LINEAR RELATIONS FOR OVERPARTITIONS INTO ODD PARTS

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Abstract

Let $\overline{p}_o(n)$ denote the number of overpartitions of n into odd parts. The partition function $\overline{p}_o(n)$ has been the subject of numerous recent studies, in which many explicit Ramanujan-like congruences have been discovered. In this paper, we provide three linear recurrence relations for $\overline{p}_o(n)$. Several connections with partitions into parts not congruent to $2 \pmod 4$, overpartitions, and partitions into distinct parts are presented in this context.

1. Introduction

A *partition* of a non-negative integer n is defined as a finite non-increasing sequence of positive integers whose sum is n. For example, there are seven partitions of 5:

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

We denote the number of partitions of n by $p(n)$. The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where

$$(a;q)_n = \begin{cases} 1, & \text{for} \quad n=0, \\ \displaystyle\prod_{k=1}^{n}(1-aq^{k-1}), & \text{for} \quad n>0 \end{cases}$$

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is the q-shifted factorial,

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \qquad |q|<1.$$

An *overpartition* of a non-negative integer n is a partition of n in which the first occurrence of parts of each size may be overlined. For example, there are eight overpartitions of the integer 3:

$$3,\ \ \overline{3},\ \ 2+1,\ \ \overline{2}+1,\ \ 2+\overline{1},\ \ \overline{2}+\overline{1},\ \ 1+1+1,\ \ \overline{1}+1+1.$$

We denote the number of overpartitions of n by $\overline{p}(n)$. The generating function of $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{1.1}$$

For more details on overpartitions, see [5], [9], [10], and [13].

Now consider that n has been overpartitioned into odd parts. We denote the number of such partitions by $\overline{p}_o(n)$. The series-product identity due to Lebesgue [12] represents the generating function of $\overline{p}_o(n)$:

$$\sum_{j=0}^{\infty}\frac{(-1;q)_j q^{j(j+1)/2}}{(q;q)_j} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.$$

For more details on overpartitions into odd parts, see [3] and [11].

Euler [1] presented the following recurrence relation for the partition function $p(n)$ involving pentagonal numbers:

$$\begin{aligned} p(n) &- p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15)\\ &+ \cdots + (-1)^k p(n-k(3k-1)/2) + (-1)^k p(n-k(3k+1)/2) + \cdots\\ &= \begin{cases} 1 & \text{if } n=0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Further, in 1973, Ewell [7] gave the following recurrence relation for the partition function $p(n)$:

$$\begin{aligned} p(n) &- p(n-1) - p(n-3) + p(n-6) + p(n-10) - p(n-15) - p(n-21)\\ &+ \cdots + (-1)^{\lceil j/2\rceil} p(n-j(j+1)/2) + (-1)^{\lceil j/2\rceil} p(n-j(j+1)/2) + \cdots\\ &= \begin{cases} 0 & \text{if } n \text{ is odd}\\ p_d(n) & \text{if } n \text{ is even,}\end{cases}\end{aligned}$$

where $p_d(n)$ denotes the number of partitions of n into distinct parts. Choliy, Kolitsch, and Sills [4] also gave two Euler-type recurrences for $p(n)$, stated below:

$$p(n) - p(n-1) - p(n-2) + p(n-4) + p(n-8) - p(n-9) - p(n-18) + \dots$$
$$\dots + (-1)^j p(n-j^2) + (-1)^j p(n-2j^2) + \dots = \begin{cases} 0 & \text{if } n \text{ is odd} \\ p_{do}(n) & \text{if } n \text{ is even,} \end{cases}$$

and

$$p(n) - 2p(n-1) + 2p(n-4) - \dots + (-1)^j 2p(n-j^2) + \dots = (-1)^i p_{do}(n),$$

where $p_{do}(n)$ denotes the number of partitions of n into distinct odd parts. In [14], Merca gave two linear recurrence relations for the partition function $p(n)$ using consequences of bisections of the pentagonal number theorem. He also established a relation showing that the partition functions $p(n)$ and $p_{do}(n)$ have the same parity. One of the main results is given below:

$$\sum_{k=0}^{\infty} p\left(n - \frac{G_k}{2}\right) - \sum_{k=0}^{\infty} p\left(\frac{n}{2} - \frac{k(k+1)}{8}\right) = 0,$$

with $p(x) = 0$ when x is a negative integer and G_k represents generalized pentagonal numbers given as:

$$G_k = \frac{1}{2}\lceil k/2 \rceil \left(3\lceil k/2 \rceil + (-1)^k\right).$$

Recently, da Silva and Sakai [6] derived several recurrence relations for $p(n)$, $p_{do}(n)$, $\overline{p}(n)$, the partition function with odd parts $p_o(n)$, and p_m^c, the partition function of n with parts congruent to $\pm c$ modulo m, by invoking classical identities and generating function manipulations.

Merca [15] also provided two recurrence relations for the function $\mathrm{ped}(n)$, where $\mathrm{ped}(n)$ denotes the number of partitions of n with distinct even parts and unrestricted odd parts. Its generating function is

$$\sum_{n=0}^{\infty} \mathrm{ped}(n) q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}.$$

The two recurrence relations for $\mathrm{ped}(n)$ due to Merca are the following.

Theorem 1 ([15]). *For $n \geq 0$,*

$$\sum_{j=0}^{\infty} (-1)^{\lceil j/2 \rceil} \ \mathrm{ped}(n - j(j+1)/2) = \begin{cases} 1 & \textit{if } n = k(k+1), \quad k \in \mathbb{N}_0, \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 2 ([15]). *For $n \geq 0$,*

$$\sum_{j=-\infty}^{\infty} (-1)^j \,\mathrm{ped}(n - 2j^2) = \begin{cases} 1 & \textit{if } n = k(k+1)/2, \quad k \in \mathbb{N}_0, \\ 0 & \textit{otherwise.} \end{cases}$$

Motivated by the above works, we establish three recurrence relations for $\overline{p}_o(n)$. The following recurrence relation is analogous to Euler's recurrence for $p(n)$, as well as to the recurrences for $\mathrm{ped}(n)$ and $pod(n)$.

Theorem 3. *For any integer $n \geq 0$,*

$$\sum_{k=-\infty}^{\infty} (-1)^k \overline{p}_o\left(n - \frac{k(3k+1)}{2}\right) = \begin{cases} (-1)^{\lceil m/2 \rceil}, & \text{if } n = \frac{m(3m+1)}{2},\ m \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

In the following, we state the recurrence relations for $\overline{p}_o(n)$ that involve the triangular numbers, $k(k+1)/2,\ k \in \mathbb{N}_0$, and the perfect square numbers, respectively.

Theorem 4. *For any integer $n \geq 0$,*

$$\sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} \overline{p}_o\left(n - \frac{k(k+1)}{2}\right) = \begin{cases} 1, & \text{if } n = \frac{m(m+1)}{2},\ m \in \mathbb{N}_0, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5. *For any integer $n \geq 0$,*

$$\overline{p}_o(n) + 2\sum_{k=1}^{\infty} (-1)^k \overline{p}_o(n - 2k^2) = \begin{cases} 2, & \text{if } n = m^2,\ m \in \mathbb{N}, \\ 1, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

If n is small then it is relatively easy to evaluate $\overline{p}_o(n)$, namely

$$\begin{aligned}
&\overline{p}_o(0) = 1, \\
&\overline{p}_o(1) = 2, \\
&\overline{p}_o(2) = 2\overline{p}_o(0) = 2, \\
&\overline{p}_o(3) = 2\overline{p}_o(1) = 4, \\
&\overline{p}_o(4) = 2 + 2\overline{p}_o(2) = 6, \\
&\overline{p}_o(5) = 2\overline{p}_o(3) = 8, \\
&\overline{p}_o(6) = 2\overline{p}_o(4) = 12, \\
&\overline{p}_o(7) = 2\overline{p}_o(5) = 16, \\
&\overline{p}_o(8) = 2\left(\overline{p}_o(6) - \overline{p}_o(0)\right) = 22, \\
&\overline{p}_o(9) = 2 + 2\left(\overline{p}_o(7) - \overline{p}_o(1)\right) = 30, \\
&\overline{p}_o(10) = 2\overline{p}_o(8) - \overline{p}_o(2) = 40.
\end{aligned}$$

We prove Theorems 3 – 5 in Section 2 and connections with $p_o(n)$, and other partition functions are established in Section 3.

2. Proofs of Theorems 3 – 5

Jacobi's triple product identity can be stated in terms of Ramanujan's theta function [2, p. 34] as follows:

$$(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\ b^{n(n-1)/2}. \tag{2.1}$$

Proof of Theorem 3. Replacing a by $-q^2$ and b by $-q$ in (2.1), we get

$$(q;q^3)_\infty(q^2;q^3)_\infty(q^3;q^3)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}, \tag{2.2}$$

which is Euler's pentagonal number theorem. The above identity can be rewritten as

$$(-q;-q^3)_\infty(q^2;-q^3)_\infty(-q^3;-q^3)_\infty = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{2.3}$$

Replacing a by $-q^2$ and b by q in (2.1), we derive

$$(-q;-q^3)_\infty(q^2;-q^3)_\infty(-q^3;-q^3)_\infty = \sum_{n=-\infty}^{\infty} (-1)^{\lceil n/2\rceil} q^{n(3n+1)/2}. \tag{2.4}$$

In view of (2.3) and (2.4), we arrive at

$$\sum_{n=-\infty}^{\infty} (-1)^{\lceil n/2\rceil} q^{n(3n+1)/2} = \sum_{n=0}^{\infty} \overline{p}_o(n) q^n \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{2.5}$$

Theorem 3 follows from the above identity. □

Proof of Theorem 4. By (2.1), with replacing a by $-q^3$ and b by $-q$, we obtain

$$(q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+n}.$$

The above identity can be rewritten as

$$(-q;q^2)_\infty(q^4;q^4)_\infty = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^{\lceil n/2\rceil} q^{n(n+1)/2}. \tag{2.6}$$

Replacing a by q^3 and b by q in (2.1), we derive the identities

$$(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{2.7}$$

$$\frac{(q^2;q^2)^2_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{2.8}$$

By (2.6) and (2.7), we deduce that

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \sum_{n=0}^{\infty} \overline{p}_o(n)q^n \sum_{n=0}^{\infty} (-1)^{\lceil n/2 \rceil} q^{n(n+1)/2}.$$

Theorem 4 follows from the last identity. □

Proof of Theorem 5. Replacing a and b by $-q^2$ in (2.1), we obtain

$$(q^2;q^4)_\infty(q^2;q^4)_\infty(q^4;q^4)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}.$$

Rewriting the above equation, we see that

$$(-q;q^2)_\infty(-q;q^2)_\infty(q^2;q^2)_\infty = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}. \tag{2.9}$$

By replacing a and b by q in (2.1), we obtain

$$(-q;q^2)^2_\infty(q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{2.10}$$

From (2.9) and (2.10), we have

$$\sum_{n=-\infty}^{\infty} q^{n^2} = \sum_{n=0}^{\infty} \overline{p}_o(n)q^n \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}.$$

Equating the coefficient of q^n on both sides of the above equation, we arrive at Theorem 5. □

3. Relation Connecting $\overline{p}_o(n)$ with Other Partition Functions

Let pod(n) denote the number of partitions of n wherein odd parts are distinct and even parts are unrestricted. The generating function for pod(n) is given by

$$\sum_{n=0}^{\infty} \text{pod}(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{3.1}$$

We shall prove that $\overline{p}_o(n)$ can be expressed in terms of pod(n).

Theorem 6. *For any integer $n \geq 0$,*

$$\overline{p}_o(n) = \sum_{k=0}^{\infty} \text{pod}\left(n - \frac{k(k+1)}{2}\right).$$

Proof. From (3.1), we have

$$\frac{(-q;q^2)_\infty}{(q;q^2)_\infty} = (q^2;q^2)_\infty(-q;q)_\infty \sum_{n=0}^{\infty} \text{pod}(n)q^n.$$

By (2.8), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_o(n)q^n = \left(\sum_{n=0}^{\infty} \text{pod}(n)q^n\right)\left(\sum_{n=0}^{\infty} q^{n(n+1)/2}\right).$$

The proof follows from the last identity. □

The parity of $\overline{p}_o(n)$ and Theorem 6 allows us to derive the following congruence.

Corollary 1. *For any integer $n > 0$,*

$$\sum_{k=0}^{\infty} \text{pod}\left(n - \frac{k(k+1)}{2}\right) \equiv 0 \pmod{2}.$$

The partition functions $p(n)$ and $\overline{p}_o(n)$ can be related as follows.

Theorem 7. *For any integer $n \geq 0$,*

$$\overline{p}_o(n) = \sum_{k=-\infty}^{\infty} (-1)^{\lceil k/2 \rceil} p\left(n - \frac{k(3k+1)}{2}\right).$$

Proof. From (2.2), we have

$$\frac{1}{(q;q)_\infty} = \left[\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}\right]^{-1}.$$

From (2.5), we see that

$$\sum_{n=0}^{\infty} \overline{p}_o(n)q^n = \left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(\sum_{n=-\infty}^{\infty} (-1)^{\lceil n/2 \rceil} q^{n(3n+1)/2}\right).$$

The proof follows from the above identity. □

Corollary 2. *For any integer $n > 0$,*

$$\sum_{k=-\infty}^{\infty} p\left(n - \frac{k(3k+1)}{2}\right) \equiv 0 \pmod{2}.$$

Proof. It easily follows from the Theorem 7 and the parity of $\overline{p}_o(n)$. □

Theorem 8. *For any integer $n \geq 0$,*

$$\overline{p}_o(n) = \sum_{k=-\infty}^{\infty} (-1)^k \overline{p}(n - 2k^2).$$

Proof. Considering the generating function of $\overline{p}_o(n)$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{p}_o(n) q^n &= (-q;q)_\infty (-q;q^2)_\infty \qquad (3.2) \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty} (-q;q^2)_\infty (q;q)_\infty. \end{aligned}$$

Using Equations (2.9) and (1.1), we see that

$$\sum_{n=0}^{\infty} \overline{p}_o(n) q^n = \left(\sum_{n=0}^{\infty} \overline{p}(n) q^n \right) \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} \right).$$

The proof follows from the final equation. □

The Equation (3.3) below appears in Theorem 2.12 of [11], while (3.4) is due to Hemanthkumar and Chandankumar [8]:

$$\sum_{n=0}^{\infty} \overline{p}_o(2n+1) q^n = 2 \frac{(q^2;q^2)_\infty (q^8;q^8)^2{}_\infty}{(q;q)^2{}_\infty (q^4;q^4)_\infty} \qquad (3.3)$$

and

$$\sum_{n=0}^{\infty} \overline{p}_o(2n) q^n = \frac{(q^4;q^4)^5{}_\infty}{(q;q)^2{}_\infty (q^2;q^2)_\infty (q^8;q^8)^2{}_\infty}. \qquad (3.4)$$

Theorem 9. *For any non-negative integer n,*

$$\overline{p}_o(2n+1) = 2 \sum_{k=0}^{\infty} \overline{p}\left(n - 2k(k+1)\right).$$

Proof. Rearranging (3.3), using the generating function of $\overline{p}(n)$ and (2.8), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{p}_o(2n+1) q^n &= 2 \sum_{n=0}^{\infty} \overline{p}(n) q^n \sum_{n=0}^{\infty} q^{\frac{4n(n+1)}{2}}, \\ &= 2 \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \overline{p}(n - 2k(k+1)) \right) q^n. \end{aligned}$$

Equating the coefficient of q^n on both sides of the last equation, we arrive at the result. □

Theorem 10. *For any non-negative integer n,*

$$\overline{p}_o(2n) = \overline{p}(n) + 2\sum_{k=0}^{\infty}\overline{p}(n-2k^2).$$

Proof. Rearranging Equations (3.4) and (1.1), we have

$$\sum_{n=0}^{\infty}\overline{p}_o(2n)q^n = \frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}\sum_{n=0}^{\infty}\overline{p}(n)q^n.$$

Invoking (2.10), we get

$$\sum_{n=0}^{\infty}\overline{p}_o(2n)q^n = \sum_{n=-\infty}^{\infty}q^{2n^2}\sum_{n=0}^{\infty}\overline{p}(n)q^n.$$

Equating the coefficient of q^n on both sides of the above equation completes the proof. □

Let $P_2(n)$ denote the partition function of n with parts not congruent to 2 (mod 4). The generating function of $P_2(n)$ is given by

$$\sum_{n=0}^{\infty}P_2(n)q^n = \frac{(q^2;q^4)_\infty}{(q;q)_\infty}.$$

Let $\overline{q}(n)$ denote the bipartition function of n into distinct parts. The generating function for $\overline{q}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{q}(n)q^n = (-q;q)_\infty^2.$$

Theorem 11. *For any positive integer n,*

$$\overline{p}_o(n) = \sum_{k=0}^{\infty}P_2\left(n-\frac{k(k+1)}{2}\right).$$

Proof. Considering the generating function of $\overline{p_o}(n)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_o(n)q^n &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\\ &= \frac{(q^2;q^4)_\infty}{(q;q)_\infty}\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.\end{aligned}$$

Invoking (2.8) in the above equation, we get

$$\sum_{n=0}^{\infty} \overline{p}_o(n)q^n = \sum_{n=0}^{\infty} P_2(n)q^n \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}},$$
$$\sum_{n=0}^{\infty} \overline{p}_o(n)q^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} P_2\left(n - \frac{k(k+1)}{2}\right)\right) q^n.$$

Equating the coefficient of q^n on both sides of the above equation completes the proof. □

The parity of $\overline{p}_o(n)$ and Theorem 11 allow us to derive the following congruence.

Corollary 3. *For any integer $n > 0$,*

$$\sum_{k=0}^{\infty} P_2\left(n - \frac{k(k+1)}{2}\right) \equiv 0 \pmod{2}.$$

The following result is a relation connecting the partition functions $\overline{q}(n)$ and $p(n)$.

Theorem 12. *For any positive integer n,*

$$\overline{q}(n) = \sum_{k=0}^{\infty} p\left(n - \frac{k(k+1)}{2}\right).$$

Proof. Consider the generating function of $\overline{p_o}(n)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{p}_o(n)q^n &= (-q;q^2)_\infty(-q;q)_\infty \\ &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}\right) \\ &= \frac{1}{(q;q)_\infty} \times \frac{1}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}.\end{aligned}$$

Rewriting the above equation, we have

$$(-q^2;q^2)_\infty \sum_{n=0}^{\infty} \overline{p}_o(n)q^n = \sum_{n=0}^{\infty} p(n)q^n \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}},$$
$$\sum_{n=0}^{\infty} \overline{q}(n)q^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} p\left(n - \frac{k(k+1)}{2}\right)\right) q^n.$$

Equating the coefficient of q^n on each side of the previous equation completes the proof. □

Let $p_d(n)$ denote the number of partitions of n into distinct parts. The number of partitions of n into distinct odd parts is denoted by $p_{do}(n)$. The generating functions for $p_d(n)$ and $p_{do}(n)$ are given by

$$\sum_{n=0}^{\infty} p_d(n)q^n = (-q;q)_\infty$$

and

$$\sum_{n=0}^{\infty} p_{do}(n)q^n = (-q;q^2)_\infty.$$

It is easy to see that

$$\overline{p}_o(n) = \sum_{k=0}^{n} p_d(k)p_{do}(n-k).$$

Theorem 13. *For any integer $n \geq 0$,*

$$\sum_{k=-\infty}^{\infty} (-1)^k \overline{p}_o\left(n - \frac{k(3k+1)}{2}\right) = \sum_{k=-\infty}^{\infty} (-1)^k p_{do}(n - k(3k+1)).$$

Proof. Rewriting (3.2), we get

$$(-q;q^2)_\infty(-q;q)_\infty(q;q)_\infty = (-q;q^2)_\infty(q^2;q^2)_\infty,$$

which implies that

$$\left(\sum_{n=0}^{\infty} \overline{p}_o(n)q^n\right)\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}\right) = \left(\sum_{n=0}^{\infty} p_{do}(n)q^n\right) \times \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)}\right).$$

□

By using Theorem 3 and Theorem 13, we obtain the following recurrence relation for $p_{do}(n)$.

Corollary 4. *For any integer $n \geq 0$,*

$$\sum_{k=-\infty}^{\infty} (-1)^k p_{do}(n - k(3k+1)) = \begin{cases} (-1)^{\lceil m/2 \rceil}, & \text{if } n = \frac{m(3m+1)}{2},\ m \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 14. *For any integer $n \geq 0$,*

$$\sum_{k=-\infty}^{\infty} (-1)^{\lceil k/2 \rceil} \overline{p}_o\left(n - \frac{k(k+1)}{2}\right) = \sum_{k=-\infty}^{\infty} (-1)^k p_d(n - k(3k+1)).$$

Proof. Rewriting relation (3.2), we see that

$$(-q;q)_\infty(-q;q^2)_\infty(q;q^2)_\infty(q^4;q^4)_\infty = (-q;q)_\infty(q^2;q^2)_\infty,$$

which implies that

$$\left(\sum_{n=0}^{\infty}\overline{p}_o(n)q^n\right)\left(\sum_{n=-\infty}^{\infty}(-1)^{\lceil n/2\rceil}q^{n(n+1)/2}\right) = \left(\sum_{n=0}^{\infty}p_d(n)q^n\right) \times \left(\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)}\right),$$

where we have used (2.6). □

By using Theorem 4 and Theorem 14, we obtain the following recurrence relation for $p_d(n)$.

Corollary 5. *For any integer $n \geq 0$,*

$$\sum_{k=-\infty}^{\infty}(-1)^k p_d(n-k(3k+1)) = \begin{cases} 1, & \text{if } n = \frac{m(m+1)}{2},\ m \in \mathbb{N}_0, \\ 0, & \text{otherwise,} \end{cases}$$

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