



A Q -WEIGHTED ANALOGUE OF THE TROLLOPE-DELANGE FORMULA

Aleksei Minabutdinov

Department of Mathematics and RiskLab, ETH Zürich, Zurich, Switzerland
aminabutdinov@ethz.ch

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Abstract

Let $s(n)$ denote the number of 1's in the dyadic (binary) representation of a positive integer n , and let $S(n) = s(1) + s(2) + \cdots + s(n-1)$. The Trollope–Delange formula is a classic result that represents the sequence $S(n)$ in terms of the Takagi function. This work extends the result by introducing a q -weighted analogue of $s(n)$, deriving a variant of the Trollope–Delange formula for this generalization. We show that for $1/2 < |q| < 1$, nondifferentiable Takagi–Landsberg functions appear, whereas for $|q| > 1$, the resulting functions are differentiable almost everywhere. We further show how the result can be used to obtain limiting curves describing fluctuations in the ergodic theorem for the dyadic odometer.

–This paper is dedicated to the memory of my teacher, Professor Anatoly Vershik. His personality, mathematical brilliance, vast knowledge, versatility, optimism, and curiosity have been deeply inspiring to me and to all who surrounded him.

1. Introduction

Let $s(n)$ denote the number of 1's in the binary (dyadic) representation of a positive integer n , and define the sequence

$$S(n) = \sum_{k=1}^{n-1} s(k).$$

The Trollope–Delange formula, established by Trollope in 1968 [24], is a fundamental result that relates the sequence $S(n)$ to a continuous, 1-periodic, nowhere differentiable function \tilde{F}_1 as follows:

$$\frac{1}{n}S(n) = \frac{1}{2} \log_2 n + \frac{1}{2} \tilde{F}_1(\log_2 n). \quad (1)$$

The Takagi function $\mathcal{T} : \mathbb{R} \rightarrow [0, +\infty)$ is defined by

$$\mathcal{T}(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \tau(2^n x),$$

where $\tau(x) = \text{dist}(x, \mathbb{Z})$ is the distance from $x \in \mathbb{R}$ to the nearest integer. It is well known (see, e.g., [9]) that \tilde{F}_1 can be expressed in terms of \mathcal{T} as

$$\tilde{F}_1(t) = 1 - t - 2^{1-t} \mathcal{T}(2^{-(1-t)}) \quad \text{for } 0 \leq t \leq 1. \tag{2}$$

Delange generalized this result to number systems with base $m \geq 3$ in 1975 [6]. Various other extensions followed, involving exponential, power, and binomial sums (see [15], [9], [2]).

A further generalization considers weighted digital sums with real-valued weights $\gamma = (\gamma_0, \gamma_1, \dots)$. For n written in binary representation as $n = \omega_0 + \omega_1 2 + \omega_2 2^2 + \dots$ with digits $\omega_i \in \{0, 1\}$, define $s(n, \gamma)$ by $s(n, \gamma) = \gamma_0 \omega_0 + \gamma_1 \omega_1 + \gamma_2 \omega_2 + \dots$. For the constant weight sequence $\gamma = (1, 1, 1, \dots)$, the definition reduces to the standard sum-of-digits function: $s(n, \gamma) = s(n)$. In 2005, an asymptotic version of the Trollope–Delange formula was derived for $S(n, \gamma) = \sum_{k=0}^{n-1} s(k, \gamma)$. For a real number c , let $[c]$ denote the integer part of c . The authors of [18] showed that there exists a continuous 1-periodic function $\tilde{G}_\gamma : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$S(n, \gamma) = \frac{n}{2} \sum_{i=0}^{\lfloor \log_2 n \rfloor} \gamma_i + n \tilde{G}_\gamma(\log_2 n) + o(n), \tag{3}$$

if and only if $\lim_{i \rightarrow \infty} \gamma_i = \tilde{\gamma}$ exists. Moreover, the $o(n)$ -term is zero if and only if $\gamma_i = \tilde{\gamma}$ for all i , and the function \tilde{G}_γ on $[0, 1]$ can be represented by

$$\tilde{G}_\gamma(x) = -\frac{\tilde{\gamma}}{2} \sum_{i=-1}^{\infty} \frac{\tau(2^{x+i})}{2^{x+i}}.$$

Unlike Equation (1), Equation (3) includes an unknown term $o(n)$. If $\lim_{i \rightarrow \infty} \gamma_i = 0$, then the function \tilde{G}_γ is identically zero, and $S(n, \gamma) - \frac{n}{2} \sum_{i=0}^{\lfloor \log_2 n \rfloor} \gamma_i$ is described only by this remainder term $o(n)$. If $\lim_{i \rightarrow \infty} \gamma_i = \infty$, then \tilde{G}_γ is not defined. Equation (3) was later generalized to bases $m > 2$ and other moments in [11]. Special cases such as $\gamma_n = (-1)^n$, $n \geq 0$, were studied in [14], yielding exact expressions for $S(\cdot, (-1)^n)$. The case $\gamma_n = (1/2)^n$ was studied in [18] in connection with the van der Corput sequence.

This work focuses on the q -weighted case $\gamma_n := q^{n+1}$ for a real q . Let $k \in \mathbb{N}_0$ be given by its dyadic expansion $k = \omega_0 + \omega_1 2 + \omega_2 2^2 + \dots$ with digits $\omega_i \in \{0, 1\}$. We denote

$$s_q(k) := s(k, (q^{i+1})_{i \geq 0}) = \sum_{i \geq 0} \omega_i q^{i+1}$$

and define

$$S_q(n) := S(n, (q^{i+1})_{i \geq 0}) = \sum_{k=0}^{n-1} s_q(k).$$

In Section 2, we show that for $|q| > \frac{1}{2}$, an exact generalized Trollope–Delange-type formula holds, involving periodic Takagi–Landsberg functions. If $1/2 < |q| < 1$, our main result, Theorem 1, refines Equation (3), providing an explicit formula for the $o(n)$ term in the case of q -weights. If $|q| > 1$, the result seems to be uncovered in the literature. Interestingly, in this case, almost everywhere differentiable functions arise¹. In Section 3, we consider complex-valued q . Sections 4 further shows how our results relate to dynamical systems, providing a description of limiting curves for the sequence $s_q(n)$. This result allows us to describe fluctuations of ergodic averages for the weighted-sum-of-digits function for the dyadic odometer in Section 5.

1.1. Takagi–Landsberg Functions

Let a be a real parameter such that $|a| < 1$. The Takagi–Landsberg function $\mathcal{T}_a : \mathbb{R} \rightarrow [0, +\infty)$ is defined by

$$\mathcal{T}_a(x) = \sum_{n=0}^{\infty} a^n \tau(2^n x), \tag{4}$$

where, as before, $\tau(x) = \text{dist}(x, \mathbb{Z})$ is the distance from $x \in \mathbb{R}$ to the nearest integer. It is easy to see that the series in Equation (4) converges uniformly², and thus defines a continuous function \mathcal{T}_a for $|a| < 1$. The family of 1-periodic functions $\{\mathcal{T}_a\}_a$ proposed in [17] can be considered a direct generalization of the well-known Takagi function \mathcal{T} , introduced by T. Takagi [23], which is obtained when a is set equal to $1/2$. For $|a| \geq \frac{1}{2}$, the functions \mathcal{T}_a are nowhere differentiable, but for $|a| < \frac{1}{2}$, they are differentiable almost everywhere³; this follows from [13] (see also the surveys [2] and [16]). The functions \mathcal{T}_a have been studied in several works; see [1, 4, 22].

From Definition (4), it follows that the function \mathcal{T}_a on the interval $[0, 1]$ satisfies the following de Rham functional equations:

$$\begin{cases} \mathcal{T}_a(x/2) = a\mathcal{T}_a(x) + x/2, \\ \mathcal{T}_a((x+1)/2) = a\mathcal{T}_a(x) + \frac{1-x}{2}. \end{cases} \tag{5}$$

Such a system of equations uniquely determines the function on dyadic rationals (that is, on points of the form $x = \frac{n}{2^k}$ with $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$; for dyadic rational

¹This contrasts, for example, with the Trollope case, where the Takagi function appears. More precisely, we show that nondifferentiable Takagi–Landsberg functions arise for $1/2 < |q| \leq 1$.

²In several works, see for example [10], more general sums of the form $\sum_{n=0}^{\infty} c_n \tau(2^n x)$ were considered where $\sum_{n=0}^{\infty} |c_n| < \infty$. Such a class of functions is called the Takagi class.

³In particular, $\mathcal{T}_{1/4}(x) = x(1-x)$.

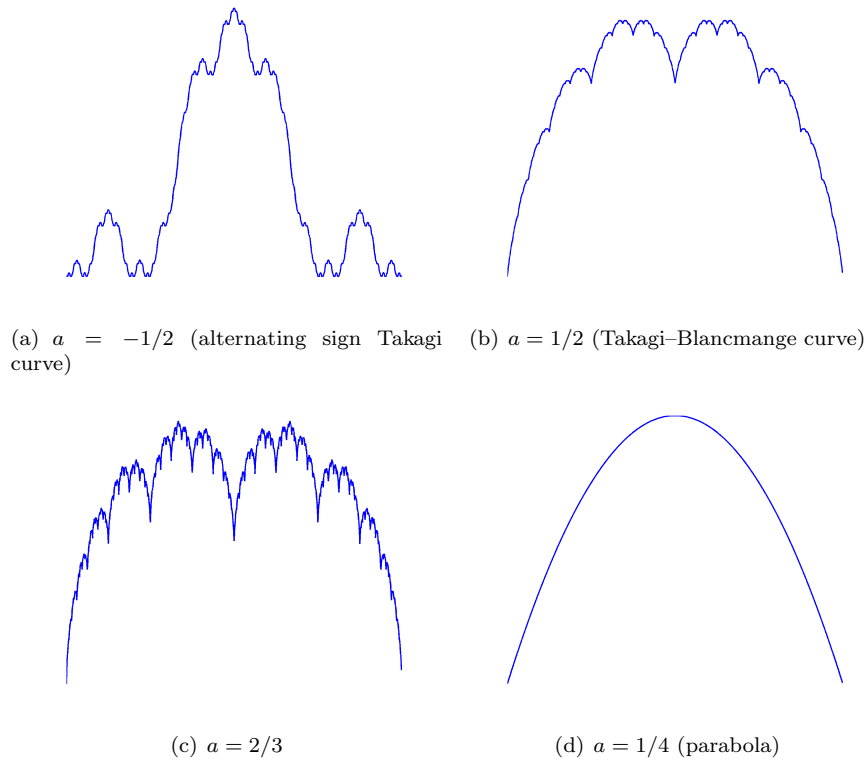


Figure 1: Takagi-Landsberg curves for different values of the parameter a

points we canonically choose the dyadic representation ending in all zeros⁴). The function can be continuously extended to the whole interval $[0, 1]$ if a contraction argument applies. More precisely, using Banach’s fixed-point theorem, it was shown in [3] (see also [8] and [9]) that any system of functional equations of the form

$$\begin{cases} f(x/2) = a_0 f(x) + g_0(x), \\ f((x+1)/2) = a_1 f(x) + g_1(x), \end{cases} \tag{6}$$

uniquely defines a continuous function on $[0, 1]$ provided that $\max\{|a_0|, |a_1|\} < 1$ and that the consistency condition⁵,

$$a_0 \frac{g_1(1)}{1 - a_1} + g_0(1) = a_1 \frac{g_0(0)}{1 - a_0} + g_1(0), \tag{7}$$

holds.

⁴Under this convention, at any dyadic rational x , the series in Equation (4) is finite irrespective of a .

⁵This condition ensures that the equations remain consistent at $x = 1/2$.

2. Main Result

Let $s_q(n)$, $q \in \mathbb{R}$, be the q -weighted sum of digits in the binary representation of a positive integer $n = \sum_{i \geq 0} \omega_i 2^i$, equal to $\sum_{i \geq 0} \omega_i q^{i+1}$, and let $S_q(n)$ denote $\sum_{j=1}^{n-1} s_q(j)$. For $q = 1$, we have $S_1(n) = S(n)$, and the classical Trollope–Delange formula in Equation (1) holds. Our result generalizes this to the q -weighted sums, as stated in the following theorem.

Theorem 1. *Let $|q| > 1/2$ and $q \neq 1$. Set $a = 1/(2q)$. For any $n \in \mathbb{N}$, the following expression holds:*

$$\frac{1}{n} S_q(n) = \frac{q}{2} \left(\frac{1 - q^{\lfloor \log_2 n \rfloor + 1}}{1 - q} - q^{\lfloor \log_2 n \rfloor} \hat{F}_q(\log_2 n) \right), \tag{8}$$

where the 1-periodic function \hat{F}_q is given by

$$\hat{F}_q(u) = 2^{1-u} \mathcal{T}_a(2^{-(1-u)}), \quad u \in [0, 1].$$

To prove Equation (8), several approaches can be used. Our approach follows Girgensohn [9]. The idea is to find the functional equations satisfied by the sequence $S_q(n)$, which leads to the function \hat{F}_q . We shall use the following lemma.

Lemma 1 ([9], Lemma 1). *Let $G : \mathbb{N} \rightarrow \mathbb{R}$ be a function. For $n \in \mathbb{N}$, set $x = \frac{n - p_n}{p_n} \in [0, 1)$ and define the function F by*

$$F(x) = F\left(\frac{n - p_n}{p_n}\right) := G(n).$$

The function F is well-defined at dyadic-rational points of the interval $[0, 1)$ if and only if $G(2n) = G(n)$ for all $n \in \mathbb{N}$.

Proof of Theorem 1. For $k \in \mathbb{N}$, set $p = 2^{k-1}$. Note that

$$s_q(2j) = q s_q(j), \tag{9}$$

$$s_q(2j + 1) = q s_q(j) + q, \tag{10}$$

$$s_q(j + p) = s_q(j) + q^k, \quad j = 0, 1, \dots, p - 1, \tag{11}$$

$$s_q(j + p) = s_q(j) - q^k(1 - q), \quad j = p, p + 1, \dots, 2p - 1. \tag{12}$$

Let $[x]$ denote the integer part of x , and let $\{x\}$ denote its fractional part. Set $k_n = [\log_2 n]$ and $u_n = \{\log_2 n\}$. Following the approach in [9], denote⁶ by $p_n = p(n) = 2^{k_n}$ the largest power to which 2 must be raised to get a number not exceeding n , and let $r_n = q^{\log_2(p_n)} = q^{k_n}$.

⁶We try to stick to the short notation p_n whenever it is sensible, that is, unless we need to consider expressions such as $p(n + p(n))$.

It is easy to see that

$$\begin{aligned} p(2n) &= 2p(n), \\ p(n + p(n)) &= 2p(n), \\ p(n + 2p(n)) &= 2p(n). \end{aligned}$$

One can verify that for any $n \in \mathbb{N}$, the following relations hold⁷:

$$S_q(n + 2p_n) = S_q(n) + S_q(2p_n) + nq^{k_n+2}, \tag{13}$$

$$S_q(n + p_n) = S_q(n) + (2q - 1)S_q(p_n) - (n - p_n)q^{k_n+1}(1 - q) + qp_n, \tag{14}$$

$$S_q(2n) = 2qS_q(n) + nq. \tag{15}$$

The last of these relations implies that at points of the form $p = 2^k$, the value of the function S_q is

$$S_q(p) = q \frac{1 - q^k}{1 - q} 2^{k-1} = q \frac{1 - q^k}{1 - q} \frac{p}{2}. \tag{16}$$

Define the function $G_q : \mathbb{N} \rightarrow \mathbb{R}$ by

$$G_q(n) = \frac{1}{p_n r_n} \left(S_q(n) - \frac{n}{p_n} S_q(p_n) \right). \tag{17}$$

The function $G_q(n)$ satisfies the following relations:

$$G_q(2n) = G_q(n), \tag{18}$$

$$G_q(n + p_n) = \frac{1}{2q} G_q(n) + \frac{p_n - n}{4p_n} (3 - 2q), \tag{19}$$

$$G_q(n + 2p_n) = \frac{1}{2q} G_q(n) + \frac{n}{4p_n} (2q - 1). \tag{20}$$

Relation (18) follows directly from Equation (15). We derive Relation (19). For brevity, here we write p for $p_n = p(n)$, $p(n + p)$ instead of $p(n + p(n))$, r for r_n , and

⁷For illustration, here we obtain Relation (13). Relations (14) and (15) can be obtained analogously. After subtracting and adding the same quantity $S_q(2p_n)$ to $S_q(n + 2p_n)$, we get

$$S_q(n + 2p_n) = s_q(2p_n + 1) + s_q(2p_n + 2) + \dots + s_q(2p_n + n) + S_q(2p_n).$$

Using Equation (11) with $p = 2p_n = 2^{k_n+1}$, the latter becomes $s_q(1) + s_q(2) + \dots + s_q(n) + nq^{k_n+2} + S_q(2p_n)$.

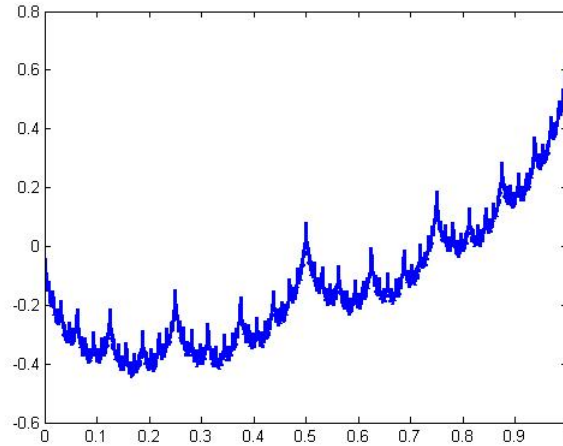


Figure 2: Graph of the function $F_{2/3}$

k for k_n .

$$\begin{aligned} G_q(n+p) &= \frac{1}{2qpr} \left(S_q(n+p) - \frac{n+p}{p(n+p)} S_q(p(n+p)) \right) = \frac{1}{2qpr} \left(S_q(n) + \right. \\ &\quad \left. + (2q-1)S_q(p) - (n-p)q^{k+1}(1-q) + pq - \frac{n+p}{2p}(2qS_q(p) + pq) \right) \\ &= \frac{1}{2q} G_q(n) + \frac{1}{2qpr} \left(\frac{p-n}{2}q + (1-q) \left(\left(\frac{n}{p} - 1 \right) S_q(p) - (n-p)q^{k+1} \right) \right) \\ &= \frac{1}{2q} G_q(n) + (3-2q) \frac{p-n}{4p}. \end{aligned}$$

Relation (20) can be derived as follows:

$$\begin{aligned} G_q(n+2p) &= \frac{1}{2qpr} \left(S_q(n+2p) - \frac{n+2p}{2p} S_q(p(n+p)) \right) \\ &= \frac{1}{2qpr} \left(S_q(n) - \frac{n}{p} S_q(p) - \frac{n}{2}q + q^{k+2} \right) \\ &= \frac{1}{2q} G_q(n) + \frac{n}{2qpr} \left((1-q) \frac{S_q(p)}{p} - \frac{q}{2} + q^{k+2} \right) \\ &= \frac{1}{2q} G_q(n) + (2q-1) \frac{n}{4p}. \end{aligned}$$

Let $x_n = x(n)$ denote

$$2^{u_n} - 1 = \frac{n - p_n}{p_n} \in [0, 1).$$

The identities

$$\frac{x(n)}{2} = \frac{n - p_n}{2p_n} = x(n + p_n), \tag{21}$$

$$\frac{x(n) + 1}{2} = \frac{n}{2p_n} = x(n + 2p_n) \tag{22}$$

also hold. By Lemma 1, the function F_q , given by $F_q(x_n) = G_q(n)$, is well-defined at dyadic-rational points of $[0, 1)$. Using that $a = 1/(2q)$, Relations (19) and (20) can be rewritten as follows for $x = x_n$:

$$\begin{cases} F_q(x/2) = aF_q(x) + (2q - 3)\frac{x}{4}, \\ F_q((x + 1)/2) = aF_q(x) + (2q - 1)\frac{x + 1}{4}. \end{cases} \tag{23}$$

System (23) satisfies the consistency condition in Equation (7) and defines F_q on dyadic rationals. The contraction condition $|a| = |1/(2q)| < 1$ allows us to extend it to the whole interval $[0, 1]$. The function $qx - \frac{1}{2}\mathcal{T}_a(x)$ also satisfies the de Rham equations in Equation (23) and thus coincides with F_q :

$$F_q(x) = qx - \frac{1}{2}\mathcal{T}_a(x) = q \left(\frac{1 + x}{2} - \mathcal{T}_a \left(\frac{x + 1}{2} \right) \right). \tag{24}$$

Using the identity $S_q(n) = r_n p_n G_q(n) + \frac{n}{p_n} S_q(p_n)$, we arrive at

$$\frac{1}{n} S_q(n) = q^{k_n} \frac{F_q(x_n)}{x_n + 1} + \frac{q}{2} \frac{1 - q^{k_n}}{1 - q}. \tag{25}$$

For brevity, we write k for k_n . Let $x = 2^u - 1$ and, accordingly, $\frac{1}{x+1} = 2^{-u}$. We can write

$$\frac{F_q(x)}{x + 1} = q \left(\frac{1}{2} - 2^{-u} \mathcal{T}_a(2^{u-1}) \right)$$

and represent Equation (25) as

$$\begin{aligned} \frac{1}{n} S_q(n) &= \frac{q}{2} \frac{1 - q^k}{1 - q} + q^{k+1} \left(\frac{1}{2} - 2^{-u} \mathcal{T}_a(2^{u-1}) \right) \\ &= \frac{q}{2} \left(\frac{1 - q^k}{1 - q} + q^k (1 - 2^{1-u} \mathcal{T}_a(2^{u-1})) \right). \end{aligned}$$

This yields the desired formula

$$\frac{1}{n} S_q(n) = \frac{q}{2} \left(\frac{1 - q^{\lceil \log_2 n \rceil + 1}}{1 - q} - q^{\lceil \log_2 n \rceil} 2^{1-u} \mathcal{T}_a(2^{u-1}) \right).$$

□

Remark 1. System (23) lets one define F_q at dyadic rational points without the assumption $|a| < 1$. This results⁸ in

$$\frac{1}{n}S_q(n) = \frac{q}{2} \frac{1 - q^{k_n+1}}{1 - q} - \frac{1}{2n} \sum_{i=1}^{k_n+1} (2q)^i \tau(n/2^i). \tag{26}$$

However, the condition $|q| > 1/2$ in Theorem 1 cannot be omitted. Banach’s fixed-point theorem, which relies on the contraction principle, cannot be applied to Equation (23) without this condition. It is then an expanding mapping instead, which results in the absence of continuous solutions on the interval $[0, 1]$.

Remark 2. The special case of Equation (8) for $q = -1$ was obtained by Krüppel in 2008; see Theorem 5.1 in [14]. In that paper, the graph of the function $\mathcal{T}_{-1/2}$ was called the alternating sign Takagi curve.

Remark 3. The case $q = 1/2$ was studied in connection with the discrepancy of the van der Corput sequence. In this case it was shown (see [18], Theorem 4) that

$$\frac{1}{n}S_{1/2}(n) = \frac{1 - D_n^*}{2} = \frac{1}{2} \left(1 - \frac{1}{n} \left(1 + \sum_{j=1}^{k_n} \tau\left(\frac{n}{2^j}\right) \right) \right),$$

where D_n^* denotes the star discrepancy of the van der Corput sequence. The expression on the right-hand side can be obtained as a special case of Equation (26).

Corollary. *Let $q > 1/2$ with $q \neq 1$, and set $a = 1/(2q)$. The following generalized Trollope–Delange formula holds.*

$$\frac{1}{n}S_q(n) = \frac{q}{2} \left(\frac{1 - q^{\log_2 n}}{1 - q} + q^{\log_2 n} \tilde{F}_q(\log_2 n) \right),$$

where the 1-periodic function \tilde{F}_q is given by

$$\tilde{F}_q(u) = \frac{1 - q^{1-u}}{1 - q} - q^{-u} 2^{1-u} \mathcal{T}_a(2^{-(1-u)}), \quad u \in [0, 1]. \tag{27}$$

Proof. We use the representation derived in Theorem 1,

$$\frac{1}{n}S_q(n) = \frac{q}{2} \left(\frac{1 - q^k}{1 - q} + q^k (1 - 2^{1-u} \mathcal{T}_a(2^{u-1})) \right),$$

where, as above, $u = \{\log_2 n\}$. Using also the fact that $q^k = q^{\log_2 n} q^{-u}$, we obtain the desired formula:

$$\frac{1}{n}S_q(n) = \frac{q}{2} \left(\frac{1 - q^{\log_2 n}}{1 - q} + q^{\log_2 n} \left(\frac{1 - q^{1-u}}{1 - q} - 2^{1-u} q^{-u} \mathcal{T}_a(2^{u-1}) \right) \right).$$

□

⁸It is straightforward to check that $\mathcal{T}_a(n/(2p_n)) = \frac{1}{a^{k_n+1}} \sum_{i=1}^{k_n+1} a^i \tau(n/2^i)$. The expression on the right-hand side of Equation (25) makes sense for any $q \equiv \frac{1}{2a} \neq 1$. Then we can use Equation (25) to obtain Equation (26).

Figure 3 further illustrates graphs of the generalized Trollope–Delange function \tilde{F}_q for different values of the weight parameter q ; in the case $q = 1$ we use Equation (2), which can also be obtained by taking the limit in Equation (27). It follows from Section 1.1 that for $q > 1$ (that is, $a < 1/2$), the functions \tilde{F}_q are almost everywhere differentiable; moreover, \tilde{F}_2 is identically zero.

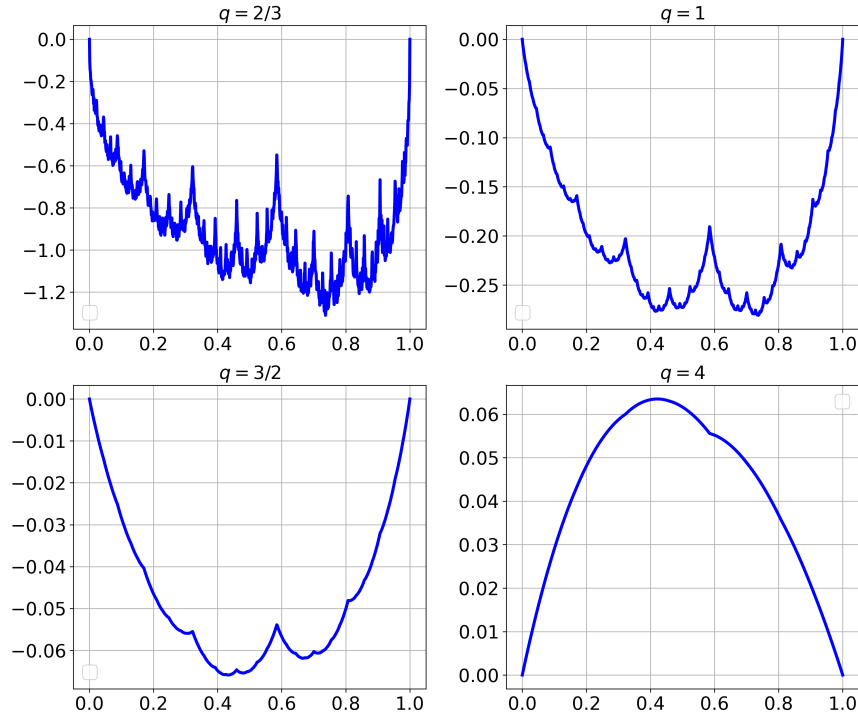


Figure 3: Graphs of the function \tilde{F}_q , $q \in \{2/3, 1, 3/2, 4\}$

3. Extension to Complex Weights

The proofs of Theorem 1 extend almost verbatim to the complex setting. Specifically, if

$$q \in \mathbb{C} \quad \text{with} \quad |q| > 1/2,$$

so that, with

$$a = \frac{1}{2q},$$

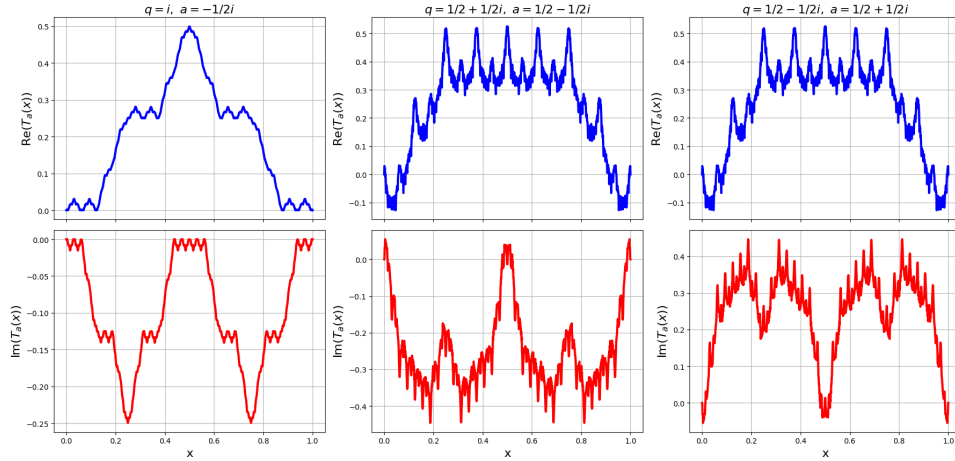


Figure 4: Takagi–Landsberg curves for complex values of a , corresponding to $q = i$, $q = \frac{1}{2} + \frac{i}{2}$, and $q = \frac{1}{2} - \frac{i}{2}$

we have $|a| < 1$, then the q -weighted sum-of-digits function

$$s_q(n) = \sum_{i \geq 0} \omega_i q^{i+1},$$

and its cumulative sum

$$S_q(n) = \sum_{k=0}^{n-1} s_q(k),$$

are well defined (with complex values), and the generalized Trollope–Delange formula

$$\frac{1}{n} S_q(n) = \frac{q}{2} \left(\frac{1 - q^{\lfloor \log_2 n \rfloor + 1}}{1 - q} - q^{\lfloor \log_2 n \rfloor} \hat{F}_q(\log_2 n) \right),$$

with

$$\hat{F}_q(u) = 2^{1-u} \mathcal{T}_a(2^{-(1-u)}), \quad a = \frac{1}{2q},$$

remains valid. Since the sequence $(|2a|^n)_{n \geq 0}$ belongs to ℓ^2 when $|a| < 1/2$, it follows from [13] that for such values of the parameter a the function \mathcal{T}_a is almost everywhere differentiable.

4. The Trollope–Delange Formula and the Limiting Curve of the Sequence $s_q(n)$

Let $(a_n)_{n \geq 0}$ be a real sequence. Extend the partial sums $S(n) = \sum_{j=0}^{n-1} a_j$ to real-valued arguments using linear interpolation and consider the sequence of continuous functions $\varphi_n : [0, 1] \rightarrow [-1, 1]$ defined by

$$\varphi_n(t) = \frac{S(t \cdot n) - t \cdot S(n)}{R_n},$$

where the normalization factor R_n is chosen to be $\max_{t \in [0,1]} |S(t \cdot n) - t \cdot S(n)|$, if this maximum is not zero, and to be 1 otherwise. Note that, due to the subtracted term $t \cdot S(n)$, the functions φ_n vanish at the endpoints of the interval $[0, 1]$.

If there exists a sequence of positive integers (l_n) such that the sequence of functions φ_{l_n} converges to a continuous function φ in the sup-metric on $[0, 1]$, then the graph of the limiting function φ is called a *limiting curve of the sequence a_n* , the sequence l_n is called a *stabilizing sequence*, and the sequence⁹ R_{l_n} is called a *normalizing sequence*. The authors of [12] proposed this definition to describe fluctuations of ergodic sums of certain functions while studying dynamical systems, particularly the Pascal adic transformation. This definition was also considered in the works of the author [19], [20]; see also the references therein. The idea is to consider the limit of the normalized difference of the partial sum and the linear function along the X and Y axes.

Consider the initial sequence a_n to be the sequence $s_q(n)$. Using the Trollope–Delange formula, we can demonstrate the following proposition, which has also been proven by an alternative method in [20].

Proposition 1. *Let $q \in \mathbb{C}$ with $|q| > \frac{1}{2}$ (in particular, this includes the real case), and set $a = 1/(2q)$. Choose any $N \in \mathbb{N}$ and fix $l = 2^N$ and $R = (2q)^{N-1}$. Consider the dyadic-rational points $t_j = \frac{j}{2^{N-1}}$ for $j = 0, 1, 2, \dots, 2^{N-1}$, and define*

$$\varphi_l(t_j) = \frac{S_q(t_j \cdot l) - t_j \cdot S_q(l)}{R}.$$

Then the following equality holds:

$$\varphi_l(t_j) = -q\mathcal{T}_a(t_j). \tag{28}$$

Proof. Define the function $\tilde{G}_q(n)$ on I by

$$\tilde{G}_q(n) = \frac{1}{R} \left(S_q(n) - \frac{n}{2p} S_q(2p) \right) = \varphi_l \left(\frac{n}{l} \right).$$

Note that r and p (and thus $rp = R$) are set here not to depend on n , which contrasts with Equation (17).

⁹What matters here is the growth rate of R_{l_n} .

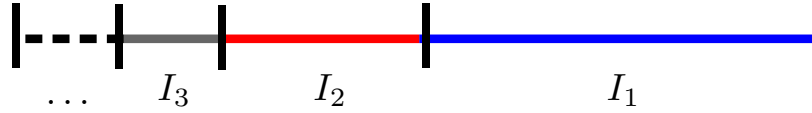


Figure 5: Partitioning of the interval $I = [1, l)$

Let $l_m = l/2^m$. Partition the interval $I = [1, l)$ into N non-overlapping sub-intervals $(I_m)_{m=1}^N$, where $I_m = [l_m, l_{m-1})$, so that $I = \bigcup_{m=1}^N I_m$; see Figure 5. To prove Equation (28) on the entire interval I , we sequentially iterate through each of the intervals I_m , $m = 1, 2, \dots, N$, and demonstrate the validity of Equation (28). Below, we verify Equation (28) first on I_1 , and then on an arbitrary interval I_m with $m \geq 1$.

On the interval I_1 (that is, for $l/2 \leq n < l$), the quantities $p = p_n$, $r = r_n$, and $x = x_n$, defined in the proof of Theorem 1, can also be given by simpler relations:

$$p = l/2 = 2^{N-1}, \quad r = q^{N-1}, \quad \frac{x + 1}{2} = \frac{n}{l}.$$

Using Relations (13) and (14), Equation (16), and Equation (24), we rewrite the expression for $\tilde{G}_q(n)$, $n \in I_1$, as follows:

$$\begin{aligned} \tilde{G}_q(n) &= \frac{1}{pr} \left(S_q(n) - \frac{nq}{2p}(2S_q(p) + p) \right) \\ &= \frac{1}{pr} \left(S_q(n) - \frac{n}{p}S_q(p) + (1 - q)\frac{n}{p}S_q(p) - q\frac{n}{2} \right) \\ &= G_q(n) - \frac{nq}{2pr} + (1 - q)\frac{n}{p}\frac{S_q(p)}{pr} \\ &= G_q(n) - \frac{qn}{2pr} + \frac{nq(1 - q^{N-1})}{2pr} \\ &= G_q(n) - \frac{qn}{2p} = F_q(x) - q\frac{x + 1}{2} = -q\mathcal{T}_a\left(\frac{n}{l}\right). \end{aligned}$$

Proceed to the interval I_m , defined by the inequalities $l_m \leq n < l_{m-1}$, where $m = 1, 2, \dots, N$. Set $N_m = N - m$, $r_m = q^{m-1}$, and $t = t_n = \frac{n}{l}$. From Equation (15) and Equation (5), it is easy to obtain that

$$\begin{aligned} S_q(l) &= (2q)^m S_q(l_m) + 2^{m-1}l_m q(1 + q + \dots + q^{m-1}), \\ \mathcal{T}_a(t) &= (2q)^m \mathcal{T}_a\left(\frac{t}{2^m}\right) - tq(1 + q + \dots + q^{m-1}). \end{aligned}$$

Then, adding and subtracting $\frac{n}{l_m} S_q(l_m)$, we can show that the following holds:

$$\begin{aligned} \tilde{G}_q(n) &= \frac{1}{(2q)^{N-1}} \left(S_q(n) - \frac{n}{l} S_q(l) \right) \\ &= \frac{1}{(2q)^{N-1}} \left(S_q(n) + \frac{n}{l_m} S_q(l_m) - \frac{n}{l_m} S_q(l_m) - \frac{n}{2^m l_m} S_q(l) \right) \\ &= \frac{1}{(2q)^{N-1}} \left((2q)^{N-m} G_q(n) + \frac{n}{l_m} S_q(l_{m-1}) \right. \\ &\quad \left. - \frac{n}{2^m l_m} \left((2q)^m S_q(l_m) + 2^{m-1} l_m (q + \dots + q^m) \right) \right) \\ &= \frac{1}{(2q)^{m-1}} G_q(n) + \frac{n}{l_m} \frac{1}{(2q)^{N-1}} \\ &\quad \times \left((1 - q^m) S_q(l_m) - \frac{2^{m-1} l_m}{2^m} (q + \dots + q^m) \right). \end{aligned}$$

Using Equation (24), which can now be written as

$$G_q(n) = F_q(x_n) = F_q(2^{m-1}t) = q(2^{m-1}t - \mathcal{T}_a(2^{m-1}t)), \quad \frac{n}{l_{m-1}} = 2^{m-1}t,$$

we then get

$$\begin{aligned} \tilde{G}_q(n) &= -\frac{1}{(2q)^{m-1}} F_q(x_n) - \frac{n}{l} \frac{1}{q^{m-1}} (q + \dots + q^m) \\ &= \frac{q}{(2q)^{m-1}} \left(2^{m-1}t - \mathcal{T}_a(2^{m-1}t) - 2^{m-1}t(1 + q + \dots + q^{m-1}) \right). \end{aligned} \tag{29}$$

This can be rewritten, by iterating Equation (5), as follows:

$$\begin{aligned} \tilde{G}_q(n) &= \frac{q}{(2q)^{m-1}} \left(2^{m-1}t - (2q)^{m-1} \mathcal{T}_a(t) + 2^{m-1}t(q + \dots + q^{m-1}) \right. \\ &\quad \left. - 2^{m-1}t(1 + q + \dots + q^{m-1}) \right) = -q \mathcal{T}_a(t) = -q \mathcal{T}_a\left(\frac{n}{l}\right). \end{aligned} \tag{30}$$

□

Define the sequence l_n as 2^n . Since, by Proposition 1, the continuous function φ_{l_n} coincides at all dyadic points of the form $t_j = \frac{j}{2^{n-1}}$, $j \in \{0, 1, 2, \dots, 2^{n-1}\}$, in the interval $[0, 1]$ with the continuous function $-q \mathcal{T}_a$, the same holds for the uniform limit $\lim_n \varphi_{l_n}$. Therefore, the function $-q \mathcal{T}_a$, where $a = 1/(2q)$, is the limiting function of the sequence $s_q(n)$ with stabilizing sequence $l_n = 2^n$ and normalizing sequence $R_n = (2q)^{n-1}$, $n \in \mathbb{N}$. We now turn to limiting curves for the dyadic odometer.

5. Limiting Curves for the Dyadic Odometer

In his 1981 work [25], Anatoly Vershik introduced into the literature the notion of adic transformations. The simplest example—the dyadic odometer—can be defined as follows. Let $\mathbb{Z}_2 = \prod_0^\infty \{0, 1\}$ be the compact additive group of dyadic integers with the Haar measure μ . The odometer transformation T is the translation $T\omega = \omega + 1$. The dyadic odometer is a well-studied dynamical system known to be uniquely ergodic, of rank one, and with a discrete dyadic spectrum. We focus on the fluctuations in the ergodic theorem for the sum-of-digits function.

Let q be a parameter (real or complex) such that

$$\frac{1}{2} < |q| < 1.$$

For $\omega = (\omega_0, \omega_1, \omega_2, \dots) \in \mathbb{Z}_2$, we denote by \mathbf{s}_q the q -weighted sum of the coordinates¹⁰ defined by

$$\mathbf{s}_q(\omega) = \sum_{i \geq 0} \omega_i q^{i+1}.$$

For $\omega \in \mathbb{Z}_2$, we define the partial sums

$$S_{q,\omega}(n) = \sum_{j=0}^{n-1} \mathbf{s}_q(T^j\omega).$$

The Birkhoff–Khinchin ergodic theorem states that for μ -almost every ω

$$\lim_{n \rightarrow \infty} \frac{1}{n} S_{q,\omega}(n) = \mathbb{E} \mathbf{s}_q.$$

To analyze fluctuations around this mean behavior, the limiting curves approach considers deviations of the form

$$\varphi_{q,\omega}^{l,R}(t) = \frac{S_{q,\omega}(t \cdot l) - t \cdot S_{q,\omega}(l)}{R}, \quad t \in [0, 1].$$

Proposition 1 can be used to prove the following result.

Theorem 2 ([20]). *For μ -almost every ω , there exists a stabilizing sequence $l_n = l_n(\omega)$ and a normalizing sequence $R_n = R_n(\omega)$ such that $\varphi_{q,\omega}^{l_n,R_n}$ converges in the sup-metric on $[0, 1]$ to the function $-\mathcal{T}_a$ with $a = \frac{1}{2q}$.*

6. Open Questions

In this paper, we derived a generalized exact Trollope–Delange formula describing the behavior of the sequence $\frac{1}{n} S_q(n)$ for $|q| > 1/2$, which can be considered the “first

¹⁰Note that the unweighted sum-of-digits function \mathbf{s}_1 can only be defined on a subset of dyadic rationals, which has measure zero. This justifies the consideration of weighted sums.

moment” of the sequence $s_q(n)$. For $|q| \leq 1/2$, our approach gives an expression defined only on dyadic rational points. It is natural to ask whether alternative representations can be obtained in this case. Additionally, exploring higher moments such as $\sum_{j=0}^{n-1} s_q^k(j)$ for $k \geq 2$, exponential sums $\sum_{j=0}^{n-1} \exp(t s_q(j))$, and binomial sums $\sum_{j=0}^{n-1} \binom{s_q(j)}{m}$ are of significant interest. These questions are left for future research.

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