#A59 INTEGERS 26 (2026)

KING PERMUTATIONS AND PARTIALLY ORDERED PATTERNS

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Received: 5/17/25, Revised: 11/19/25, Accepted: 3/10/26, Published: 5/1/26

Abstract

Finding distributions of statistics in pattern-avoiding permutations has attracted significant attention in the literature. In particular, partially ordered patterns (POPs) have been extensively studied in various contexts. In this paper, we extend the study of POPs to the domain of king permutations, introduced by Riordan in 1965 and later explored in a series of papers. A permutation $\sigma_1\sigma_2\cdots\sigma_n$ is called a king permutation if $|\sigma_{i+1}-\sigma_i|>1$ for each $1\leq i\leq n-1$. As the main results of this paper, we derive closed-form expressions for the generating functions that simultaneously account for four permutation statistics: ascents, descents, left-to-right maxima (or left-to-right minima), and right-to-left maxima (or right-to-left minima), on king permutations avoiding any flat POP of size 4. As a special case, we provide distributions of descents over length-4 flat POP-avoiding king permutations. Moreover, we discuss several restrictions that result in the non-existence of king permutations satisfying them. In particular, we note that no separable king permutations exist.

1. Introduction

A permutation of length n is a rearrangement of the set $[n]=\{1,2,\ldots,n\}$. Let S_n be the set of all permutations of $[n]$, and ε be the empty permutation. A permutation $\sigma=\sigma_1\sigma_2\cdots\sigma_n\in S_n$ avoids a pattern $P=P_1P_2\cdots P_k\in S_k$ if no subsequence $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ has the property that $\sigma_{i_j}<\sigma_{i_m}$ exactly when $P_j<P_m$. For example, the permutation 41235 avoids the pattern 321. Let $S_n(P)$ denote

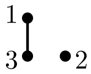
DOI: 10.5281/zenodo.19949822

the set of permutations in S_n that avoid pattern P. Let $\sigma^r = \sigma_n\sigma_{n-1}\cdots\sigma_1$ and $\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\cdots(n+1-\sigma_n)$ denote the *reverse* and *complement* of σ, respectively. For example, $(2341)^r = 1432$ and $(2341)^c = 3214$. If σ is any sequence of distinct numbers, then the *reduced form* of σ, denoted as $\mathrm{red}(\sigma)$, is constructed as follows: we replace the smallest element in σ with 1, the second smallest element with 2, and continue this process inductively. For instance, if $\sigma = 3524$, then $\mathrm{red}(\sigma) = 2413$.

For a permutation $\sigma = \sigma_1\cdots\sigma_n \in S_n$ and $1 \le i \le n-1$, we say that i is an *ascent* (resp., *descent*) in σ if $\sigma_i < \sigma_{i+1}$ (resp., $\sigma_i > \sigma_{i+1}$). We denote the number of ascents (resp., descents) in σ by $\mathrm{asc}(\sigma)$ (resp., $\mathrm{des}(\sigma)$). Moreover, an element σ_i in σ is a *right-to-left maximum* (resp., *left-to-right maximum*) if σ_i is greater than any element to the right (resp., to the left) of it. We use $\mathrm{rmax}(\sigma)$ and $\mathrm{lmax}(\sigma)$ to denote the number of right-to-left maxima and left-to-right maxima in σ, respectively. Analogously, an element σ_i in σ is a *right-to-left minimum* (resp., *left-to-right minimum*) if σ_i is smaller than any element to the right (resp., to the left) of it. We use $\mathrm{rmin}(\sigma)$ and $\mathrm{lmin}(\sigma)$ to denote the number of right-to-left minima and left-to-right minima in σ, respectively. In this paper, "g.f." stands for "generating function".

1.1. Partially Ordered Patterns

A *partially ordered pattern* (*POP*) P of length k is defined by a k-element partially ordered set (poset) M labeled by the elements in $\{1,\ldots,k\}$. An occurrence of the POP P within a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is a subsequence $\sigma_{i_1}\cdots\sigma_{i_k}$, subject to the conditions that $1 \le i_1 < \cdots < i_k \le n$ and $\sigma_{i_j} < \sigma_{i_m}$ if and only if the label j is less than the label m in the poset M. For example, the POP

$P =$ 

occurs two times in the permutation 24135, namely, as the subsequences 241 and 413. Clearly, avoiding P is the same as avoiding the patterns 231, 312, and 321 at the same time. POPs of the form in Figure 1 are *flat POPs*, and they are of interest in our paper. POPs have been studied in different contexts in permutations; see [4, 6, 8, 12, 13, 15, 14, 17, 20, 21].

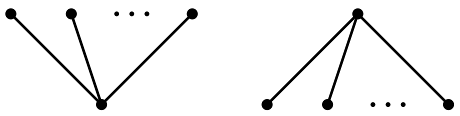


Figure 1: The form of flat POPs.

1.2. Separable Permutations

The *separable permutations* are those which can be built from the permutation 1 by repeatedly applying the *direct sum* $\oplus$ and *skew sum* $\ominus$ operations [7]. Bose, Buss and Lubiw [5] introduced the notion of separable permutation in 1998, and it is well-known [16] that the set of all separable permutations of length $n \geq 1$ is precisely $S_n(2413, 3142)$. Separable permutations appear in the literature in various contexts (see references in [7]).

Any $\sigma \in S_n(2413, 3142)$ has the following structure (see also Figure 2 for a schematic representation, where a permutation is viewed as a diagram with a dot in position (i, σ_i) for each element σ_i of σ):

$$\sigma = L_1 L_2 \cdots L_m n R_m R_{m-1} \cdots R_1$$

where

- for $1 \leq i \leq m$, L_i and R_i are non-empty ($\neq \varepsilon$), with possible exception for L_1 and R_m, separable permutations which are intervals in σ (that is, consist of all elements in $\{a, a+1, \dots, b\}$ for some a and b);
- $L_1 < R_1 < L_2 < R_2 < \cdots < L_m < R_m$, where $A < B$, for two permutations A and B, means that each element of A is less than every element of B. In particular, L_1, if it is not empty, contains 1.

For example, if $\sigma = 2165743$ then $L_1 = 21$, $L_2 = 65$, $R_1 = 43$, and $R_2 = \varepsilon$.

1.3. King Permutations

A permutation $\sigma \in S_n$ is a *king permutation*, or a *king n-permutation*, if $|\sigma_{i+1} - \sigma_i| > 1$ for each $1 \leq i \leq n-1$. Let K_n (resp., $K_n(P)$) be the set of all king n-permutations (resp., avoiding a pattern P). Let $K(P) := \cup_{n \geq 0} K_n(P)$. King permutations were studied in the literature in [1, 2, 9, 11, 18, 19]. Let A_n be the number of all king n-permutations. Riordan [19] derived a recurrence relation for A_n in 1965: $A_0 = A_1 = 1$, $A_2 = A_3 = 0$, and for $n \geq 4$,

$$A_n = (n+1)A_{n-1} - (n-2)A_{n-2} - (n-5)A_{n-3} + (n-3)A_{n-4}.$$

The initial values for A_n are

$$1, 1, 0, 0, 2, 14, 90, 646, 5242, 47622, 479306, 5296790, 63779034, \dots.$$

It is known [9] that for $n \geq 4$,

$$A_n = n! + \sum_{k=1}^{n} (-1)^k \sum_{i=1}^{k} \binom{k-1}{i-1} \binom{n-k}{i} 2^i (n-k)!.$$

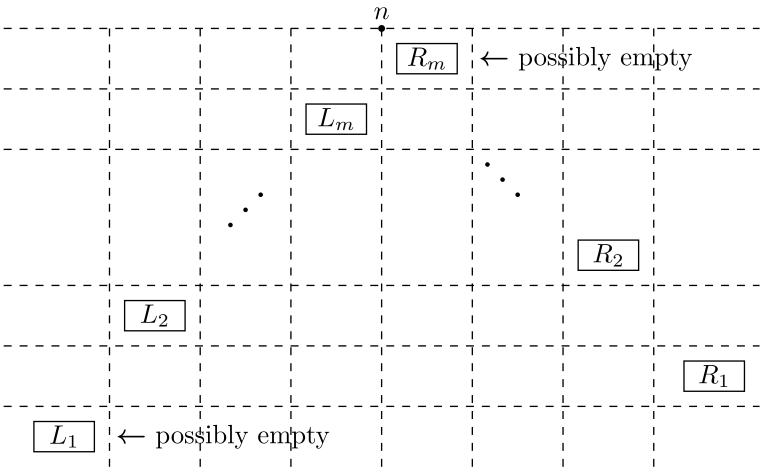


Figure 2: A schematic view of the permutation diagrams corresponding to separable permutations. Each L_i and R_j is a separable permutation.

Moreover, Flajolet and Sedgewick [11] showed that

$$A(t) = \sum_{n\geq 0} A_n t^n = \sum_{n\geq 0} \frac{n! t^n (1-t)^n}{(1+t)^n}.$$

1.4. Results in This Paper

As the main results of this paper, we derive closed-form expressions for the generating functions

$$A_P := A_P(x,p,q,u,v) = \sum_{n\geq 0} \sum_{\sigma \in K_n(P)} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)}$$

and

$$\bar{A}_P := \bar{A}_P(x,p,q,s,t) = \sum_{n\geq 0} \sum_{\sigma \in K_n(P)} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} s^{\mathrm{lmin}(\sigma)} t^{\mathrm{rmin}(\sigma)},$$

where $P \in \{P_1, P_2, P_3, P_4\}$ for A_P (resp., $P \in \{Q_1, Q_2, Q_3, Q_4\}$ for $\bar{A}_P$), and the patterns are given in Table 1 along with references. We only need to consider two cases given by Theorems 3 and 4; all other cases can be obtained by applying reverse and complement operations and renaming variables in A_P. As numerous specializations of our results, in particular, we give distributions of descents over $K(P)$ for length-4 POPs. Moreover, in Section 2.1, we discuss several restrictions that result in the non-existence of king permutations satisfying them. In particular, we show that no separable king permutations exist.

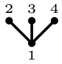
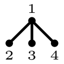
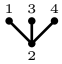
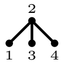
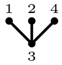
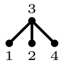
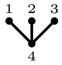
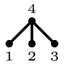
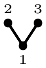
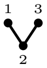
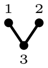
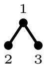
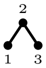
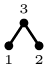
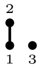
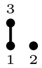
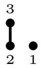
Name	POP P	Reference	Name	POP P	Reference
P_1		Theorem 3	Q_1		Corollary 2
P_2		Theorem 4	Q_2		Corollary 5
P_3		Corollary 4	Q_3		Corollary 6
P_4		Corollary 1	Q_4		Corollary 3

Table 1: The POPs for which we derive joint distributions of 4 statistics.

2. Pattern-Avoiding King Permutations

2.1. Non-Existence Results for King Permutations

Theorem 1. *For $n \geq 2$, we have $K_n(P) = \emptyset$ for P in the set*

$\{12, 21, 132, 213, 231, 312,$ , , , , , , , ,  $\}$.

Proof. The statement is clear for a length-2 pattern, since the only permutation avoiding it is not a king permutation.

Next, note that the reverse and/or complement of a king permutation is a king permutation. Additionally, if P is a POP in the set, then its avoidance involves avoidance of a non-monotone length-3 pattern. Hence, it is sufficient to prove the statement for $P = 132$. We do it by induction on n with easily checkable base cases $n = 2, 3, 4$ (no king permutation exists for $n = 2, 3$ and each king permutation of length 4 contains one occurrence of P). Now, if $n \geq 5$ and $\sigma = \sigma' n \sigma'' \in K_n(132)$ then every element in σ' is larger than any element in σ''. Hence, σ' (after reducing its elements) and σ'' are 132-avoiding king permutations of smaller size, and at least one of them is of size at least 2. However, by the induction hypothesis, such permutations do not exist, which completes the proof of our theorem. ☐

Remark 1. Note that P-avoiding king permutations for $P \in \{123, 321\}$ exist (see Problem 1 in Section 3).

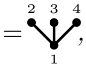
The following result appears as Theorem 3.10 in [1], but we provide an alternative proof.

Theorem 2 ([1])**.** *No separable king permutations of length $n \geq 2$ exist.*

Proof. We prove the statement by induction on n. No king permutation of length $n = 2, 3$ exists, and both king permutations 3142 and 2413 are the forbidden patterns in separable permutations. Suppose now that $n \geq 5$. Referring to the structure of separable permutations in Figure 2, we can assume that in a given separable king permutation σ, R_m is non-empty (if $R_m = \emptyset$, we can proceed similarly with $L_m \neq \emptyset$). The length of R_m must be at least 4, or else σ cannot be a king permutation. But by the induction hypothesis, no R_m can exist, which completes the proof of our theorem. $\square$

Remark 2. Note that Baxter king permutations exist (see Problem 2). Baxter permutations are a natural superset of non-separable permutations (see Section 3 for a precise definition).

2.2. King Permutations in $K_n(P_1)$, $K_n(P_4)$, $K_n(Q_1)$, $K_n(Q_4)$

Theorem 3. *For* $P_1 =$ 

, we have

$$A_{P_1} = \frac{A}{B}, \tag{1}$$

where

$$\begin{aligned}
A =& -1 - uvx + pqvx^2 + (pq^2v + pquv^2)x^3 + (p^2q^2v^2 + pq^3v^2 + pq^2uv^2 - p^2qu^2v^2 - pq^2u^2v^2)x^4\\
&+ (p^2q^3v^2 - p^2q^2u^2v^2 + 2p^2q^3v^3 + pq^4v^3 - 2p^2q^2u^2v^3 - pq^3u^2v^3)x^5\\
&+ (2p^2q^4v^3 - 2p^2q^3u^2v^3 + p^2q^4v^4 - p^2q^3u^2v^4)x^6 + (p^2q^5v^4 - p^2q^4u^2v^4)x^7;\\
B =& -1 + pqvx^2 + pq^2vx^3 + (p^2q^2v^2 + pq^3v^2)x^4 + (p^2q^3v^2 + 2p^2q^3v^3 + pq^4v^3)x^5\\
&+ (2p^2q^4v^3 + p^2q^4v^4)x^6 + p^2q^5v^4x^7.
\end{aligned}$$

Proof. We define the following g.f.'s:

$$B_{P_1} = \sum_{n\geq 1} \sum_{\substack{\sigma\in K_n(P_1)\\ \sigma_n=1}} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)};$$

$$C_{P_1} = \sum_{n\geq 2} \sum_{\substack{\sigma\in K_n(P_1)\\ \sigma_{n-1}=1}} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)};$$

$$D_{P_1} = \sum_{n\geq 3} \sum_{\substack{\sigma\in K_n(P_1)\\ \sigma_{n-2}=1}} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)}.$$

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in K_n(P_1)$. When $n = 0$, the empty permutation ε makes a contribution of 1 to the function A_{P_1}. When $n = 1$, the only permutation 1 contributes the term uvx to A_{P_1}. For the lengths $n = 2$ and $n = 3$, no king

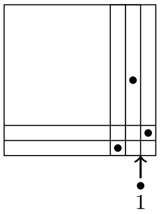


Figure 3: Related to case (b) in the proof of Theorem 3

permutation exists. For $n \geq 4$, to avoid the POP P_1, it is necessary that either $\sigma_n = 1$, $\sigma_{n-1} = 1$, or $\sigma_{n-2} = 1$. Otherwise, the element 1, along with some elements to its right, will form an occurrence of P_1. Hence, we have

$$A_{P_1} = 1 + uvx + B_{P_1} - uvx + C_{P_1} + D_{P_1},$$

which leads to

$$A_{P_1} - B_{P_1} - C_{P_1} - D_{P_1} = 1. \tag{2}$$

For $n \geq 1$, we now proceed to consider three distinct scenarios.

`Case 1:` $\sigma_n = 1$. We have $\sigma = \sigma' 1$, where $\mathrm{red}(\sigma') \in A_{n-1}(P_1)$ and σ' does not end with 2. For $n = 1$, the sole permutation results in the term uvx. Let $n \geq 2$. Clearly, king permutations either end with the smallest element or do not end with the smallest element. So, the g.f. for σ' is $A_{P_1} - B_{P_1} - 1$. Also, σ_n contributes qvx, since it is a right-to-left maximum and forms a descent with σ_{n-1}. Hence, the g.f. for such σ's is

$$B_{P_1} = uvx + qvx(A_{P_1} - B_{P_1} - 1),$$

which leads to

$$qvxA_{P_1} - (1 + qvx)B_{P_1} = qvx - uvx. \tag{3}$$

`Case 2:` $\sigma_{n-1} = 1$. We consider the following way to form σ. Take a permutation $\pi = \pi_1\pi_2 \cdots \pi_{n-1} \in S_{n-1}$ and insert the element 1 into it as shown in Figure 3. As a result, we obtain $\sigma = (\pi_1 + 1)(\pi_2 + 1) \cdots (\pi_{n-2} + 1)1(\pi_{n-1} + 1)$.

For σ to be a king permutation avoiding P_1, π must avoid P_1. Also, we must have $\pi_{n-2} \neq 1$ and $\pi_{n-1} \neq 1$, which implies that $\pi_{n-3} = 1$.

Moreover, we can conclude that $\pi_{n-1} = 2$. Indeed, otherwise, because 1 and 2 cannot be adjacent in π, $2\pi_{n-4}\pi_{n-2}\pi_{n-1}$ forms an occurrence of P_1.

When $n = 4$, the term corresponding to $\sigma = 2413$ is $p^2qu^2v^2x^4$. We consider two subcases for $n \geq 5$.

`Subcase (i):` π is a king permutation. Given that $\pi_{n-3} = 1$ and $\pi \in K_{n-1}$, the g.f. for such π's is D_{P_1}. When we insert the element 1 into π, the element 1

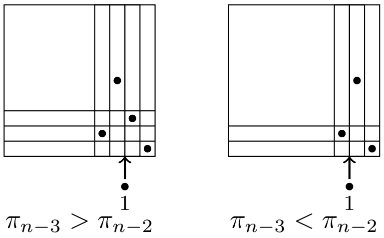


Figure 4: Related to case (c) in the proof of Theorem 3

contributes a factor of px. Consequently, the corresponding g.f. for σ is

$$\sum_{n\geq 5} x^n p^{\text{asc}(\sigma)} q^{\text{des}(\sigma)} u^{\text{lmax}(\sigma)} v^{\text{rmax}(\sigma)} = pxD_{P_1}.$$

`Subcase (ii):` π is almost a king permutation, that is, $|\pi_{i+1} - \pi_i| > 1$ for each $1 \leq i \leq n-3$ and $|\pi_{n-1} - \pi_{n-2}| = 1$. We have that $\pi = \pi_1\pi_2\cdots\pi_{n-4}132$. The contribution of the last four elements 2413 in σ is $p^2q^2v^2x^4$, and the g.f. for $\pi_1\pi_2\cdots\pi_{n-4}$ is $A_{P_1} - 1$. Thus, the corresponding g.f. for σ is

$$\sum_{n\geq 5} x^n p^{\text{asc}(\sigma)} q^{\text{des}(\sigma)} u^{\text{lmax}(\sigma)} v^{\text{rmax}(\sigma)} = p^2q^2v^2x^4(A_{P_1} - 1).$$

Summarizing the two cases above for $\sigma_{n-1} = 1$, we have

$$C_{P_1} = p^2qu^2v^2x^4 + pxD_{P_1} + p^2q^2v^2x^4(A_{P_1} - 1).$$

After rearranging, we obtain:

$$p^2q^2v^2x^4A_{P_1} - C_{P_1} + pxD_{P_1} = p^2q^2v^2x^4 - p^2qu^2v^2x^4. \tag{4}$$

`Case 3:` $\sigma_{n-2} = 1$. Consider inserting 1 into $\pi = \pi_1\pi_2\cdots\pi_{n-1} \in S_{n-1}$ as shown in Figure 4. After this insertion, the permutation becomes

$$\sigma = (\pi_1 + 1)(\pi_2 + 1)\cdots(\pi_{n-3} + 1)1(\pi_{n-2} + 1)(\pi_{n-1} + 1).$$

Since π must avoid P_1, and $\pi_{n-2} \neq 1$ and $\pi_{n-3} \neq 1$, it follows that $\pi_{n-1} = 1$.

For $n = 4$, the term associated with σ is $pq^2u^2v^2x^4$. For $n = 5$, we have $\sigma = 35142$ or $\sigma = 53142$ contributing the terms $p^2q^2u^2v^3x^5$ and $pq^3uv^3x^5$, respectively. For $n \geq 6$, we consider the following two subcases depending on whether π is a king permutation or not.

`Subcase (a):` For a king permutation π, assume $\pi = \pi'1$, and the contribution of the element 1 to π is qvx.

When $\pi_{n-3} > \pi_{n-2}$ (as shown schematically in Figure 4 to the left), it is required that $\pi_{n-4} = 2$. Otherwise, the element 2, along with the three elements to its right, will form an occurrence of the pattern P_1. The g.f. for such π' is D_{P_1}. Thus, the g.f. for π is $qvxD_{P_1}$. After inserting the element 1 into π, the contribution of the inserted element 1 to π is px. Therefore, the corresponding g.f. for σ is

$$\sum_{n\geq 6} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)} = pqvx^2 D_{P_1}.$$

Likewise, when $\pi_{n-3} < \pi_{n-2}$ (as shown schematically in Figure 4 to the right), we can conclude that $\pi_{n-3} = 2$. Then, the g.f. for π' is C_{P_1}. As a result, the g.f. of π is $qvxC_{P_1}$. After inserting the element 1, the contribution of this 1 to the permutation π is qx. So, the g.f. for σ is

$$\sum_{n\geq 6} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)} = q^2 vx^2 C_{P_1}.$$

`Subcase (b):` π is almost a king permutation. Specifically, π satisfies $|\pi_{i+1} - \pi_i| > 1$ for every $1 \leq i \leq n-2$, with the sole exception that $|\pi_{n-2} - \pi_{n-3}| = 1$.

When $\pi_{n-3} > \pi_{n-2}$ (as shown schematically in Figure 4 to the left), it is necessary that $\pi_{n-4} = 2$ and $\pi_{n-2} = 3$. Otherwise, the element 3 will combine with three elements to its right to form the pattern P_1. In this situation, π can be expressed as $\pi = \pi_1\pi_2\cdots\pi_{n-5}2431$. The g.f. for $\pi_1\pi_2\cdots\pi_{n-5}$ is $A_{P_1} - 1$. Hence, after inserting the element 1 into π, the contribution of the last five elements 35142 in σ is $p^2q^3v^3x^5$. Consequently, the corresponding g.f. for σ is

$$\sum_{n\geq 6} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)} = p^2q^3v^3x^5(A_{P_1} - 1).$$

Likewise, when $\pi_{n-3} < \pi_{n-2}$ (as shown schematically in Figure 4 to the right), we have $\pi_{n-3} = 2$. Then, π takes the form $\pi = \pi_1\pi_2\cdots\pi_{n-4}231$. The g.f. for $\pi_1\pi_2\cdots\pi_{n-4}$ is $A_{P_1} - 1 - uvx$. Hence, the contribution of the last four elements 3142 in σ is $pq^3v^2x^4$. Thus, the g.f. for σ is

$$\sum_{n\geq 6} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)} = pq^3v^2x^4(A_{P_1} - 1 - uvx).$$

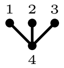
Summarizing the cases above for $n \geq 4$, we have

$$\begin{aligned} D_{P_1} &= pq^2u^2v^2x^4 + (pq^3uv^3 + p^2q^2u^2v^3)x^5 + q^2vx^2C_{P_1} \\ &\quad + pqvx^2D_{P_1} + p^2q^3v^3x^5(A_{P_1} - 1) + pq^3v^2x^4(A_{P_1} - 1 - uvx). \end{aligned}$$

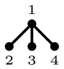
After rearranging, we obtain:

$$\begin{aligned} &(pq^3v^2x^4 + p^2q^3v^3x^5)A_{P_1} + q^2vx^2C_{P_1} + (pqvx^2 - 1)D_{P_1} \\ &\qquad = (pq^3v^2 - pq^2u^2v^2)x^4 + (p^2q^3v^3 - p^2q^2u^2v^3)x^5. \end{aligned} \tag{5}$$

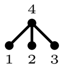
By simultaneously solving Equations (2)–(5), we obtain the desired result. ☐

Corollary 1. *For* $P_4 =$  *, we have* $A_{P_4}(x,p,q,u,v) = A_{P_1}(x,q,p,v,u)$.

Proof. By reversing P_1-avoiding permutations, we obtain P_4-avoiding permutations and observe that ascents and descents, as well as right-to-left maxima and left-to-right maxima, are interchanged under the reversal. ☐

Corollary 2. *For* $Q_1 =$  *, we have* $\bar{A}_{Q_1}(x,p,q,s,t) = A_{P_1}(x,q,p,s,t)$.

Proof. By complementing P_1-avoiding permutations, we obtain Q_1-avoiding permutations and observe that ascents and descents are interchanged under the complement, while right-to-left maxima (resp., left-to-right maxima) are transformed into right-to-left minima (resp., left-to-right minima). ☐

Corollary 3. *For* $Q_4 =$  *, we have* $\bar{A}_{Q_4}(x,p,q,s,t) = A_{P_1}(x,p,q,t,s)$.

Proof. By reversing and complementing P_1-avoiding permutations, we obtain Q_4-avoiding permutations and observe that ascents and descents remain unchanged under the composition of these operations, while right-to-left maxima (resp., left-to-right maxima) are transformed into left-to-right minima (resp., right-to-left minima). ☐

We end this subsection by providing several specializations obtained by setting the respective variables in Equation (1) to 1 starting with the generating function for P_1-avoiding (equivalently, P_4-avoiding, Q_1-avoiding, or Q_4-avoiding) king permutations:

$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_1)} x^n = \frac{-1-x+x^2+2x^3+x^4}{-1+x^2+x^3+2x^4+4x^5+3x^6+x^7}$$

$$= 1+x+2x^4+6x^5+9x^6+12x^7+20x^8+41x^9+80x^{10}+\cdots;$$

$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_1)} x^n q^{\mathrm{des}(\sigma)} =$$

$$\frac{-1-x+qx^2+q(1+q)x^3+q(q^2+q-1)x^4+q^2(q^2+2q-3)x^5+3q^3(q-1)x^6+q^4(q-1)x^7}{-1+qx^2+q^2x^3+q^2(q+1)x^4+q^3(q+3)x^5+3q^4x^6+q^5x^7};$$

$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_1)} x^n u^{\mathrm{lmax}(\sigma)} =$$

$$\frac{-1-ux+x^2+(u+1)x^3+(2+u-2u^2)x^4+4(1-u^2)x^5+3(1-u^2)x^6+(1-u^2)x^7}{-1+x^2+x^3+2x^4+4x^5+3x^6+x^7};$$

$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_1)} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} =$$

$$\frac{\begin{array}{l}-1-x+pqx^2+pq(1+q)x^3+pq(p(q-1)+q^2)x^4\\ \qquad +p(q-1)q^2(3p+q)x^5+3p^2(q-1)q^3x^6+p^2(q-1)q^4x^7\end{array}}{-1+p^2q^2x^4(1+qx)^3+pqx^2(1+qx+q^2x^2+q^3x^3)};$$

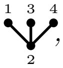
$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_1)} x^n u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)} =$$

$$\frac{\begin{array}{l}-1-uvx+vx^2+v(1+uv)x^3+v^2(2+u-2u^2)x^4+v^2(1+3v-3u^2v-u^2)x^5\\ \qquad +v^3(2-2u^2+v-u^2v)x^6+v^4(1-u^2)x^7\end{array}}{-1+vx^2+vx^3+2v^2x^4+v^2(3v+1)x^5+v^3(2+v)x^6+v^4x^7};$$

$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_1)} x^n q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} =$$

$$\frac{\begin{array}{l}-1-ux+qx^2+q(q+u)x^3+q\left(q+q^2+qu-u^2-qu^2\right)x^4\\ \qquad +q^2\left(q^2+3q-3u^2-qu^2\right)x^5+3q^3(q-u^2)x^6+q^4(q-u^2)x^7\end{array}}{-1+qx^2+q^2x^3+q^2(1+q)x^4+q^3(q+3)x^5+3q^4x^6+q^5x^7}.$$

2.3. King Permutations in $K_n(P_2)$, $K_n(P_3)$, $K_n(Q_2)$, $K_n(Q_3)$

Theorem 4. *For* $P_2 =$ 

, we have

$$A_{P_2} = \frac{A}{B}, \tag{6}$$

where

$$\begin{aligned} A =& -1-uvx+(pqu+pquv)x^2+(p^2quv+pqu^2v+pqu^2v^2)x^3\\ &+(p^2q^2uv+p^3qu^2v-p^2q^2u^2v+p^2q^2uv^2)x^4\\ &+(p^3q^2u^2v+p^2q^3uv^2+p^3q^2u^2v^2-p^2q^2u^3v^2)x^5+p^3q^3u^2v^2x^6;\\ B =& -1+(pqu+pquv)x^2+p^2quvx^3+(p^2q^2uv+p^3qu^2v-p^2q^2u^2v+p^2q^2uv^2)x^4\\ &+(p^3q^2u^2v+p^2q^3uv^2+p^3q^2u^2v^2)x^5+p^3q^3u^2v^2x^6. \end{aligned}$$

Proof. We define the following g.f.'s:

$$B_{P_2} = \sum_{n\geq 1} \sum_{\substack{\sigma\in K_n(P_2)\\ \sigma_n=1}} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)};$$

$$C_{P_2} = \sum_{n\geq 2} \sum_{\substack{\sigma\in K_n(P_2)\\ \sigma_{n-1}=1}} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)};$$

$$D_{P_2} = \sum_{n\geq 1} \sum_{\substack{\sigma\in K_n(P_2)\\ \sigma_1=1}} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)}.$$

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in K_n(P_2)$. When $n = 0$, the empty permutation ε contributes a value of 1 to A_{P_2}. When $n = 1$, the only permutation with the single element 1 contributes the term uvx to A_{P_2}. For the lengths $n = 2$ and $n = 3$, there are no king permutations. For $n \geq 4$, to avoid the POP P_2, we must have $\sigma_n = 1$, or $\sigma_{n-1} = 1$, or $\sigma_1 = 1$. Otherwise, the element 1, along with some elements to its right, will result in an occurrence of P_2. Note that both B_{P_2} and D_{P_2} count the permutation 1. Hence, we have

$$A_{P_2} = 1 + B_{P_2} + C_{P_2} + D_{P_2} - uvx,$$

which results in

$$A_{P_2} - B_{P_2} - C_{P_2} - D_{P_2} = 1 - uvx. \tag{7}$$

For $n \geq 1$, we now proceed to consider three distinct scenarios.

`Case 1:` $\sigma_n = 1$. As discussed in case (a) of Theorem 3, we derive

$$B_{P_2} = uvx + qvx\big(A_{P_2} - B_{P_2} - 1\big),$$

which leads to

$$qvxA_{P_2} - (1 + qvx)B_{P_2} = qvx - uvx. \tag{8}$$

`Case 2:` $\sigma_{n-1} = 1$. We consider a permutation $\pi = \pi_1\pi_2\cdots\pi_{n-1} \in S_{n-1}$ and insert the element 1 into π as shown in Figure 5. Consequently, we obtain $\sigma = (\pi_1 + 1)(\pi_2 + 1)\cdots(\pi_{n-2} + 1)1(\pi_{n-1} + 1)$.

To ensure that σ is a king permutation that avoids P_2, it is necessary for π to also avoid P_2. Additionally, we require that $\pi_{n-2} \neq 1$ and $\pi_{n-1} \neq 1$. This set of conditions implies that $\pi_1 = 1$.

When inserting the element 1, we need to consider two distinct cases: $\pi_{n-2} < \pi_{n-1}$ and $\pi_{n-2} > \pi_{n-1}$. To facilitate our analysis, we introduce the following notation: for $\sigma = \sigma_1\cdots\sigma_n \in K_n(P_2)$, let $A_n^1(P_2)$ (resp., $A_n^2(P_2)$) denote the set of P_2-avoiding king n-permutations where $\sigma_{n-1} < \sigma_n$ (resp., $\sigma_{n-1} > \sigma_n$); let $D_n^1(P_2)$

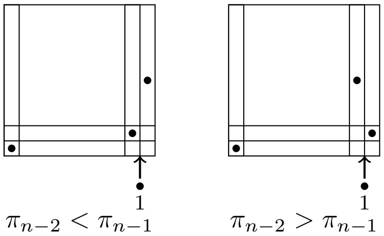


Figure 5: Related to case (b) in the proof of Theorem 4

(resp., $D_n^2(P_2)$) denote the set of P_2-avoiding king n-permutations where $\sigma_{n-1} < \sigma_n$ (resp., $\sigma_{n-1} > \sigma_n$) and $\sigma_1 = 1$. The respective g.f.'s (with five variables) are denoted $A_{P_2}^1$, $A_{P_2}^2$, $D_{P_2}^1$, and $D_{P_2}^2$. Note that

$$A_{P_2} = 1 + uvx + A_{P_2}^1 + A_{P_2}^2.$$

We will now derive $A_{P_2}^1$ and $A_{P_2}^2$.

Suppose $n \geq 4$. For $\sigma \in A_n^1(P_2)$, to avoid P_2, we only need to consider the two cases of $\sigma_1 = 1$ and $\sigma_{n-1} = 1$. When $\sigma_1 = 1$, let $\sigma = 1\sigma'$. Since $\sigma_2 \neq 2$ and $\mathrm{red}(\sigma') \in A_{n-1}^1(P_2)$, we must have $\sigma_{n-1} = 2$. Hence, the corresponding g.f. for $\mathrm{red}(\sigma')$ is C_{P_2}. The element $\sigma_1 = 1$ contributes a factor of pux. Then, the corresponding g.f. for σ is $puxC_{P_2}$. When $\sigma_{n-1} = 1$, clearly, the corresponding g.f. for σ is C_{P_2}. Consequently, we obtain

$$A_{P_2}^1 = puxC_{P_2} + C_{P_2},$$

or

$$A_{P_2}^1 = (1 + pux)C_{P_2}. \tag{9}$$

Similarly, for $\sigma \in A_n^2(P_2)$, we have to focus only on two cases: $\sigma_1 = 1$ and $\sigma_n = 1$. When $\sigma_1 = 1$, let $\sigma = 1\sigma'$. Since $\sigma_2 \neq 2$ and $\mathrm{red}(\sigma') \in A_{n-1}^1(P_2)$, we must have $\sigma_n = 2$. Hence, the corresponding g.f. for $\mathrm{red}(\sigma')$ is $B_{P_2} - uvx$. The element $\sigma_1 = 1$ contributes a factor of pux. Then, the corresponding g.f. for σ is $pux(B_{P_2} - uvx)$. When $\sigma_n = 1$, clearly, the corresponding g.f. for σ is $B_{P_2} - uvx$. Consequently, we have

$$A_{P_2}^2 = pux(B_{P_2} - uvx) + B_{P_2} - uvx,$$

or

$$A_{P_2}^2 = (1 + pux)(B_{P_2} - uvx). \tag{10}$$

Next, we will derive $D_{P_2}^1$ and $D_{P_2}^2$.

Suppose $n \geq 4$. For $\sigma \in D_n^1(P_2)$, because $\sigma_1 = 1$, we have $\sigma_2 \neq 2$ and hence the g.f. for $\sigma_2\sigma_3 \cdots \sigma_n$ is $A_{P_2}^1 - D_{P_2}^1$. The element $\sigma_1 = 1$ contributes a factor of pux. Hence,

$$D_{P_2}^1 = pux(A_{P_2}^1 - D_{P_2}^1). \tag{11}$$

Similarly, for $\sigma \in D^2_n(P_2)$, we have

$$D^2_{P_2} = pux(A^2_{P_2} - D^2_{P_2}). \tag{12}$$

When $n = 4$, the only permutation, $\sigma = 2413$, contributes the term $p^2qu^2v^2x^4$ to C_{p_2}. We consider two subcases for $n \geq 5$:

`Subcase (i):` π is a king permutation. Given that $\pi_1 = 1$, the contribution of the inserted element 1 to $D^1_{P_2}$ (resp., $D^2_{P_2}$) is qx (resp., px). Consequently, the corresponding g.f for σ is

$$\sum_{n\geq 5} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)} = qxD^1_{P_2} + pxD^2_{P_2}.$$

`Subcase (ii):` π is almost a king permutation, that is, if $|\pi_{i+1} - \pi_i| > 1$ for each $1 \leq i \leq n-3$ and $|\pi_{n-1} - \pi_{n-2}| = 1$. We consider the following subcases.

When $\pi_{n-2} < \pi_{n-1}$ (as shown schematically in Figure 5 to the left), we can conclude that $\pi_{n-2} = 2$. Otherwise, the element 2, along with the three elements to its right, will form an occurrence of the pattern P_2. Then, π can be expressed as $\pi = 1\pi_2 \cdots \pi_{n-3}23$. The g.f. for $\pi_2\pi_3 \cdots \pi_{n-3}$ is $A_{P_2} - 1$. Hence, after inserting the element 1 into π, the contribution of the last three elements 314 in σ is pq^2vx^3 and the contribution of the first number $\sigma_1 = 2$ is pux. Thus, the corresponding g.f. for σ is

$$\sum_{n\geq 5} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)} = p^2q^2uvx^4(A_{P_2} - 1).$$

Likewise, when $\pi_{n-2} > \pi_{n-1}$ (as shown schematically in Figure 5 to the right), it is necessary that $\pi_{n-1} = 2$. Hence, π can be expressed as $\pi = 1\pi_2 \cdots \pi_{n-3}32$. The contribution of the last two elements 13 in σ is $pqvx^2$, and the contribution of the first number $\sigma_1 = 2$ is pux. The g.f. for $\sigma_2\sigma_3 \cdots \sigma_{n-2}4$ is $B_{P_1} - uvx$. Consequently, the corresponding g.f. for σ is

$$\sum_{n\geq 5} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)} = p^2quvx^3(B_{P_2} - uvx).$$

Therefore, when $n \geq 4$,

$$C_{P_2} = p^2qu^2v^2x^4 + qxD^1_{P_2} + pxD^2_{P_2} + p^2q^2uvx^4(A_{P_2} - 1) + p^2quvx^3(B_{P_2} - uvx),$$

which leads to

$$p^2q^2uvx^4A_{P_2} + p^2quvx^3B_{P_2} - C_{P_2} + qxD^1_{P_2} + pxD^2_{P_2} = p^2q^2uvx^4. \tag{13}$$

By simultaneously solving Equations (9)–(13), we obtain

$$\begin{aligned} &\left(p^2q^2uvx^4 + \frac{pqux^2}{1+pux}\right)A_{P_2} + (p^2ux^2 + p^2quvx^3 - pqux^2)B_{P_2} - C_{P_2} \\ &\qquad = p^2u^2vx^3 + p^2q^2uvx^4 - \frac{p^2qu^3vx^4 - pqux^2}{1+pux}. \end{aligned} \tag{14}$$

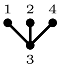
`Case 3:` $\sigma_1 = 1$. We have $\sigma = 1\sigma'$, where $\mathrm{red}(\sigma') \in A_{n-1}(P_2)$ and σ' does not begin with 2. For $n = 1$, the sole permutation results in the term uvx. Let $n \geq 2$. Clearly, king permutations either begin with the smallest element or do not. So, the g.f. for σ' is $A_{P_2} - D_{P_2} - 1$. Also, σ_1 contributes pux, since it is a left-to-right maximum and forms an ascent with σ_2. Hence, we obtain:

$$D_{P_2} = uvx + pux\big(A_{P_2} - D_{P_2} - 1\big),$$

which leads to

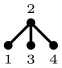
$$puxA_{P_2} - (1 + pux)D_{P_2} = pux - uvx. \tag{15}$$

By simultaneously solving Equations (7), (8), (14), and (15), we obtain the desired result. □

Corollary 4. *For* $P_3 =$ 

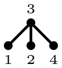
, we have $A_{P_3}(x, p, q, u, v) = A_{P_2}(x, q, p, v, u)$.

Proof. By reversing P_2-avoiding permutations, we obtain P_3-avoiding permutations and observe that ascents and descents, as well as right-to-left maxima and left-to-right maxima, are interchanged under the reversal. □

Corollary 5. *For* $Q_2 =$ 

, we have $\bar{A}_{Q_2}(x, p, q, s, t) = A_{P_2}(x, q, p, s, t)$.

Proof. By complementing P_2-avoiding permutations, we obtain Q_2-avoiding permutations and observe that ascents and descents are interchanged under the complement, while right-to-left maxima (resp., left-to-right maxima) are transformed into right-to-left minima (resp., left-to-right minima). □

Corollary 6. *For* $Q_3 =$ 

, we have $\bar{A}_{Q_3}(x, p, q, s, t) = A_{P_2}(x, p, q, t, s)$.

Proof. By reversing and complementing P_2-avoiding permutations, we obtain Q_3-avoiding permutations and observe that ascents and descents remain unchanged under the composition of these operations, while right-to-left maxima (resp., left-to-right maxima) are transformed into left-to-right minima (resp., right-to-left minima). □

We end this subsection by providing several specializations obtained by setting the respective variables in Equation (6) to 1, starting with the generating function for P_2-avoiding (equivalently, P_3-avoiding, Q_2-avoiding, or Q_3-avoiding) king permutations:

$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_2)} x^n = \frac{-1 + x + x^2 + x^4}{-1 + 2x - x^2 + x^3 + x^4}$$

$$= 1 + x + x^4 + 3x^5 + 5x^6 + 8x^7 + 15x^8 + 30x^9 + 58x^{10} + \cdots;$$

$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_2)} x^n q^{\mathrm{des}(\sigma)} = \frac{(1+x)(-1+q^2x^4+q^3x^5+qx^2(2+x))}{-1+q^3x^5(1+x)+q^2x^4(1+2x)+qx^2(2+x+x^2)};$$

$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_2)} x^n u^{\mathrm{lmax}(\sigma)} = \frac{-1-u^3x^5+u^2x^3(2+2x^2+x^3)+ux(-1+2x+x^2+2x^3+x^4)}{-1+u^2x^5(2+x)+ux^2(2+x+2x^2+x^3)};$$

$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_2)} x^n p^{\mathrm{asc}(\sigma)} q^{\mathrm{des}(\sigma)} =$$

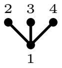
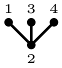
$$\frac{-1-x+2pqx^2+p(2+p)qx^3+p^2q(p+q)x^4+p^2q^2(-1+2p+q)x^5+p^3q^3x^6}{-1+2pqx^2+p^3qx^4(1+qx)^2+p^2qx^3(1+qx+q^2x^2)};$$

$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_2)} x^n u^{\mathrm{lmax}(\sigma)} v^{\mathrm{rmax}(\sigma)} =$$

$$\frac{-1-u^3v^2x^5+u^2vx^3(1+x^2+v(1+x^2+x^3))+ux(x+v^2x^3(1+x)+v(-1+x+x^2+x^3))}{-1+u^2vx^5(1+v+vx)+ux^2(1+v^2x^2(1+x)+v(1+x+x^2))};$$

$$\sum_{n\geq 0}\sum_{\sigma\in K_n(P_2)} x^n q^{\mathrm{des}(\sigma)} u^{\mathrm{lmax}(\sigma)} = \frac{(1+ux)(-1-q^2(-2+u)ux^4+q^3ux^5+qux^2(2+x))}{-1+q^3ux^5(1+ux)+qux^2(2+x+ux^2)+q^2ux^4(2+u(-1+2x))}.$$

3. Concluding Remarks

In this paper, we derive generating functions involving four statistics for POP-avoiding king permutations. In particular, from our results we learn that the POP $P_1 =$  appears to be easier to avoid than the POP $P_2 =$ , in the sense that the number of P_1-avoiding permutations is greater than that of P_2-avoiding permutations. We note that controlling additional statistics proved to be challenging in the context of pattern-avoiding king permutations. In particular, using our approach for the patterns P_i's, we were unable to control the statistics *left-to-right minima*, *right-to-left minima*, *valleys*, *peaks*, *double ascents*, and *double descents*. We leave adding additional statistics to our enumerative results as an open direction for future research.

Additionally, we state the following open problems, the first one of which is related to Remark 1. Note that the existence of an arbitrarily 321-avoiding king permutation of length $n \geq 4$ follows from the fact that 2413 is such a permutation that does not begin (resp., end) with the smallest (resp., largest) element, allowing us to use the direct sum $\oplus$ to concatenate any number of such permutations, which will result in a 321-avoiding king permutation.

Problem 1. Enumerate 123-avoiding (equivalently, 321-avoiding) king permutations. The respective sequence is $1, 1, 0, 0, 2, 6, 13, 30, 76, 198, 518, \ldots$, for $n \geq 0$.

For the following problem, recall that *Baxter permutations* are permutations that avoid the vincular patterns $2\underline{41}3$ and $3\underline{14}2$ simultaneously, where the elements corresponding to the underlined elements must stay together in any occurrence of such a pattern. Clearly, non-separable permutations are a subset of Baxter permutations. Note that the existence of an arbitrarily long Baxter king permutation of length $n \geq 6$ follows from the fact that 25314 is such a permutation that does not begin (resp., end) with the smallest (resp., largest) element, allowing us to use the direct sum $\oplus$ to concatenate any number of such permutations, which will result in a Baxter king permutation.

Problem 2. Enumerate Baxter king permutations. For $n \geq 0$, the corresponding sequence is $1, 1, 0, 0, 0, 2, 8, 32, 120, 468, 1858, \ldots$.

The following problem deals with a subset of Baxter permutations that is a superset of non-separable permutations. This set of permutations has been considered in [10] in connection with the so-called $\beta(1, 0)$-trees, which are in turn linked to planar maps. Arbitrarily long permutations in the set exist for the same reason as above for Baxter permutations.

Problem 3. Enumerate king permutations that avoid simultaneously the patterns $2\underline{41}3$ and 3142. The corresponding sequence begins, for $n \geq 0$, with $1, 1, 0, 0, 0, 1, 4, 14, 46, 151, 500, \ldots$.

Finally, the following problem deals with *Schröder permutations*, which are permutations that avoid both the patterns 4132 and 4231 simultaneously [3].

Problem 4. Enumerate king permutations that avoid simultaneously the patterns 4132 and 4231. The corresponding sequence begins, for $n \geq 0$, with $1, 1, 0, 0, 2, 8, 26, 88, 310, 1116, 4078, \ldots$.

Acknowledgements. We would like to express our gratitude to Philip B. Zhang for the useful discussions related to our paper.

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