#A63 INTEGERS 26 (2026)

FORMULAS FOR THE GENERALIZED FROBENIUS NUMBER OF TRIANGULAR NUMBERS

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Received: 3/4/25, Revised: 8/5/25, Accepted: 4/30/26, Published: 6/8/26

Abstract

For $k \geq 2$, let $A = (a_1, a_2, \ldots, a_k)$ be a k-tuple of positive integers with $\gcd(A) = 1$. For a non-negative integer s, the generalized Frobenius number of A, denoted as $\mathtt{g}(A; s) = \mathtt{g}(a_1, a_2, \ldots, a_k; s)$, represents the largest integer that has at most s representations in terms of $a_1, a_2, \ldots, a_k$ with non-negative integer coefficients. In this article, we provide a formula for the generalized Frobenius number of three consecutive triangular numbers, $\mathtt{g}(t_n, t_{n+1}, t_{n+2}; s)$, which is valid for all $s \geq 0$ where t_n is given by $\binom{n+1}{2}$. Furthermore, we present a proof of Komatsu's conjecture.

1. Introduction

The purpose of this paper is to provide a confirmation of Komatsu's conjecture from [10]. He proposed a conjecture for the explicit formula for the generalized Frobenius numbers of three consecutive triangular numbers. Before reaching the conclusion, we introduce some notation and background information. Afterward, we present the full statement of Komatsu's conjecture.

Let $A = (a_1, a_2, \ldots, a_k)$ be a k-tuple of positive integers with $\gcd(a_1, \ldots, a_k) = 1$. For $n \in \mathbb{Z}_{\geq 0}$, let $\mathrm{d}(n; A) = \mathrm{d}(n; a_1, a_2, \ldots, a_k)$ be the number of representations of $a_1x_1 + a_2x_2 + \cdots + a_kx_k = n$. Its generating series is given by

$$\sum_{n \geq 0} \mathrm{d}(n; a_1, \ldots, a_k)x^n = \frac{1}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_k})}.$$

The study of the generalized Frobenius number has gained attention due to its connection to number theory and combinatorial structures. Sylvester [15] and Cayley [6] demonstrated that the function $\mathrm{d}(n; a_1, a_2, \ldots, a_k)$ can be expressed as the sum of two components: a polynomial in n of degree $k-1$ and a periodic function with a period of $a_1a_2 \cdots a_k$. Beck, Gessel, and Komatsu [1] refined this result by providing

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DOI: 10.5281/zenodo.20597423

an explicit formula for the polynomial component using Bernoulli numbers. For the two-variable case, Tripathi [17] derived a specific formula for $\mathrm{d}(n; a_1, a_2)$. Extending to three variables, Komatsu [8] showed that the periodic component can be expressed using trigonometric functions when the integers a_1, a_2, and a_3 are pairwise coprime. Additionally, Binner [4] derived a formula for the number of non-negative integer solutions to the equation $ax + by + cz = n$ and established a connection between the number of solutions and quadratic residues.

We remind readers of the well-known linear Diophantine problem introduced by Sylvester, known as the *Frobenius problem*[1]: "Given positive integers $a_1, a_2, \dots, a_k$ which are relatively prime, find the largest integer that *cannot* be expressed as a non-negative integer linear combination of these numbers." This largest integer is called the *Frobenius number* of the tuple $A = (a_1, a_2, \dots, a_k)$, and is denoted by $\mathsf{g}(A) = \mathsf{g}(a_1, a_2, \dots, a_k)$. Then, with the above notation, the Frobenius number is given by

$$\mathsf{g}(A) = \max\{n \in \mathbb{Z} \mid \mathrm{d}(n; A) = 0\}.$$

Note that if all non-negative integers can be expressed as a non-negative integer linear combination of A, then $\mathsf{g}(A) = -1$. For example, $\mathsf{g}(1, 2) = -1$. For two variables $A = (a, b) \subset \mathbb{Z}_{>0}$, it is shown by Sylvester [16] that

$$\mathsf{g}(a, b) = ab - a - b. \tag{1}$$

For instance, if $A = (a, b) = (7, 11)$, then the Frobenius number of A is given by $\mathsf{g}(7, 11) = 77 - 7 - 11 = 59$. This means that every integer $n > 59$ can be expressed as a non-negative integer linear combination of 7 and 11.

Several computational techniques and partial formulas for the Frobenius number in three variables have been discussed by Tripathi [18], although a complete closed-form formula remains unknown. Some results therein were later clarified and reformulated in [14]. Unfortunately, it is important to recognize that deriving closed-form solutions becomes more challenging as the number of variables increases beyond three ($k > 3$). Despite these difficulties, various formulas have been developed for Frobenius numbers in specific contexts or particular cases. For example, explicit formulas have been established for certain sequences, including arithmetic, geometric-like, Fibonacci, Mersenne, and triangular numbers (see [12] and references therein).

In this work, we focus on a generalization of the Frobenius number. For a given integer $s \geq 0$, let

$$\mathsf{g}(A; s) = \mathsf{g}(a_1, a_2, \dots, a_k; s) = \max\{n \in \mathbb{Z} \mid \mathrm{d}(n; A) \leq s\}$$

be the largest integer such that the number of representations by $a_1, a_2, \dots, a_k$ is at most s ways. That means all integers greater than $\mathsf{g}(A; s)$ have at least $s + 1$

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[1] It is also known as the coin problem, postage stamp problem, or Chicken McNugget problem.

representations. The integer $\mathsf{g}(A;s)$ is called *the generalized Frobenius number.* Furthermore, $\mathsf{g}(A;s)$ is well-defined (that is, bounded above) (see [7]). Notice that $\mathsf{g}(a_1,a_2,\dots,a_k;0)=\mathsf{g}(a_1,a_2,\dots,a_k)$. As a generalization of (1), for $A=(a,b)$ and $s\in\mathbb{Z}_{\ge 0}$ (see [3]), an exact formula for $\mathsf{g}(A,s)=\mathsf{g}(a,b;s)$ is given by

$$\mathsf{g}(a,b;s)=(s+1)ab-a-b.$$

In general, $\mathrm{d}\big(\mathsf{g}(A,s);A\big)\le s$, but for the case of two variables, by Theorem 1 in [3], one can see that $\mathrm{d}\big(\mathsf{g}(A,s);A\big)=s$ when $|A|=2$. While exact formulas for the generalized Frobenius number in cases where $k\ge 3$ remain unknown, specific results exist for $k=3$ in particular cases. Examples include explicit formulas for triangular numbers [10], repunits [9], and Fibonacci numbers [11]. Recently, Binner [5] derived bounds for the number of solutions to the equation $a_1x_1+a_2x_2+a_3x_3=n$ and leveraged these bounds to find $\mathsf{g}(a_1,a_2,a_3;s)$ for large values of s. Woods [19] also provided formulas and asymptotic results for the generalized Frobenius problem by utilizing the restricted partition function.

One of the special cases is to calculate the Frobenius number for three consecutive triangular numbers, where the nth triangular number t_n is defined as $t_n=\frac{n(n+1)}{2}$ for $n\ge 1$. The explicit formula for $s=0$ (the original Frobenius number) is provided by Robles-Pérez and Rosales [12],

$$\mathsf{g}(t_n,t_{n+1},t_{n+2};0)=\begin{cases}\frac{(n+1)(n+2)}{4}(3n)-1, & \text{for even } n,\\ \frac{(n+1)(n+2)}{4}(3n-3)-1, & \text{for odd } n,\end{cases}$$

and, for $s=1,2,\dots,10$, they are presented by Komatsu [10]. Furthermore, Komatsu formulated a conjecture for the explicit formula for $s\ge 0$ as follows. “For some non-negative integer s, there exists an odd integer q and integers n_j ($j=1,2,3,4$) such that

$$\mathsf{g}\left(t_n,t_{n+1},t_{n+2};s\right)+1=\begin{cases}\frac{(qn)(n+1)(n+2)}{4}, & \text{for even } n\ge n_2,\\ \frac{(qn-3)(n+1)(n+2)}{4}, & \text{for odd } n\ge n_1,\end{cases}$$

and

$$\mathsf{g}\left(t_n,t_{n+1},t_{n+2};s+1\right)+1=\begin{cases}\frac{(qn+6)(n+1)(n+2)}{4}, & \text{for even } n\ge n_4,\\ \frac{(qn+3)(n+1)(n+2)}{4}, & \text{for odd } n\ge n_3,\end{cases}$$

and so on. For some non-negative integer s', there exists an even integer q' and integers n_5 and n_6 such that

$$\mathsf{g}\left(t_n,t_{n+1},t_{n+2};s'\right)+1=\frac{(q'n)\,(n+1)(n+2)}{4}\qquad(n\ge n_5)$$

and

$$\mathsf{g}\left(t_n,t_{n+1},t_{n+2};s'+1\right)+1=\frac{(q'n+6)\,(n+1)(n+2)}{4}\qquad(n\ge n_6)$$

and so on." In other words, for many integers $s \geq 0$ (or $s' \geq 0$), we can determine $\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s+1)$ from $\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s)$ and the difference between those two is $\frac{6(n+1)(n+2)}{4}$. We make this precise in our main results, which are summarized in Theorem 1 and Theorem 2. Theorem 1 presents an explicit formula for the generalized Frobenius number involving three consecutive triangular numbers for all $s \geq 0$. Theorem 2 confirms Komatsu's conjecture and provides a precise statement for the above-mentioned phenomena.

Theorem 1. *The Frobenius number $\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s)$ is given for all $s \geq 0$ as follows:*

(i) For even $n > 6\lfloor\sqrt{s+1}\rfloor - 6$, we have

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s) = \frac{(n+1)(n+2)}{4}(q_s n + 6c_s) - 1. \tag{2}$$

(ii) For odd $n > 6\left\lfloor\frac{\sqrt{4s+5}-1}{2}\right\rfloor - 3$, we have

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s) = \frac{(n+1)(n+2)}{4}(q_s n + 6c_s - 3\delta_s) - 1. \tag{3}$$

Here the q_s, c_s, and δ_s are given by

$$q_s = 2\lfloor\sqrt{s}\rfloor + 2 + \delta_s, \quad c_s = s - \lfloor\sqrt{s}\rfloor^2 - \delta_s\lfloor\sqrt{s}\rfloor, \quad \delta_s = \begin{cases} 1, & \textit{if } s \geq \lfloor\sqrt{s}\rfloor^2 + \lfloor\sqrt{s}\rfloor, \\ 0, & \textit{otherwise.} \end{cases}$$

Some initial values of q_s, c_s, and δ_s are shown in Table 1.

s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
q_s	3	4	5	5	6	6	7	7	7	8	8	8	9	9	9	9	10	10	10	10	11
c_s	0	0	0	1	0	1	0	1	2	0	1	2	0	1	2	3	0	1	2	3	0
δ_s	1	0	1	1	0	0	1	1	1	0	0	0	1	1	1	1	0	0	0	0	1

Table 1: q_s, c_s, and δ_s when $s = 0, \ldots, 20$

For $s \geq 0$, we write for the bounds appearing in the above theorem

$$N_s^{even} := 6\lfloor\sqrt{s+1}\rfloor - 6 \qquad \text{and} \qquad N_s^{odd} := 6\left\lfloor\frac{\sqrt{4s+5}-1}{2}\right\rfloor - 3. \tag{4}$$

We also define

$$\mathbb{B} = \{n \in \mathbb{N} \mid n = k^2 \text{ or } n = k(k+1) \text{ for some } k \geq 1\} = \{1, 2, 4, 6, 9, 12, 16, \ldots\}.$$

We show later that if $s \notin \mathbb{B}$, then the condition for n in Theorem 1 becomes $n \geq N_s^{even}$ and $n \geq N_s^{odd}$.

Remark 1. In Komatsu's conjecture, two types of linear coefficients appear in the formula: odd integers q and even integers q', depending on the structure of the value s. In Theorem 1, both cases are unified under the formula

$$q_s = 2\lfloor\sqrt{s}\rfloor + 2 + \delta_s,$$

where δ_s determines the parity of q_s: it is odd when $\delta_s = 1$ (matching q in Komatsu's conjecture), and even when $\delta_s = 0$ (corresponding to q'). This parity behavior is closely related to the structure of the set $\mathbb{B}$. In fact, we have the following characterization: if $k \geq 1$ such that $k(k+1) \leq s < (k+1)(k+2)$, then

$$\delta_s = \begin{cases} 1 & \text{if } \ s < \ k^2, \\ 0 & \text{if } \ s \geq \ k^2. \end{cases}$$

As a consequence of Theorem 1, we prove the following result, which gives a proof of Komatsu's conjecture.

Theorem 2. *Let $s, n \in \mathbb{Z}_{\geq 0}$. Suppose that $n > N_{s+1}^{even}$ if n is even, and $n > N_{s+1}^{odd}$ if n is odd. Then, the following statements hold:*

(i) If $s+1 \notin \mathbb{B}$, we have

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s+1) - \mathsf{g}(t_n, t_{n+1}, t_{n+2}; s) = \frac{6(n+1)(n+2)}{4}.$$

(ii) If n is even and $s+1 \in \mathbb{B}$, then

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s+1) - \mathsf{g}(t_n, t_{n+1}, t_{n+2}; s) = \frac{(n-6k+6)(n+1)(n+2)}{4}.$$

(iii) If n is odd and $s+1 \in \mathbb{B}$, then

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s+1) - \mathsf{g}(t_n, t_{n+1}, t_{n+2}; s) = \begin{cases} \frac{(n-6k+9)(n+1)(n+2)}{4} & \textit{if } s+1 = k^2, \\ \frac{(n-6k+3)(n+1)(n+2)}{4} & \textit{otherwise.} \end{cases}$$

The proofs of Theorem 1 and Theorem 2 are given in Section 3.

Corollary 1. *We have*

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; 11) = \frac{(n+1)(n+2)}{4}(8n+12) - 1, \quad \textit{for } n = 12 \textit{ and } n \geq 14,$$

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; 12) = \begin{cases} \frac{(n+1)(n+2)}{4}(9n) - 1, & \textit{for even } n \geq 14, \\ \frac{(n+1)(n+2)}{4}(9n-3) - 1, & \textit{for odd } n \geq 17, \end{cases}$$

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; 13) = \begin{cases} \frac{(n+1)(n+2)}{4}(9n+6) - 1, & \textit{for even } n \geq 12, \\ \frac{(n+1)(n+2)}{4}(9n+3) - 1, & \textit{for odd } n \geq 15, \end{cases}$$

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; 14) = \begin{cases} \frac{(n+1)(n+2)}{4}(9n+12) - 1, & \textit{for even } n \geq 12, \\ \frac{(n+1)(n+2)}{4}(9n+9) - 1, & \textit{for odd } n \geq 15, \end{cases}$$

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; 15) = \begin{cases} \frac{(n+1)(n+2)}{4}(9n+18) - 1, & \textit{for even } n \geq 18, \\ \frac{(n+1)(n+2)}{4}(9n+15) - 1, & \textit{for odd } n \geq 15, \end{cases}$$

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; 16) = \frac{(n+1)(n+2)}{4}(10n) - 1, \quad \textit{for } n = 17 \textit{ and } n \geq 19,$$

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; 17) = \frac{(n+1)(n+2)}{4}(10n+6) - 1, \quad \textit{for } n = 15 \textit{ and } n \geq 17,$$

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; 18) = \frac{(n+1)(n+2)}{4}(10n+12) - 1, \quad \textit{for } n = 17, \textit{ and } n \geq 19,$$

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; 19) = \frac{(n+1)(n+2)}{4}(10n+18) - 1, \quad \textit{for } n = 20 \textit{ and } n \geq 22,$$

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; 20) = \begin{cases} \frac{(n+1)(n+2)}{4}(11n) - 1, & \textit{for even } n \geq 18, \\ \frac{(n+1)(n+2)}{4}(11n-3) - 1, & \textit{for odd } n \geq 21. \end{cases}$$

2. A Reformulation and Preliminary Lemmas

In this section, we give a reformulation of Theorem 1, which is Theorem 3. To facilitate this reformulation, we introduce the following variables: x_s^{even}, y_s^{even}, x_s^{odd}, and y_s^{odd}.

Definition 1. Let s be a non-negative integer and let k be the non-negative integer such that $s = k(k+1) + i$, for some $0 \leq i \leq 2k+1$. Then, we define integers x_s^{even}, y_s^{even}, x_s^{odd}, and y_s^{odd} as follows:

$$\big(x_s^{even}, y_s^{even}\big) = \begin{cases} \big(i, 2(k-i)\big), & \text{if } 0 \leq i \leq k, \\ \big(i-k-1, 4k-2i+3\big), & \text{if } k+1 \leq i \leq 2k+1, \end{cases}$$

$$\big(x_s^{odd}, y_s^{odd}\big) = \begin{cases} \big(2i, k-i\big), & \text{if } 0 \leq i \leq k, \\ \big(2(i-k)-1, 2k-i+1\big), & \text{if } k+1 \leq i \leq 2k+1. \end{cases}$$

s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
x_s^{even}	0	0	0	1	0	1	0	1	2	0	1	2	0	1	2	3	0	1	2	3	0
y_s^{even}	0	1	2	0	3	1	4	2	0	5	3	1	6	4	2	0	7	5	3	1	8
x_s^{odd}	0	1	0	2	1	3	0	2	4	1	3	5	0	2	4	6	1	3	5	7	0
y_s^{odd}	0	0	1	0	1	0	2	1	0	2	1	0	3	2	1	0	3	2	1	0	4

Table 2: x_s^{even}, y_s^{even}, x_s^{odd} and y_s^{odd} when $s = k(k+1) + i$ for $0 \leq i \leq 2k+1$

With the same parameters as defined in Definition 1, we can express N_s^{even} and N_s^{odd}, introduced in (4), as follows:

$$N_s^{even} = \begin{cases} 6k-6, & \text{if } 0 \le i \le k-1, \\ 6k, & \text{if } k \le i \le 2k+1, \end{cases} \quad \text{and} \quad N_s^{odd} = \begin{cases} 6k-3, & \text{if } 0 \le i \le 2k, \\ 6k+3, & \text{if } i = 2k+1. \end{cases}$$

The values of integers x_s^{even}, y_s^{even}, x_s^{odd}, and y_s^{odd} are presented in Table 2, while the values for N_s^{even} and N_s^{odd} can be found in Table 3.

i	$0, 1, \ldots, k-1$	k	$k+1, \ldots, 2k$	$2k+1$
N_s^{even}	$6k-6$	$6k$	$6k$	$6k$
N_s^{odd}	$6k-3$	$6k-3$	$6k-3$	$6k+3$

Table 3: N_s^{even} and N_s^{odd} for $k \ge 1$ when $s = k(k+1) + i$ for $0 \le i \le 2k+1$

Theorem 3. *Let s be a non-negative integer. Then, for an integer $n \ge 2$,*

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s) + 1 = \begin{cases} \frac{t_{n+1}}{2}\big((2x_s^{even} + y_s^{even} + 3)n + 6x_s^{even}\big), & \text{for even } n > N_s^{even}, \\ \frac{t_{n+1}}{2}\big((x_s^{odd} + 2y_s^{odd} + 3)n + 3x_s^{odd} - 3\big), & \text{for odd } n > N_s^{odd}. \end{cases}$$

The next section shows how Theorem 3 implies Theorem 1. To prove Theorem 3, we first introduce several lemmas.

For a positive integer n, let t_n denote the nth triangular number, which is given by $t_n = \binom{n+1}{2} = \frac{n(n+1)}{2}$. We also define

$$d_1 \coloneqq \gcd(t_{n+1}, t_{n+2}) = \begin{cases} \frac{n+2}{2}, & \text{if } n \text{ is even}, \\ n+2, & \text{if } n \text{ is odd}. \end{cases}$$

Then,

$$\frac{t_{n+1}}{d_1} = \begin{cases} n+1, & \text{if } n \text{ is even}, \\ \frac{n+1}{2}, & \text{if } n \text{ is odd}, \end{cases} \quad \text{and} \quad \frac{t_{n+2}}{d_1} = \begin{cases} n+3, & \text{if } n \text{ is even}, \\ \frac{n+3}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

Beck and Kifer [2] show the following result on $\mathsf{g}(a_1, a_2, \ldots, a_k; s)$ in terms of $\ell = \gcd(a_2, a_3, \ldots, a_k)$.

Lemma 1 ([2, Lemma 4])**.** *For $k \ge 2$, let $A = (a_1, \ldots, a_k)$ be a k-tuple of positive integers with $\gcd(A) = 1$. Let $\ell = \gcd(a_2, a_3, \ldots, a_k)$ and $a_j = \ell a'_j$ for $2 \le j \le k$. Then, for $s \ge 0$,*

$$\mathsf{g}(a_1, a_2, \ldots, a_k; s) = \ell\, \mathsf{g}\big(a_1, a'_2, a'_3, \ldots, a'_k; s\big) + a_1(\ell - 1).$$

Our main approach involves applying Lemma 1 to three consecutive triangular numbers t_n, t_{n+1}, and t_{n+2} by setting $\ell = d_1 := \gcd(t_{n+1}, t_{n+2})$. For $s \geq 0$, this leads to the expression:

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s) = d_1 \mathsf{g}\Big(t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; s\Big) + t_n(d_1 - 1).$$

By Theorem 4, it follows that

$$\mathsf{g}\Big(t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; s\Big) = \mathsf{g}\Big(\frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; x_s\Big) + y_s t_n,$$

where $(x_s, y_s) = (x_s^{\text{even}}, y_s^{\text{even}})$ for even $n > N_s^{\text{even}}$, and $(x_s, y_s) = (x_s^{\text{odd}}, y_s^{\text{odd}})$ for odd $n > N_s^{\text{odd}}$. This leads directly to the formulas presented in Theorem 3. In Section 3, we establish a connection between Theorem 3 and our main result in Theorem 1. The proof of these claims involves introducing several supporting lemmas.

Lemma 2. *For integers* $m \geq 0$*,*

$$\mathrm{d}\Big(m; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\Big) = \sum_{j=0}^{\lfloor m/t_n \rfloor} \mathrm{d}\Big(m - j t_n; \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\Big).$$

Proof. This follows immediately from the definition of d. □

The detailed proofs of the following lemmas are discussed and presented in [13].

Lemma 3 ([13, Lemma 6])**.** *Let* $a, b \in \mathbb{Z}_{>0}$ *with* $\gcd(a, b) = 1$*, and let* $i, s \in \mathbb{Z}_{\geq 0}$*. Suppose that* c *is a positive integer such that* $c \equiv 0 \pmod{a}$ *or* $c \equiv 0 \pmod{b}$*, and* $j \in \mathbb{Z}$*. Then,*

$$\mathrm{d}\big(\mathsf{g}(a, b; s) + jc; a, b\big) = i,$$

if and only if,

$$\mathsf{g}(a, b; i-1) < \mathsf{g}(a, b; s) + jc \leq \mathsf{g}(a, b; i).$$

Here we set $\mathsf{g}(a, b; -1)$ *to be* $-\infty$*.*

Lemma 4 ([13, Lemma 7])**.** *Let* $a, b \in \mathbb{Z}_{>0}$ *with* $a < b$ *and* $\gcd(a, b) = 1$*, and let* $s, k \in \mathbb{Z}_{\geq 0}$*. If* m *is an integer such that* $m > \mathsf{g}(a, b; s) + ka$*, then, for all* $j \in \mathbb{Z}_{\geq 0}$*, we have*

$$\mathrm{d}\big(m - ja; a, b\big) \geq \mathrm{d}\big(\mathsf{g}(a, b; s) + (k - j)a; a, b\big).$$

The next lemma shows that, for $s \geq 0$, the number of representations for $\mathsf{g}\Big(\frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; x_s\Big) + y_s t_n$ in terms of t_n, $\frac{t_{n+1}}{d_1}$, and $\frac{t_{n+2}}{d_1}$ is equal to s.

Lemma 5. *Let k be a non-negative integer and $i \in \{0, 1, \ldots, 2k+1\}$. We let $s := s_{k,i} = k(k+1) + i$. Then, for each even $n > N_s^{even}$ and odd $n > N_s^{odd}$,*

$$\mathrm{d}\left(\mathsf{g}\Big(\frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; x_s\Big) + y_s t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) = k(k+1) + i = s, \tag{5}$$

where $(x_s, y_s) = (x_s^{even}, y_s^{even})$ if n is even and $(x_s, y_s) = (x_s^{odd}, y_s^{odd})$ if n is odd.

Proof. Throughout the proof, we define

$$\mathsf{g}(m) := \mathsf{g}\Big(\frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; m\Big) \quad \text{and} \quad \tilde{\mathrm{d}}(m) := \mathrm{d}\Big(m; \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\Big).$$

Then, the left-hand side of (5), making use of Lemma 2, can be written as

$$\mathrm{d}\left(\mathsf{g}(x_s) + y_s t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) = \sum_{j=0}^{\lfloor(\mathsf{g}(x_s)) + y_s t_n)/t_n\rfloor} \tilde{\mathrm{d}}\big(\mathsf{g}(x_s) + (y_s - j)t_n\big). \tag{6}$$

Recall that for two variables, $\tilde{\mathrm{d}}(\mathsf{g}(m)) = m$ for all $m \geq 0$. We prove this lemma by induction on the non-negative integer k. Let $k = 0$. Hence, $i \in \{0, 1\}$. Suppose that $i = 0$. Then, $s = 0$. We have $N_0^{even} = 0$, $N_0^{odd} = -3$, and $x_0^{even} = y_0^{even} = x_0^{odd} = y_0^{odd} = 0$. From Lemma 2 and Lemma 3, we have that $\mathrm{d}\big(\mathsf{g}(0); t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\big) = 0$.

If $i = 1$, then $s = 1$ with $N_1^{even} = 0$, and $N_1^{odd} = 3$. For each even integer $n > 0$, we have

$$x_1 = x_1^{even} = 0 \quad \text{and} \quad y_1 = y_1^{even} = 1.$$

Observe that, for all even $n > 0 = N_1^{even}$,

$$\mathsf{g}(0) < \mathsf{g}(0) + t_n \leq \mathsf{g}(1).$$

Thus, by Lemma 3, $\tilde{\mathrm{d}}\big(\mathsf{g}(0) + t_n\big) = 1$. Hence, for all even numbers $n > 0$,

$$\mathrm{d}\Big(\mathsf{g}(0) + t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\Big) = \tilde{\mathrm{d}}\big(\mathsf{g}(0) + t_n\big) + \sum_{j=0}^{\lfloor(\mathsf{g}(0)+t_n)/t_n\rfloor} \tilde{\mathrm{d}}\big(\mathsf{g}(0) - jt_n\big) = 1.$$

For each odd integer $n > 3 = N_1^{odd}$, we have

$$x_1 = x_1^{odd} = 1 \quad \text{and} \quad y_1 = y_1^{odd} = 0.$$

It can be shown that, for odd $n \geq 3$, we have $\mathsf{g}(1) - t_n \leq \mathsf{g}(0)$. Therefore, by (6) and Lemma 3, we obtain that, for odd $n > 3$,

$$\mathrm{d}\Big(\mathsf{g}(1); t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\Big) = \tilde{\mathrm{d}}\big(\mathsf{g}(1)\big) + \sum_{j=1}^{\lfloor \mathsf{g}(1)/t_n\rfloor} \tilde{\mathrm{d}}\big(\mathsf{g}(1) - jt_n\big) = 1.$$

For the induction hypothesis, let $P(m)$ be the property that, for an integer $m \geq 0$ and an integer $i \in \{0, 1, \ldots, 2m+1\}$, if we set

$$s_i := m(m+1) + i,$$

then we have

$$\mathrm{d}\Big(\mathsf{g}(x_{s_i}) + y_{s_i} t_n;\, t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\Big) \;=\; s_i,$$

provided that n is even with $n > N_{s_i}^{\text{even}}$ or is odd with $n > N_{s_i}^{\text{odd}}$.

Assume that $P(k-1)$ holds. In other words, for $r \in \{0, 1, \ldots, 2k-1\}$ and $v := v_r = (k-1)k + r$, the following condition is satisfied:

$$\mathrm{d}\left(\mathsf{g}(x_v) + y_v t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) = (k-1)k + r = v,$$

for even $n > N_v^{even}$ or odd $n > N_v^{odd}$, where N_v^{even} and N_v^{odd} are shown in Table 4. The goal is to show that $P(k)$ holds. To prove this, we divide the argument into eight cases, as outlined in Table 3. We begin with the scenario when n is even.

r	$0, 1, \ldots, k-2$	$k-1$	$k, \ldots, 2k-2$	$2k-1$
N_v^{even}	$6k-12$	$6k-6$	$6k-6$	$6k-6$
N_v^{odd}	$6k-9$	$6k-9$	$6k-9$	$6k-3$

Table 4: N_v^{even} and N_v^{odd} for $k \geq 1$ when $v = (k-1)k + r$ for $0 \leq r \leq 2k-1$

`Case 1:` n is even. In this case, we consider the following subcases.

`Subcase (a):` Let $i \in \{0, 1, \ldots, k-1\}$ and $s = k(k+1) + i$. Thus, $N_s^{even} = 6k - 6$,

$$x_s^{even} = i \quad \text{and} \quad y_s^{even} = 2(k-i).$$

Observe that, by the induction hypothesis, we have, for each $r = 0, 1, \ldots, k-1$ and for $n > 6k-6$ and $x_v^{even} = r$, and $y_v^{even} = 2((k-1) - r)$. Then,

$$\begin{aligned}\sum_{j=0}^{\lfloor \frac{\mathsf{g}(r)}{t_n} \rfloor + 2(k-r-1)} \tilde{\mathrm{d}}\big(\mathsf{g}(r) + (2k-2r-2-j)t_n\big) &= \mathrm{d}\Big(\mathsf{g}(r) + 2(k-1-r)t_n; t_n, \tfrac{t_{n+1}}{d_1}, \tfrac{t_{n+2}}{d_1}\Big) \\ &= (k-1)k + r.\end{aligned}$$

Thus, it follows that

$$\mathrm{d}\left(\mathsf{g}(i) + 2(k-i)t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right)$$

$$= \tilde{\mathrm{d}}\Big(\mathsf{g}(i) + (2k-2i)t_n\Big) + \tilde{\mathrm{d}}\Big(\mathsf{g}(i) + (2k-2i-1)t_n\Big) + \sum_{j=2}^{\left\lfloor \frac{\mathsf{g}(i)}{t_n} \right\rfloor + 2k-2i} \tilde{\mathrm{d}}\Big(\mathsf{g}(i) + (2k-2i-j)t_n\Big)$$

$$= \tilde{\mathrm{d}}\Big(\mathsf{g}(i) + (2k-2i)t_n\Big) + \tilde{\mathrm{d}}\Big(\mathsf{g}(i) + (2k-2i-1)t_n\Big) + (k-1)k + i. \tag{7}$$

We claim that for even $n > 6k-6$,

$$\mathsf{g}(k-1) < \mathsf{g}(i) + (2k-2i-1)t_n < \mathsf{g}(i) + (2k-2i)t_n \leq \mathsf{g}(k). \tag{8}$$

To determine the condition for an even integer n such that $\mathsf{g}(k-1) < \mathsf{g}(i) + (2k-2i-1)t_n$, we can equivalently examine the inequality:

$$\mathsf{g}(i) - \mathsf{g}(k-1) + (2k-2i-1)t_n > 0.$$

Since n is even, $\mathsf{g}(m) = \mathsf{g}\big(\frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; m\big) = (m+1)(n+1)(n+3) - (n+1) - (n+3)$. Consequently, the above inequality becomes

$$(i-k+1)(n+1)(n+3) + (2k-2i-1)\frac{n(n+1)}{2} > 0,$$

which is equivalent to $n > 6(k-i-1)$. Therefore, the inequality holds for even $n > 6k-6$ when $i = 0$, and remains valid even for even $n \geq 6k-6$ when $i \in \{1, \ldots, k-1\}$.

On the other hand, to show that $\mathsf{g}(i) + (2k-2i)t_n \leq \mathsf{g}(k)$ for even $n > 6k-6$, it suffices to verify that

$$(k-i)(n+1)(n+3) - (2k-2i)\frac{n(n+1)}{2} \geq 0.$$

This is equivalent to $6(k-i) \geq 0$, which is always true since $i \in \{0, 1, \ldots, k-1\}$. By (8) and Lemma 3, we get

$$\tilde{\mathrm{d}}\Big(\mathsf{g}(i) + (2k-2i)t_n\Big) = \tilde{\mathrm{d}}\Big(\mathsf{g}(i) + (2k-2i-1)t_n\Big) = k.$$

Hence, (7) becomes

$$\mathrm{d}\left(\mathsf{g}(i) + 2(k-i)t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) = 2k + (k-1)k + i = k(k+1) + i = s.$$

`Subcase (b):` Let $i = k$ and $s = k(k+1) + k$. We have $N_s^{even} = 6k$, $x_s^{even} = k$, and $y_s^{even} = 0$. We obtain

$$\mathrm{d}\left(\mathsf{g}(k); t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) = \tilde{\mathrm{d}}\Big(\mathsf{g}(k)\Big) + \sum_{j=1}^{\left\lfloor \frac{\mathsf{g}(k)}{t_n} \right\rfloor} \tilde{\mathrm{d}}\Big(\mathsf{g}(k) - jt_n\Big) = k + \sum_{j=1}^{\left\lfloor \frac{\mathsf{g}(k)}{t_n} \right\rfloor} \tilde{\mathrm{d}}\Big(\mathsf{g}(k) - jt_n\Big). \tag{9}$$

We claim that, for all $n > 6k$ and $\ell = 0, 1, \ldots, k$,

$$\mathsf{g}\big(k-\ell-1\big) < \mathsf{g}(k) - (2\ell+2)t_n < \mathsf{g}(k) - (2\ell+1)t_n \le \mathsf{g}\big(k-\ell\big). \tag{10}$$

The second inequality is obvious. To establish the first condition in (10), we examine

$$(\ell+1)(n+1)(n+3) - (2\ell+2)t_n > 0.$$

This inequality is equivalent to $(2\ell+2)(n+3) - (2\ell+2)n > 0$, and hence to $6(\ell+1) > 0$, which holds for $\ell \ge 0$. Therefore, the first inequality in (10) holds. In the same way, the second inequality in (10), $\mathsf{g}(k) - (2\ell+1)t_n \le \mathsf{g}\big(k-\ell\big)$, holds if and only if $n \ge 6\ell$, which is true since $n > 6k \ge 6\ell$. Thus, (10) holds. Additionally, it is not challenging to demonstrate that $\mathsf{g}(k) \ge 2kt_n$ for all $n \ge 1$. By (10) and Lemma 3, (9) becomes

$$\mathrm{d}\left(\mathsf{g}(k); t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) = k + 2\big(k + (k-1) + \cdots + 1\big) = k + k(k+1) = s.$$

`Subcase (c):` Let $i \in \{k+1, k+2, \ldots, 2k\}$ and $s = k(k+1) + i$. In this subcase, we have $N_s^{even} = 6k$,

$$x_s^{even} = i - (k+1), \quad \text{and} \quad y_s^{even} = 4k - 2i + 3.$$

By the induction hypothesis, for each $r \in \{k, k+1, \ldots, 2k-1\}$, we have $v := (k-1)k + r$ with $x_v^{even} = r - (k-1+1) = r - k$, and $y_v^{even} = 4(k-1) - 2r + 3$. Then, for even $n > 6k - 6$,

$$\begin{aligned}(k-1)k + r &= \mathrm{d}\left(\mathsf{g}(r-k) + \big(4(k-1) - 2r + 3\big)t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right)\\ &= \sum_{j=0}^{\left\lfloor \frac{\mathsf{g}(r-k)}{t_n} \right\rfloor + (4k-2r-1)} \tilde{\mathrm{d}}\Big(\mathsf{g}(r-k) + (4k - 2r - 1 - j)t_n\Big).\end{aligned}$$

Observe that

$$\begin{aligned}&\mathrm{d}\left(\mathsf{g}(i-(k+1)) + (4k-2i+3)t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right)\\ &= \tilde{\mathrm{d}}\Big(\mathsf{g}(i-k-1) + (4k-2i+3)t_n\Big) + \tilde{\mathrm{d}}\Big(\mathsf{g}(i-k-1) + (4k-2i+2)t_n\Big)\\ &\qquad + \sum_{j=0}^{\left\lfloor \frac{\mathsf{g}(i-k-1)}{t_n} \right\rfloor + 4k-2i+1} \tilde{\mathrm{d}}\Big(\mathsf{g}(i-k-1) + (4k-2i+1-j)t_n\Big).\end{aligned} \tag{11}$$

By the induction hypothesis, since $i-1 \in \{k, k+1, \ldots, 2k-1\}$, we find that the last summation yields

$$\sum_{j=0}^{\lfloor \frac{\mathsf{g}(i-k-1)}{t_n} \rfloor + 4k-2i+1} \tilde{\mathrm{d}}\Big(\mathsf{g}(i-k-1) + (4k-2i+1-j)t_n\Big) = (k-1)k + i - 1.$$

Therefore, (11) becomes

$$\begin{aligned}\mathrm{d}&\left(\mathsf{g}(i-(k+1)) + (4k-2i+3)t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right)\\ &= \tilde{\mathrm{d}}\Big(\mathsf{g}(i-k-1) + (4k-2i+3)t_n\Big) + \tilde{\mathrm{d}}\Big(\mathsf{g}(i-k-1) + (4k-2i+2)t_n\Big)\\ &\quad + (k-1)k + i - 1.\end{aligned}$$

We claim that, for even $n > 6k$, we have

$$\mathsf{g}(k-1) < \mathsf{g}(i-k-1) + (4k-2i+2)t_n \le \mathsf{g}(k) \tag{12}$$

and

$$\mathsf{g}(k) < \mathsf{g}(i-k-1) + (4k-2i+3)t_n \le \mathsf{g}(k+1). \tag{13}$$

A direct calculation shows that $\mathsf{g}(k-1) < \mathsf{g}(i-k-1) + (4k-2i+2)t_n$ is equivalent to $n > 3(2k-i)$, which holds since $n > 6k$ and $i \le 2k$. Hence, the first inequality in (12) is satisfied for $i = k+1, \ldots, 2k$. In the same way, $\mathsf{g}(i-k-1) + (4k-2i+2)t_n \le \mathsf{g}(k)$ holds if and only if $3(4k-2i+2) \ge 0$. This condition follows since $i \le 2k$. Therefore, (12) holds for $n > 6k$. To show the left inequality in (13), we examine whether $\mathsf{g}(k) < \mathsf{g}(i-k-1) + (4k-2i+3)t_n$ holds. This is equivalent to $n > 6(2k-i+1)$. Since $i \ge k+1$ and $n > 6k$, this condition is satisfied. Note that for $i \in \{k+2, \ldots, 2k-1\}$, the inequality $n > 6(2k-i+1)$ remains valid even for even $n \ge 6k$. On the other hand, $\mathsf{g}(i-k-1) + (4k-2i+3)t_n \le \mathsf{g}(k+1)$ holds if and only if $n + 6(2k-i+2) > 0$, which is true since $i \le 2k$. Then, (13) holds. Therefore,

$$\mathrm{d}\left(\mathsf{g}(i-(k+1)) + (4k-2i+3)t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) = k(k+1) + i = s.$$

`Subcase (d):` Let $i = 2k+1$ and $s = k(k+1) + (2k+1)$. Then, $N_s^{even} = 6k$,

$$x_s^{even} = 2k+1-(k+1) = k, \quad \text{and} \quad y_s^{even} = 4k - 2(2k+1) + 3 = 1.$$

Then, by (10) in Subcase (b), it follows that, for even $n > 6k$,

$$\mathrm{d}\left(\mathsf{g}(k) + t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) = \tilde{\mathrm{d}}\Big(\mathsf{g}(k) + t_n\Big) + \tilde{\mathrm{d}}\Big(\mathsf{g}(k)\Big) + \sum_{j=1}^{\lfloor \frac{\mathsf{g}(k)}{t_n} \rfloor} \tilde{\mathrm{d}}\Big(\mathsf{g}(k) - jt_n\Big)$$

$$= \tilde{\mathrm{d}}\Big(\mathsf{g}(k) + t_n\Big) + k + k(k+1). \tag{14}$$

It remains to show that $\tilde{\mathrm{d}}\Big(\mathsf{g}(k) + t_n\Big) = k + 1$ which is equivalent to showing that

$$\mathsf{g}(k) < \mathsf{g}(k) + t_n \leq \mathsf{g}(k+1).$$

The first condition is obvious. For the second, a direct calculation shows that it is equivalent to $n \geq -6$, which is true under the assumption $n > 6k$. Therefore, for even $n > 6k$, (14) becomes

$$\mathrm{d}\left(\mathsf{g}(k) + t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) = (k+1) + k + k(k+1) = k(k+1) + (2k+1) = s.$$

Hence, we have completed the proof for the case when $n > N_s^{even}$ is even. Lastly, we prove the case when $n > N_s^{odd}$ is odd.

`Case 2:` n is odd. In this case, we consider the following subcases.

`Subcase (a):` Let $i \in \{0, 1, \ldots, k-1\}$ and $s = k(k+1) + i$. In this subcase, we have $N_s^{odd} = 6k - 3$,

$$x_s^{odd} = 2i, \quad \text{and} \quad y_s^{odd} = k - i.$$

By the induction hypothesis, for all $r \in \{0, 1, \ldots, k-1\}$, we have $v = (k-1)k + r$ with $x_v^{odd} = 2r$ and $y_v^{odd} = k - 1 - r$, so that, for odd $n > 6k - 9$,

$$\begin{aligned}\mathrm{d}\Big(\mathsf{g}(2r) + (k-1-r)t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\Big) &= \sum_{j=0}^{\lfloor \frac{\mathsf{g}(2r)}{t_n} \rfloor + (k-r-1)} \tilde{\mathrm{d}}\Big(\mathsf{g}(2r) + (k-r-1-j)t_n\Big) \\ &= (k-1)k + r.\end{aligned}$$

Observe that

$$\begin{aligned}&\mathrm{d}\left(\mathsf{g}(2i) + (k-i)t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) \\ &= \tilde{\mathrm{d}}\Big(\mathsf{g}(2i) + (k-i)t_n\Big) + \sum_{j=0}^{\lfloor \frac{\mathsf{g}(2i)}{t_n} \rfloor + k-i-1} \tilde{\mathrm{d}}\Big(\mathsf{g}(2i) + (k-i-1-j)t_n\Big) \\ &= \tilde{\mathrm{d}}\Big(\mathsf{g}(2i) + (k-i)t_n\Big) + (k-1)k + i. \end{aligned} \tag{15}$$

We need to show that, for odd $n > 6k - 3$,

$$\mathsf{g}(2k-1) < \mathsf{g}(2i) + (k-i)t_n \leq \mathsf{g}(2k). \tag{16}$$

Since n is odd, $\mathtt{g}(m) = \mathtt{g}\big(\frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; m\big) = (m+1)\frac{(n+1)(n+3)}{4} - \frac{(n+1)}{2} - \frac{(n+3)}{2}$. Thus,

$$(2i-2k+1)\frac{(n+1)(n+3)}{4} + (k-i)t_n > 0$$

is equivalent to $(2i-2k+1)(n+3)+(2k-2i)n > 0$, and hence to $n > 6k-6i-3$. Therefore, the left inequality in (16) holds since $i \geq 0$ and $n > 6k-3$. Note that for $i \in \{1,\ldots,k-1\}$, the inequality $n > 6k-6i-3$ remains valid even for odd $n \geq 6k-3$. On the other hand, the right inequality in (16), $\mathtt{g}(2i)+(k-i)t_n \leq \mathtt{g}(2k)$ holds if and only if $3(2k-2i) > 0$. Since $0 \leq i \leq k-1$, the condition is satisfied. Therefore, by (15), (16), and Lemma 3, we obtain the result,

$$\mathrm{d}\left(\mathtt{g}(2i) + (k-i)t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) = 2k + (k-1)k + i = k(k+1) + i = s.$$

`Subcase (b):` Let $i = k$ and $s = k(k+1)+k$. Then, $N_s^{odd} = 6k-3$, $x_s^{odd} = 2k$, and $y_s^{odd} = 0$. We first claim that, for all $\ell \in \{1,\ldots,k\}$ and for odd $n > 6k-3$, the inequality

$$\mathtt{g}(2k-2\ell) < \mathtt{g}(2k) - \ell t_n \leq \mathtt{g}(2k-2\ell+1) \tag{17}$$

holds. The left inequality, $\mathtt{g}(2k-2\ell) < \mathtt{g}(2k) - \ell t_n$, holds if and only if $6\ell > 0$, which is true since $\ell \geq 1$. Hence, $\mathtt{g}(2k-2\ell) < \mathtt{g}(2k) - \ell t_n$ holds for $\ell = 1,\ldots,k$. On the other hand, we have that $\mathtt{g}(2k) - \ell t_n \leq \mathtt{g}(2k-2\ell+1)$ holds if and only if $n \geq 6\ell - 3$. This condition follows since $n > 6k-3$ and $k \geq \ell$. We also observe that $\mathtt{g}(2k) \geq k t_n$ for $n \geq 1$. Therefore, by applying Lemma 3 to (17), we obtain the result that

$$\begin{aligned}\mathrm{d}\left(\mathtt{g}(2k); t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) &= \tilde{\mathrm{d}}\Big(\mathtt{g}(2k)\Big) + \sum_{j=1}^{\left\lfloor \frac{\mathtt{g}(2k)}{t_n} \right\rfloor} \tilde{\mathrm{d}}\Big(\mathtt{g}(2k) - j t_n\Big) \\ &= 2k + \Big((2k-1) + (2k-3) + \cdots + 1\Big) = 2k + k^2 = s.\end{aligned}$$

`Subcase (c):` Let $i \in \{k+1,\ldots,2k\}$ and $s = k(k+1)+i$. Thus, we have $N_s^{odd} = 6k-3$,

$$x_s^{odd} = 2(i-(k+1)) + 1 = 2i-2k-1, \quad \text{and} \quad y_s^{odd} = 2k-i+1.$$

By the induction hypothesis, if $r \in \{k, k+1, \ldots, 2k-1\}$ and $v \coloneqq (k-1)k + r$, we have $x_v^{odd} = 2(r-k)+1 = 2r-2k+1$ and $y_v^{odd} = 2(k-1)-r+1 = 2k-r-1$. Therefore, for odd $n > 6k-3 = N_v^{odd}$, we have that

$$(k-1)k + r = \mathrm{d}\left(\mathtt{g}(2r-2k+1) + (2k-r-1)t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right)$$

$$= \sum_{j=0}^{\left\lfloor \frac{\mathtt{g}(2r-2k+1)}{t_n} \right\rfloor + 2k-r-1} \tilde{\mathrm{d}}\Big(\mathtt{g}(2r-2k+1) + (2k-r-1-j)t_n\Big).$$

We consider the expression

$$\mathrm{d}\left(\mathtt{g}(2i-2k-1) + (2k-i+1)t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right)$$
$$= \tilde{\mathrm{d}}\Big(\mathtt{g}(2i-2k-1) + (2k-i+1)t_n\Big) + \sum_{j=0}^{\left\lfloor \frac{\mathtt{g}(2i-2k-1)}{t_n} \right\rfloor + 2k-i} \tilde{\mathrm{d}}\Big(\mathtt{g}(2i-2k-1) + (2k-i-j)t_n\Big).$$

Notice that $i-1 \in \{k, k+1, \ldots, 2k-1\}$. Therefore, in the last summation, by the induction hypothesis, we obtain that the right-hand side of the above equation is equal to the expression

$$\tilde{\mathrm{d}}\Big(\mathtt{g}(2i-2k-1) + (2k-i+1)t_n\Big) + (k-1)k + i - 1.$$

It remains to show that, for odd $n > 6k-3$, we have

$$\mathtt{g}(2k) < \mathtt{g}(2i-2k-1) + (2k-i+1)t_n \le \mathtt{g}(2k+1). \tag{18}$$

Again, $\mathtt{g}(2k) < \mathtt{g}(2i-2k-1) + (2k-i+1)t_n$ holds if and only $n > 3(4k-2i+1)$. Since $i \ge k+1$, we have $4k-2i+1 \le 2k-1$. Hence, we have $n > 6k-3 \ge 3(4k-2i+1)$. Note that, for $i \in \{k+2, \ldots, 2k\}$, the inequality $n > 3(4k-2i+1)$ remains valid even for $n \ge 6k-3$. Similarly,

$$\mathtt{g}(2i-2k-1) + (2k-i+1)t_n \le \mathtt{g}(2k+1)$$

holds if and only if $6(2k-i+1) \ge 0$. This condition holds for all $i \in \{k+1, k+2, \ldots, 2k\}$ and all n. Therefore, (18) holds, and we are done in this subcase.

`Subcase (d):` As in the previous subcases, we let $i = 2k+1$ and $s = k(k+1) + 2k+1 = (k+1)(k+2)-1$. We have $N_s^{odd} = 6k+3$ and $x_s^{odd} = 2k+1$, and $y_s^{odd} = 0$.

We first show that, for odd $n > 6k+3$ and $\ell \in \{1, 2, \ldots, k+1\}$, we have

$$\mathtt{g}\big(2k-2\ell+1\big) < \mathtt{g}(2k+1) - \ell t_n \le \mathtt{g}\big(2k-2\ell+2\big). \tag{19}$$

To establish the left inequality in (19), we consider whether $\mathtt{g}(2k-2\ell+1) < \mathtt{g}(2k+1) - \ell t_n$ holds. This is equivalent to $3\ell > 0$, which is true since $\ell \ge 1$. Therefore, the inequality holds. On the other hand, $\mathtt{g}(2k+1) - \ell t_n \le \mathtt{g}\big(2k-2\ell+2\big)$ holds if and only if $n \ge 6\ell - 3$. The condition is satisfied since $\ell \le k+1$ and $n > 6k+3 = 6(k+1)-3$.

One can also show that $\mathtt{g}(2k+1) \ge kt_n$ for all $n \ge 1$. Using (19) and Lemma 3, we obtain that

$$\mathrm{d}\left(\mathtt{g}(2k+1); t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\right) = \tilde{\mathrm{d}}\Big(\mathtt{g}(2k+1)\Big) + \sum_{j=1}^{\left\lfloor \frac{\mathtt{g}(2k+1)}{t_n} \right\rfloor} \tilde{\mathrm{d}}\Big(\mathtt{g}(2k+1) - jt_n\Big)$$

$$\begin{aligned} &= (2k+1) + \big(2k + 2(k-1) + \cdots + 2\big) \\ &= (2k+1) + k(k+1) = s, \end{aligned}$$

for all odd integers $n > 6k+3 = N_s^{odd}$. Hence, we have completed the proof of the induction step. ☐

Remark 2. Note that, in the proof of Lemma 5, particularly in Subcases (a) and (c) for both even and odd cases, if $s \notin \mathbb{B}$ ($i \neq 0$ or $k+1$), Equation (5) holds for even $n \geq N_s^{even}$ and odd $n \geq N_s^{odd}$.

Next, we present Theorem 4, which is a consequence of Lemma 5 and Lemma 4. Following this, we show the proof for Theorem 3.

Theorem 4. *Let s be a non-negative integer with $s = k(k+1)+i$ for some integers $k \geq 0$ and $0 \leq i \leq 2k+1$. Then, for even $n > N_s^{even}$ and odd $n > N_s^{odd}$,*

$$\begin{aligned} \mathsf{g}\Big(t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; s\Big) &= \mathsf{g}\Big(\frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; x_s\Big) + y_s t_n \\ &= (x_s+1)\frac{t_{n+2}t_{n+1}}{d_1^2} - \frac{t_{n+2}}{d_1} - \frac{t_{n+1}}{d_1} + y_s t_n, \end{aligned}$$

where $(x_s, y_s) = (x_s^{even}, y_s^{even})$ if n is even and $(x_s, y_s) = (x_s^{odd}, y_s^{odd})$ if n is odd.

Proof. Again, for convenience, we let $\mathsf{g}(m)$ and $\tilde{\mathrm{d}}(m)$ denote

$$\mathsf{g}\Big(\frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; m\Big) \quad \text{and} \quad \mathrm{d}\Big(m; \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\Big),$$

respectively. By Lemma 5, we have $\mathrm{d}\Bigg(\mathsf{g}(x_s) + y_s t_n; t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\Bigg) = s$. Therefore, by the definition of $\mathsf{g}\Big(t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; s\Big)$, it follows that

$$\mathsf{g}(x_s) + y_s t_n \leq \mathsf{g}\Big(t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; s\Big).$$

Suppose that m is an integer such that $m > \mathsf{g}(x_s) + y_s t_n$. Then, $m - y_s t_n > \mathsf{g}(x_s)$, and

$$\mathrm{d}\Big(m - y_s t_n; \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\Big) > x_s = \mathrm{d}\Big(\mathsf{g}\Big(\frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; x_s\Big); \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}\Big).$$

Therefore, $\tilde{\mathrm{d}}\big(m - y_s t_n\big) > \tilde{\mathrm{d}}\big(\mathsf{g}(x_s)\big)$. Since t_{n+1}/d_1 divides t_n for all positive integers n, following the result in Lemma 4, we have that for $j \in \mathbb{Z}_{\geq 0}$,

$$\tilde{\mathrm{d}}\Big(m - j t_n\Big) \geq \tilde{\mathrm{d}}\Big(\mathsf{g}(x_s) + (y_s - j)t_n\Big).$$

So, by comparing term by term, we obtain that

$$\begin{aligned}
&\mathrm{d}\Big(m;t_n,\frac{t_{n+1}}{d_1},\frac{t_{n+2}}{d_1}\Big)\\
&\quad=\sum_{j=0}^{\lfloor m/t_n\rfloor}\tilde{\mathrm{d}}\Big(m-jt_n\Big)\geq\sum_{j=0}^{\lfloor \mathsf{g}(x_s)/t_n\rfloor+y_s}\tilde{\mathrm{d}}\Big(m-jt_n\Big)\\
&\quad>\sum_{j=0}^{\lfloor \mathsf{g}(x_s)/t_n\rfloor+y_s}\tilde{\mathrm{d}}\Big(\mathsf{g}(x_s)+(y_s-j)t_n\Big)=\mathrm{d}\Big(\mathsf{g}(x_s)+y_st_n;t_n,\frac{t_{n+1}}{d_1},\frac{t_{n+2}}{d_1}\Big)=s.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathsf{g}\Big(t_n,\frac{t_{n+1}}{d_1},\frac{t_{n+2}}{d_1};s\Big)&=\mathsf{g}\Big(\frac{t_{n+1}}{d_1},\frac{t_{n+2}}{d_1};x_s\Big)+y_st_n\\
&=(x_s+1)\frac{t_{n+2}t_{n+1}}{d_1^2}-\frac{t_{n+2}}{d_1}-\frac{t_{n+1}}{d_1}+y_st_n. \qquad\square
\end{aligned}$$

We are now ready to prove Theorem 3.

Proof of Theorem 3. By Lemma 1 and Theorem 4, we have that, for $s\geq 0$,

$$\begin{aligned}
&\mathsf{g}(t_n,t_{n+1},t_{n+2};s)\\
&\quad=d_1\mathsf{g}\Big(t_n,\frac{t_{n+1}}{d_1},\frac{t_{n+2}}{d_1};s\Big)+t_n(d_1-1)\\
&\quad=\begin{cases}(x_s^{even}+1)\frac{t_{n+1}t_{n+2}}{d_1}+(y_s^{even}+1)t_nd_1-t_n-t_{n+1}-t_{n+2},\\ (x_s^{odd}+1)\frac{t_{n+1}t_{n+2}}{d_1}+(y_s^{odd}+1)t_nd_1-t_n-t_{n+1}-t_{n+2},\end{cases}\\
&\quad=\begin{cases}\frac{(n+1)(n+2)}{4}\Big((2x_s^{even}+y_s^{even}+3)n+6x_s^{even}\Big)-1, & \text{for even } n>N_s^{even},\\ \frac{(n+1)(n+2)}{4}\Big((x_s^{odd}+2y_s^{odd}+3)n+3x_s^{odd}-3\Big)-1, & \text{for odd } n>N_s^{odd}.\end{cases}
\end{aligned}$$

$\square$

Following Remark 2, if $s\notin\mathbb{B}$, Theorem 3 and Theorem 4 are valid for all even numbers $n\geq N_s^{even}$ and odd numbers $n\geq N_s^{odd}$.

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let $s\geq 0$ be an integer. Assume that $s=k(k+1)+i$ for some $k\geq 0$ and $i\in\{0,1,\dots,2k+1\}$.

`Case 1:` If $i\in\{0,1,\dots,k\}$, then by Definition 1 and Theorem 3, we have that

$$\big(x_s^{even},y_s^{even}\big)=\big(i,\,2(k-i)\big)\quad\text{and}\quad\big(x_s^{odd},y_s^{odd}\big)=\big(2i,\,k-i\big),$$

and the value of $\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s)$ is given by

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s) = \begin{cases} \frac{(n+1)(n+2)}{4}\Big((2i + 2(k-i) + 3)n + 6i\Big) - 1, \\ \frac{(n+1)(n+2)}{4}\Big((2i + 2(k-i) + 3)n + 3(2i) - 3\Big) - 1, \end{cases}$$
$$= \begin{cases} \frac{(n+1)(n+2)}{4}\Big((2k+3)n + 6i\Big) - 1, & \text{for even } n > N_s^{even}, \\ \frac{(n+1)(n+2)}{4}\Big((2k+3)n + 6i - 3\Big) - 1, & \text{for odd } n > N_s^{odd}. \end{cases}$$

In this case, one can see that $\lfloor\sqrt{s}\rfloor = k$ and $\delta_s = 1$ since $s = k(k+1) + i \geq k^2 + k$. It follows that

$$q_s = 2\lfloor\sqrt{s}\rfloor + 2 + \delta_s = 2k + 2 + 1 = 2k + 3,$$
$$c_s = s - \lfloor\sqrt{s}\rfloor^2 - \delta_s\lfloor\sqrt{s}\rfloor = k(k+1) + i - k^2 - k = i.$$

Thus, Equation (2) is proved for this case.

`Case 2:` If $i \in \{k+1, k+2, \ldots, 2k+1\}$, then by Definition 1 and Theorem 3, we have that

$$\big(x_s^{even}, y_s^{even}\big) = \big(i - k - 1,\, 4k - 2i + 3\big) \quad \text{and}$$
$$\big(x_s^{odd}, y_s^{odd}\big) = \big(2(i-k) - 1,\, 2k - i + 1\big),$$

and the value of $\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s)$ is given, for both even $n > N_s^{even}$ and odd $n > N_s^{odd}$, by

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s)$$
$$= \begin{cases} \frac{(n+1)(n+2)}{4}\Big(\big((2i-2k-2) + (4k-2i+3) + 3\big)n + 6(i-k-1)\Big) - 1, \\ \frac{(n+1)(n+2)}{4}\Big(\big((2i-2k-1) + (4k-2i+2) + 3\big)n + (6i-6k-3) - 3\Big) - 1, \end{cases}$$
$$= \begin{cases} \frac{(n+1)(n+2)}{4}\Big((2k+4)n + 6(i-k-1)\Big) - 1, & \text{for even } n > N_s^{even}, \\ \frac{(n+1)(n+2)}{4}\Big((2k+4)n + 6(i-k-1)\Big) - 1, & \text{for odd } n > N_s^{odd}. \end{cases}$$

Since $s = k(k+1)+i \geq k(k+1)+(k+1) = (k+1)^2$, it follows that $\lfloor\sqrt{s}\rfloor = k+1$ but $\delta_s = 0$ since $s = k(k+1) + i < (k+1)^2 + (k+1) = \lfloor\sqrt{s}\rfloor^2 + \lfloor\sqrt{s}\rfloor$. In this case, we have

$$q_s = 2\lfloor\sqrt{s}\rfloor + 2 + \delta_s = 2(k+1) + 2 + 0 = 2k + 4,$$
$$c_s = s - \lfloor\sqrt{s}\rfloor^2 - \delta_s\lfloor\sqrt{s}\rfloor = k(k+1) + i - (k+1)^2 = i - k - 1.$$

Therefore, Equation (3) follows. This concludes the proof of Theorem 1. □

Next, we give the proof of Theorem 2.

Proof of Theorem 2. Let $s \geq 0$. We divide the proof into two cases: $s+1 \notin \mathbb{B}$ and $s+1 \in \mathbb{B}$.

`Case 1:` $s+1 \notin \mathbb{B}$. Suppose that $s \geq 2$ and that there exist integers $k \geq 1$ and $i \in \{0, 1, \ldots, 2k+1\}$ such that $s = k(k+1)+i$. Since $s+1 \notin \mathbb{B}$, the integer $s+1$ is neither a square nor of the form $k(k+1)$. Therefore, exactly one of the following statements hold:

(i) $k(k+1) \leq s < s+1 \leq k(k+1)+k$.

(ii) $k(k+1)+(k+1) \leq s < s+1 \leq k(k+1)+(2k+1)$.

By the proof of Theorem 1, both cases have $\lfloor\sqrt{s+1}\rfloor = \lfloor\sqrt{s}\rfloor$ and $\delta_{s+1} = \delta_s$. Thus, $q_{s+1} = 2\lfloor\sqrt{s+1}\rfloor + 2 + \delta_{s+1} = 2\lfloor\sqrt{s}\rfloor + 2 + \delta_s = q_s$ and we have

$$\begin{aligned}
&\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s+1) - \mathsf{g}(t_n, t_{n+1}, t_{n+2}; s)\\
&= \frac{(n+1)(n+2)}{4}\Big((q_{s+1}-q_s)n + 6(c_{s+1}-c_s)\Big)\\
&= \frac{6(n+1)(n+2)}{4}\Big((s+1-\lfloor\sqrt{s+1}\rfloor^2 - \delta_{s+1}\lfloor\sqrt{s+1}\rfloor) - (s-\lfloor\sqrt{s}\rfloor^2 - \delta_s\lfloor\sqrt{s}\rfloor)\Big).
\end{aligned}$$

Therefore,

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s+1) - \mathsf{g}(t_n, t_{n+1}, t_{n+2}; s) = \frac{6(n+1)(n+2)}{4}.$$

`Case 2:` $s+1 \in \mathbb{B}$. There exists an integer $k \geq 1$ such that one of the following statements hold:

(i) $s+1 = k^2 = (k-1)k + k$ and $s = k^2 - 1 = (k-1)k + (k-1)$,

(ii) $s+1 = k(k+1) = (k-1)k + 2k$ and $s = k(k+1) - 1 = (k-1)k + (2k-1)$.

If n is even and (i) holds, then following the proof in Theorem 1, we obtain that $\lfloor\sqrt{s+1}\rfloor = k$, $\lfloor\sqrt{s}\rfloor = k-1$, $\delta_{s+1} = 0$, and $\delta_s = 1$. Thus,

$$\begin{aligned}
q_{s+1} - q_s &= \big(2\lfloor\sqrt{s+1}\rfloor + 2 + \delta_{s+1}\big) - \big(2\lfloor\sqrt{s}\rfloor + 2 + \delta_s\big)\\
&= \big(2k+2\big) - \big(2(k-1)+2+1\big) = 1,
\end{aligned}$$

and

$$\begin{aligned}
c_{s+1} - c_s &= \big(s+1-\lfloor\sqrt{s+1}\rfloor^2 - \delta_{s+1}\lfloor\sqrt{s+1}\rfloor\big) - \big(s-\lfloor\sqrt{s}\rfloor^2 - \delta_s\lfloor\sqrt{s}\rfloor\big)\\
&= \big(s+1-k^2\big) - \big(s-(k-1)^2-(k-1)\big)\\
&= 1 - k^2 + k(k-1) = 1-k.
\end{aligned}$$

If n is even and (ii) holds, we obtain $\lfloor\sqrt{s+1}\rfloor = k = \lfloor\sqrt{s}\rfloor$, $\delta_{s+1} = 1$, and $\delta_s = 0$. In this case, we have

$$q_{s+1} - q_s = (2k+2+1) - (2k+2+0) = 1$$

$$c_{s+1} - c_s = \big(s+1-k(k+1)\big) - \big(s-k^2\big) = 1-k.$$

Hence, if n is even and $s+1 \in \mathbb{B}$, then

$$\begin{aligned}&\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s+1) - \mathsf{g}(t_n, t_{n+1}, t_{n+2}; s)\\ &= \frac{(n+1)(n+2)}{4}\Big((q_{s+1}-q_s)n + 6(c_{s+1}-c_s)\Big) = \frac{(n+1)(n+2)}{4}\big(n-6k+6\big).\end{aligned}$$

If n is odd and (i) holds, similar to the case where n is even, we have $\lfloor\sqrt{s+1}\rfloor = k$, $\lfloor\sqrt{s}\rfloor = k-1$, $\delta_{s+1} = 0$, and $\delta_s = 1$. Thus, $q_{s+1} - q_s = 1$, $c_{s+1} - c_s = 1-k$, and $\delta_{s+1} - \delta_s = -1$.

If n is odd and (ii) holds, we have $\lfloor\sqrt{s+1}\rfloor = k = \lfloor\sqrt{s}\rfloor$, $\delta_{s+1} = 1$, and $\delta_s = 0$. Thus, $q_{s+1} - q_s = 1$, $c_{s+1} - c_s = 1-k$, and $\delta_{s+1} - \delta_s = 1-0 = 1$.

Hence, if n is odd and $s+1 \in \mathbb{B}$, we have that

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s+1) - \mathsf{g}(t_n, t_{n+1}, t_{n+2}; s) = \frac{(n+1)(n+2)}{4}(n-6k+9)$$

if $s+1 = k^2$, and

$$\mathsf{g}(t_n, t_{n+1}, t_{n+2}; s+1) - \mathsf{g}(t_n, t_{n+1}, t_{n+2}; s) = \frac{(n+1)(n+2)}{4}(n-6k+3)$$

if $s+1 = k(k+1)$. This completes the proof of Theorem 2. □

Remark 3. We introduce Theorem 4, a reformulation of Theorem 3, using the sequence (x_s, y_s) from Definition 1. Moreover, the proof of Theorem 4 extends beyond triangular numbers to any three integers that satisfy certain conditions. Let s be a positive integer, and assume that $s = k(k+1)+i$ for some $i \in \{0, 1, \dots, 2k+1\}$. Let $A_1, A_2, A_3 \in \mathbb{Z}_{>1}$ satisfy $\gcd(A_1, A_2, A_3) = 1$ and $A_1 \equiv 0 \pmod{A_2}$. Let $K_s^{ev} = \lfloor\sqrt{s+1}\rfloor - 1$ and $K_s^{od} = \lfloor\frac{\sqrt{4s+5}-1}{2}\rfloor$. If $2 < \frac{A_2A_3}{A_1}$ and $\frac{A_2A_3}{A_1} < 2 + \frac{1}{K_s^{ev}}$ (if $k \ge 1$, i.e., if $s \ge 2$), then we have

$$\mathsf{g}\Big(A_1, A_2, A_3; s\Big) = \mathsf{g}\Big(A_2, A_3; x_s^{even}\Big) + y_s^{even}A_1.$$

If $\frac{1}{2} < \frac{A_2A_3}{A_1} < 1$ (if $k = 0$) and $\frac{1}{2} < \frac{A_2A_3}{A_1} < \frac{K_s^{od}}{2K_s^{od}-1}$ (if $k \ge 1$), then we have

$$\mathsf{g}\Bigg(A_1, A_2, A_3; s\Bigg) = \mathsf{g}\Bigg(A_2, A_3; x_s^{odd}\Bigg) + y_s^{odd}A_1.$$

Acknowledgements. This project was supported by the Development and Promotion of Science and Technology Talents Project (DPST), Thailand. I sincerely appreciate the invaluable guidance and support of my supervisors, Prof. Henrik Bachmann and Prof. Kohji Matsumoto. I would also like to thank Prof. Takao Komatsu for his insightful comments.

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