$. This gives

$$[1, n-1] \cap \Big(\bigcup_{i=0}^{m} (x + id - I_n) \Big) = \emptyset.$$

Thus, $|A-A| \geq 2(n+c+mn)+1$. Now, using Relation (1), we get

$$\begin{aligned}|A-A|-|A+A| &= (2n+2c+2mn+1)-(3n+mn+2m)\\ &> m(n-2)-1>0.\end{aligned}$$

`Case 2.` $x<\frac{n-1}{2}$. We have

$$\{i-x : c+1 \leq i \leq n-1\} = (x-I_n)^+$$

and $n-1-x<n-1<x+id-(n-1)$ for $i\in[2,m]$. Also,

$$\{i-x : c+1 \leq i \leq n-1\} \cap (x+d-I_n) = \emptyset.$$

Otherwise, $d\in\mathbb{N}$, a contradiction. Thus,

$$|A-A| \geq 4n-2c+2mn-3.$$

Now, using Relation (1), we get

$$\begin{aligned}|A-A|-|A+A| &= (4n-2c+2mn-3)-(3n+mn+2m)\\ &> m(n-2)-2\geq 0.\end{aligned}$$

This proves that A is not an MSTD set. $\square$

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